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RESEARCH ARTICLE

Biconservative Riemannian Submanifolds in Minkowski 5-Space

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(Dedicated to Professor Bang-Yen CHEN on the occasion of his 80th birthday)

ABSTRACT

In this paper, we focus on biconservative Riemannian submanifolds with parallel normalized mean curvature vector field (PNMCV) in $\mathbb{E}_1^5$. We obtain explicit classifications for the biconservative PNMCV submanifolds with exactly two distinct principal curvatures of the shape operator along the mean curvature vector field. In particular, we investigate these submanifolds which have time-like and space-like mean curvature vector field in $\mathbb{E}_1^5$.

Keywords: *Biconservative submanifolds, submanifolds with parallel normalized mean curvature vector, Minkowski space.*
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1. Introduction

Biharmonic maps are the critical points of the bienergy functional

$$E_2 : C^\infty : (M,N) \to \mathbb{R}, \quad E_2(\psi) = \frac{1}{2}\int_M |\tau(\psi)|^2 v_g,$$

where $\tau(\psi) = \operatorname{tr} \nabla d\psi$ is the tension field of ψ and v_g is the volume element of g. By considering the first variation of E_2, it was showed that ψ is biharmonic if and only if the associated Euler-Lagrange equation

$$\tau_2(\psi) = -\Delta\tau(\psi) - \operatorname{tr} \tilde{R}(d\psi, \tau(\psi))d\psi = 0 \tag{1.1}$$

is satisfied where τ_2 is the bitension field of ψ, Δ is the rough Laplacian and $\tilde{R}$ denotes the curvature tensor field of N, [6].

The study of biharmonic submanifolds has been an active research since the well-known conjecture was proposed by B.-Y. Chen in 1991.

Chen's conjecture: Any biharmonic submanifold in the Euclidean space is minimal.

The conjecture is still open although it was proved to be true when additional geometric properties for these submanifolds were assumed, (see, for example, [4, 7, 10, 14]).

On the other hand, if $\psi : (M,g) \to (N,\tilde{g})$ is an isometric immersion satisfying the condition

$$\langle \tau_2(\psi), d\psi \rangle = 0, \tag{1.2}$$

then it is called biconservative immersion, [1, 7].

Assume that the ambient manifold N is flat. Then by splitting of the bitension field $\tau_2(\psi)$ from (1.1) with respect to its tangential and normal components, we obtain that ψ is biconservative if and only if the equation

$$n\nabla \|H\|^2 + 4\operatorname{tr} A_{\nabla^\perp_\cdot H}(\cdot) = 0 \tag{1.3}$$

is satisfied, where H is the mean curvature, $\nabla^\perp$ is the normal connection, A is the shape operator of M and n is the dimension of M, [3, 8].

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Obviously, biharmonic submanifolds are always biconservative. Therefore, biconservative submanifolds form a much bigger family that includes the biharmonic submanifolds. One interesting problem is the classification of biconservative submanifolds. There are many articles in the literature about biconservative submanifolds, (see [5], [8]-[14]). In [5], Y. Fu studied for space-like and time-like biconservative surfaces in $\mathbb{E}^3_1$. In [14], the author and N.C. Turgay studied biconservative surfaces in $\mathbb{E}^4$ with parallel normalized mean curvature vector field. They completely classified the biconservative meridian surfaces. Moreover, the author studied biconservative m-dimensional submanifolds with parallel normalized mean curvature vector field (PNMCV) in $\mathbb{E}^{m+2}$, [13]. Recently, the author and N.C. Turgay have investigated biconservative and biharmonic submanifolds of $\mathbb{E}^5$ in [10]. They have showed that Chen's Biharmonic Conjecture is true for these submanifolds.

In this paper, we investigate 3-dimensional biconservative PNMCV submanifolds in $\mathbb{E}^5_1$. We find the shape operators of such submanifolds under the assumption that their shape operator in the direction of the mean curvature vector field has two distinct principal curvatures. We investigate such submanifolds for which the mean curvature vector field is either time-like or space-like in $\mathbb{E}^5_1$ by separately.

2. Preliminaries

Let $\mathbb{E}^m_s$ denote the pseudo-Euclidean m-space with the canonical pseudo-Euclidean metric tensor $\tilde{g}$ of index s given by

$$\tilde{g} = -\sum_{i=1}^{s} dx_i^2 + \sum_{i=s+1}^{m} dx_i^2,$$

where $(x_1, x_2, \ldots, x_m)$ is a rectangular coordinate system in $\mathbb{E}^m_s$.

Consider an isometric immersion ψ from a 3-dimensional Riemannian manifold (M^3, g) into a Minkowski 5-space $\mathbb{E}^5_1$. We denote the Levi-Civita connections of M^3 and $\mathbb{E}^5_1$ by ∇ and $\tilde{\nabla}$, respectively. For any tangent vector field X, Y of M^3 in $\mathbb{E}^5_1$, the Gauss formula is given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.1}$$

where h is the the second fundamental form. For any tangent vector field X and normal vector field ξ of M^3 in $\mathbb{E}^5_1$, the Weingarten formula is given by

$$\widetilde{\nabla}_X \xi = -A_\xi(X) + \nabla^\perp_X \xi, \tag{2.2}$$

where $\nabla^\perp$ is the normal connection and A is the shape operator. Moreover, it is well known that h and A_ξ are related by

$$g(A_\xi(X), Y) = \tilde{g}(h(X, Y), \xi). \tag{2.3}$$

The Ricci tensor of M^3 denoted by Ric is defined as follows:

$$\mathrm{Ric}(X, Y) = \mathrm{tr}\ \{Z \hookrightarrow R(Z, X)Y)\} \tag{2.4}$$

for any tangent vector field X, Y, Z of M^3. For a unit vector e_i, the Ricci curvature $\mathrm{Ric}(e_i)$ is defined by $\mathrm{Ric}(e_i) = \mathrm{Ric}(e_i, e_i)$.

The scalar curvature S of M^3 is defined by

$$S = \mathrm{tr}\ (\mathrm{Ric}(e_i)\,). \tag{2.5}$$

The mean curvature vector field H of M^3 is given by

$$H = \frac{1}{3}\mathrm{tr}\, h \tag{2.6}$$

and the mean curvature function is expressed as $f = |\langle H, H\rangle|^{1/2}$. A submanifold M^3 is called minimal, if H vanishes identically.

A normal vector field η is called parallel if $\nabla^\perp_X \eta = 0$ whenever X is a tangent vector field. Assume that M^3 is not minimal, i.e., $f > 0$. If the unit normal vector field H/f is parallel, then M^3 is called a PNMCV submanifold.

Let $\tilde{R}$ and R be the curvature tensor of M^3 and $\mathbb{E}^5_1$, respectively. Since the ambient space $\mathbb{E}^5_1$ is flat, $\tilde{R} = 0$ and then the Gauss equation is given by

$$\langle R(X,Y)Z,W\rangle = \langle h(X,W),h(Y,Z)\rangle - \langle h(Y,W),h(X,Z)\rangle \tag{2.7}$$

for X,Y,Z,W tangent to M^3.

The Codazzi equation is given by

$$(\bar{\nabla}_X h)(Y,Z) = (\bar{\nabla}_Y h)(X,Z) \tag{2.8}$$

where $\bar{\nabla}h(Y,Z)$ is defined by

$$(\bar{\nabla}_X h)(Y,Z) = \nabla^{\perp}_X h(Y,Z) - h(\nabla_X Y,Z) - h(Y,\nabla_X Z)$$

for any tangent vector field X,Y and Z on M^3.

On the other hand, we suppose that ψ has parallel normalized mean curvature vector e_4. By using the Ricci equation, we have $\left(\tilde{R}(X,Y)e_4\right)^T = 0$ which gives that all the shape operators of ψ can be diagonalized simultaneously.

3. Biconservative PNMCV Riemannian Submanifolds

In this section, we focus biconservative PNMCV submanifolds in $\mathbb{E}^5_1$.

Let ψ be a biconservative PNMCV isometric immersion of a Riemannian 3-manifold M^3 into $\mathbb{E}^5_1$.

Remark 3.1. To avoid trivial cases, we assume that M^3 is completely contained in $\mathbb{E}^5_1$.

We choose the frame field $\{e_4,e_5\}$ of the normal bundle of M^3 with e_4 is defined by $e_4 = \frac{H}{f}$. Since e_4 is parallel and co-dimension of M^3 is 2, e_5 is also parallel. Therefore, we have

$$\nabla^{\perp}_X e_4 = 0, \quad \nabla^{\perp}_X e_5 = 0 \tag{3.1}$$

for any tangent vector field X. Then, we obtain that biconservativity condition (1.3) is equivalent to

$$A_{e_4}(\operatorname{grad} f) = -\varepsilon_4 \frac{3f}{2}(\operatorname{grad} f), \tag{3.2}$$

where A_{e_4} is the shape operator with respect to e_4 and

$$\varepsilon_4 = \begin{cases} 1, & \text{if } H \text{ is space-like} \\ -1, & \text{if } H \text{ is time-like.} \end{cases}$$

Remark 3.2. In order to avoid trivial cases, we will suppose that a biconservative PNMCV immersion in $\mathbb{E}^5_1$ has positive mean curvature function on M^3 and its gradient is nowhere vanishing. Moreover, these submanifolds are called be as proper. In this paper, we investigate smooth, connected and proper submanifolds unless otherwise is stated.

Since H is proportional to e_4, we have

$$H = \varepsilon_4 f e_4. \tag{3.3}$$

By using (3.2) and (3.3), we have

$$\operatorname{tr} A_4 = 3f, \qquad \operatorname{tr} A_5 = 0. \tag{3.4}$$

On the other hand, if we consider $e_1 = \frac{\operatorname{grad} f}{|\operatorname{grad} f|}$, we obtain

$$e_1(f) \neq 0,\ e_2(f) = e_3(f) = 0 \tag{3.5}$$

and $k_1 = -\varepsilon_4 \frac{3f}{2}$ from (3.2). Therefore, the matrix representations of the shape operators along e_4 and e_5 are given respectively by

$$A_{e_4} = \begin{pmatrix} -\varepsilon_4\frac{3f}{2} & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{pmatrix}, \quad A_{e_5} = \begin{pmatrix} l_1 & 0 & 0 \\ 0 & l_2 & 0 \\ 0 & 0 & l_3 \end{pmatrix} \tag{3.6}$$

for some smooth functions $f, k_2, k_3, l_1, l_2, l_3$. By using (3.4), we get

$$k_2 + k_3 = 3f + \varepsilon_4 \frac{3f}{2} \text{ and } l_1 + l_2 + l_3 = 0. \tag{3.7}$$

Let l_f be the number of distinct principle curvatures of A_{e_4}. Then, we have two cases: $l_f = 2$ or $l_f = 3$. In this paper, we investigate only $l_f = 2$.

On the other hand, since M^3 is a Riemannian manifold in $\mathbb{E}^5_1$, the mean curvature vector H is time-like or space-like.

3.1. *H is time-like in* $\mathbb{E}^5_1$

In this subsection, we focus on biconservative PNMCV submanifolds having *time-like* mean curvature vector field H in $\mathbb{E}^5_1$ and $l_f = 2$; that is, $k_2 = k_3$. By using (3.2) and (3.4), the shape operators corresponding to e_4 and e_5 are given, with respect to $\{e_1, e_2, e_3\}$, by

$$A_{e_4} = \begin{pmatrix} \frac{3f}{2} & 0 & 0 \\ 0 & \frac{3f}{4} & 0 \\ 0 & 0 & \frac{3f}{4} \end{pmatrix}, \quad A_{e_5} = \begin{pmatrix} l_1 & 0 & 0 \\ 0 & l_2 & 0 \\ 0 & 0 & l_3 \end{pmatrix}, \tag{3.8}$$

for some smooth functions l_1, l_2, l_3 satisfying $l_1 + l_2 + l_3 = 0$.

By applying Codazzi equation for $(X, Y, Z) = (e_1, e_A, e_1)$ such that $A = 2, 3$, we have

$$\omega_{1A}(e_1) = 0, \; e_A(l_1) = 0. \tag{3.9}$$

If we apply Codazzi equation for $(X, Y, Z) = (e_A, e_1, e_A)$ such that $A = 2, 3$, we obtain

$$\omega_{1A}(e_A) = \frac{e_1(f)}{f}, \tag{3.10a}$$

$$e_1(l_A) = \omega_{1A}(e_A)(l_1 - l_A), \tag{3.10b}$$

which give $l_1 = 2cf^{-3}$ for a smooth function c such that $e_1(c) = 0$. Considering (3.9), we get $e_2(c) = e_3(c) = 0$ which gives c is a constant.

On the other hand, using (3.10a) and (3.10b), we find

$$l_2 = -cf^{-3} + f_2 f^{-1} \tag{3.11}$$

for a smooth function f_2 satisfying $e_1(f_2) = 0$. Moreover, using (3.7) we get

$$l_3 = -cf^{-3} - f_2 f^{-1}. \tag{3.12}$$

Consequently, we have the following.

Proposition 3.1. *Let $\psi : (M^3, g) \hookrightarrow \mathbb{E}^5_1$ be a biconservative PNMCV immersion with two distinct principal curvatures in the direction of the mean curvature vector. Then, H is time-like vector in $\mathbb{E}^5_1$ if and only if there exists an orthonormal frame field $\{e_1, e_2, e_3; e_4, e_5\}$ such that shape operators along e_4 and e_5 given by the matrices*

$$A_{e_4} = \begin{pmatrix} \frac{3f}{2} & 0 & 0 \\ 0 & \frac{3f}{4} & 0 \\ 0 & 0 & \frac{3f}{4} \end{pmatrix}, A_{e_5} = \begin{pmatrix} 2cf^{-3} & 0 & 0 \\ 0 & -cf^{-3} + f_2 f^{-1} & 0 \\ 0 & 0 & -cf^3 - f_2 f^{-1} \end{pmatrix} \tag{3.13}$$

and

$$\nabla^\perp e_4 = \nabla^\perp e_5 = 0$$

where c is a non-zero constant and f, f_2 are some smooth functions satisfy $e_1(f_2) = e_2(f) = e_3(f) = 0$.

Proof. From the above results, the proof of the necessary condition is completed. The converse of this proposition is trivial. □

Next, by using Proposition 3.1, we give the following theorem for biconservative PNMCV immersions.

Theorem 3.1. *Let $\psi : (M^3, g) \hookrightarrow \mathbb{E}^5_1$ be a biconservative PNMCV immersion with two distinct principal curvatures in the direction of the mean curvature vector. Then, H is time-like vector in $\mathbb{E}^5_1$ if and only if ψ is one of the following immersions given below.*

Case I. An isometric immersion ψ_1 which has an orthonormal frame field $\{e_1, e_2, e_3; e_4, e_5\}$ such that

$$A_{e_4} = \begin{pmatrix} \frac{3f}{2} & 0 & 0 \\ 0 & \frac{3f}{4} & 0 \\ 0 & 0 & \frac{3f}{4} \end{pmatrix}, \quad A_{e_5} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & f_2 f & 0 \\ 0 & 0 & -f_2 f \end{pmatrix}, \tag{3.14}$$

and

$$\begin{aligned} &\omega_{12}(e_1) = \omega_{13}(e_1) = \omega_{12}(e_3) = \omega_{13}(e_2) = \omega_{23}(e_1) = 0, \\ &\omega_{12}(e_2) = \omega_{13}(e_3) = \frac{e_1(f)}{f}, \\ &\omega_{23}(e_2) = \frac{1}{2}\frac{e_3(f_2)}{f_2}, \quad \omega_{23}(e_3) = -\frac{1}{2}\frac{e_2(f_2)}{f_2} \end{aligned} \tag{3.15}$$

for some smooth functions f and f_2 such that $e_2(f) = e_3(f) = e_1(f_2) = 0$ and f does not vanish.

Case II. An isometric immersion ψ_2 which has an orthonormal frame field $\{e_1, e_2, e_3; e_4, e_5\}$ such that

$$A_{e_4} = \begin{pmatrix} \frac{3f}{2} & 0 & 0 \\ 0 & \frac{3f}{4} & 0 \\ 0 & 0 & \frac{3f}{4} \end{pmatrix}, \quad A_{e_5} = \begin{pmatrix} 2cf^{-3} & 0 & 0 \\ 0 & -cf^{-3} & 0 \\ 0 & 0 & -cf^{-3} \end{pmatrix}, \tag{3.16}$$

and

$$\begin{aligned} &\omega_{12}(e_1) = \omega_{13}(e_1) = \omega_{12}(e_3) = \omega_{13}(e_2) = \omega_{23}(e_2) = \omega_{23}(e_3) = 0, \\ &\omega_{12}(e_2) = \omega_{13}(e_3) = \frac{e_1(f)}{f} \end{aligned} \tag{3.17}$$

for a smooth non-vanishing function f satisfying $e_2(f) = e_3(f) = 0$.

Proof. Suppose that ψ is a biconservative PNMCV immersion. By using Proposition 3.1, we obtain that the shape operators of ψ have the matrix representations given in (3.13). If we apply Codazzi equations for $(X, Y, Z) = (e_2, e_3, e_3)$ and $(X, Y, Z) = (e_3, e_2, e_2)$, we obtain

$$\omega_{23}(e_3) = -\frac{e_2(f_2)}{2f_2} \tag{3.18}$$

and

$$\omega_{23}(e_2) = \frac{e_3(f_2)}{2f_2}, \tag{3.19}$$

respectively. Moreover, by considering the Codazzi equations for $(X, Y, Z) = (e_1, e_2, e_3)$ and $(X, Y, Z) = (e_1, e_3, e_2)$, we have

$$\omega_{13}(e_2) = \omega_{12}(e_3) = 0$$

and

$$f_2\omega_{23}(e_1) = 0. \tag{3.20}$$

Firstly, assume that $f_2 \neq 0$. Then, (3.20) gives $\omega_{23}(e_1) = 0$. From the Gauss equation we find

$$R(e_1, e_2, e_2, e_1) = -e_1(\alpha) - \alpha^2 = -\frac{9f^2}{8} - 2c^2 f^{-6} + 2cf_2 f^{-4}, \tag{3.21}$$

where $\alpha = \omega_{12}(e_2)$. Also,

$$R(e_1, e_2, e_2, e_1) = R(e_1, e_3, e_3, e_1) = -e_1(\alpha) - \alpha^2 \tag{3.22}$$

and

$$\langle h(e_1, e_1), h(e_2, e_2) - h(e_3, e_3)\rangle = 0. \tag{3.23}$$

By using equations (3.21), (3.22) and (3.23), we find $c = 0$. Therefore, we obtain the Case I of the theorem.

Now, we are going to consider the case $f_2 = 0$. From (3.18) and (3.19) we obtain $\omega_{23}(e_2) = \omega_{23}(e_3) = 0$. Substituting $f_2 = 0$ in (3.13), the shape operators along e_4 and e_5 become (3.16). Hence, we have the Case II of the theorem. □

Next, we have the following lemma.

Lemma 3.1. *Let (M^3, g) be a Riemannian manifold described in Theorem 3.1.*

Case I. If M^3 admits a biconservative PNMCV isometric immersion $\psi = \psi_1$, then

i. The scalar curvature of M^3 satisfies

$$S = -(\frac{45f^2}{16} + f_2^2 f^{-2}), \tag{3.24}$$

ii. The Ricci tensor of M^3 satisfies $\mathrm{Ric}(e_i) = \lambda_i e_i$ and

$$\lambda_1 = -\frac{9f^2}{4}, \lambda_2 = \lambda_3 = \frac{-27f^2}{16} - f_2^2 f^{-2}. \tag{3.25}$$

iii. The set span$\{\nabla\lambda_1, \nabla\lambda_2, \nabla\lambda_3\}$ has dimension 2.

Case II. If M^3 admits a biconservative PNMCV isometric immersion $\psi = \psi_2$, then

i. The scalar curvature of M^3 satisfies

$$S = -(\frac{45f^2}{16} + 3c^2 f^{-6}), \tag{3.26}$$

ii. The Ricci tensor of M^3 satisfies $\mathrm{Ric}(e_i) = \lambda_i e_i$ and

$$\lambda_1 = -\frac{9f^2}{4} - 4c^2 f^{-6}, \lambda_2 = \lambda_3 = \frac{-27f^2}{16} - c^2 f^{-6}. \tag{3.27}$$

iii. The set span$\{\nabla\lambda_1, \nabla\lambda_2, \nabla\lambda_3\}$ has dimension 1.

Proof. We suppose that M^3 admits the isometric immersion ψ_1 introduced in Case I of Theorem 3.1. It follows from (2.3) and (3.14), we find

$$h(e_1, e_1) = -\frac{3f}{2} e_4, \quad h(e_2, e_2) = \frac{-3f}{4} e_4 + f_2 f^{-1} e_5,$$
$$h(e_3, e_3) = \frac{-3f}{4} e_4 - f_2 f^{-1} e_5.$$

By a straightforward computation with using the Gauss equation (2.7), we have

$$\begin{aligned} R(e_1, e_2, e_2, e_1) &= R(e_1, e_3, e_3, e_1) = -\frac{9f^2}{8}, \\ R(e_2, e_3, e_3, e_2) &= \frac{-9f^2}{16} - f_2^2 f^{-2}. \end{aligned} \tag{3.28}$$

It follows from the Ric of M^3 satisfies $\mathrm{Ric}(e_i) = \lambda_i e_i$, we obtain functions λ_i as (3.25).

From (2.5), we calculate the scalar curvature of M^3 as (3.24). On the other hand, by using (3.25), we get

$$\nabla\lambda_1 = \frac{-9}{2} f e_1(f) e_1, \tag{3.29}$$
$$\nabla\lambda_2 = \nabla\lambda_3 = \left(\frac{-27f}{8} + 2f_2^2 f^{-3} 5\right) e_1(f) e_1 - 2f_2 f^{-2}(e_2(f_2)e_2 + e_3(f_2)e_3) \tag{3.30}$$

which implies that the set span$\{\nabla\lambda_1, \nabla\lambda_2, \nabla\lambda_3\}$ has dimension 2.

Moreover, we assume that M^3 admits the isometric immersion ψ_2 given in Case II of Theorem 3.1. By making similar calculations, we find the eigenvalues of Ric as (3.27).

From (2.5), the scalar curvature of M^3 is

$$S = -\left(\frac{45f^2}{16} + 3c^2 f^{-6}\right).$$

Therefore, we find f^2 and c^2 in terms of λ_1, λ_2. Also, from (3.27) we have

$$\dim\left(\operatorname{span}\{\nabla\lambda_1, \nabla\lambda_2, \nabla\lambda_3\}\right) = 1.$$

As an immediate consequence of Theorem 3.1, we would like to state the following corollary.

Corollary 3.1. *Let $\psi : (M^3, g) \hookrightarrow \mathbb{E}^5_1$ be a biconservative PNMCV immersion with two distinct principal curvatures in the direction of the mean curvature vector field. Then, the vector field e_1 and the functions f, f_2, c appearing in Theorem 3.1 can be computed intrinsically.*

3.2. H is space-like in $\mathbb{E}^5_1$

In this subsection, we investigate biconservative PNMCV submanifolds having *space-like* mean curvature vector field H in $\mathbb{E}^5_1$ and $l_f = 2$; that is, $k_2 = k_3$. From (3.2) and (3.4), the matrix representations of the shape operators e_4 and e_5 take the form

$$A_{e_4} = \begin{pmatrix} \frac{-3f}{2} & 0 & 0 \\ 0 & \frac{9f}{4} & 0 \\ 0 & 0 & \frac{9f}{4} \end{pmatrix}, \quad A_{e_5} = \begin{pmatrix} l_1 & 0 & 0 \\ 0 & l_2 & 0 \\ 0 & 0 & l_3 \end{pmatrix} \tag{3.31}$$

with respect to $\{e_1, e_2, e_3\}$ for some smooth functions l_1, l_2, l_3 satisfying $l_1 + l_2 + l_3 = 0$. By making similar calculations, we give the following proposition.

Proposition 3.2. *Let $\psi : (M^3, g) \hookrightarrow \mathbb{E}^5_1$ be a biconservative PNMCV immersion with two distinct principal curvatures in the direction of the mean curvature vector. Then, H is space-like vector if and only if there exists an orthonormal frame field $\{e_1, e_2, e_3; e_4, e_5\}$ such that shape operators along e_4 and e_5 given by the matrices*

$$A_{e_4} = \begin{pmatrix} \frac{-3f}{2} & 0 & 0 \\ 0 & \frac{9f}{4} & 0 \\ 0 & 0 & \frac{9f}{4} \end{pmatrix}, A_{e_5} = \begin{pmatrix} 2cf^{9/5} & 0 & 0 \\ 0 & -cf^{9/5} + f_2 f^{3/5} & 0 \\ 0 & 0 & -cf^{9/5} - f_2 f^{3/5} \end{pmatrix} \tag{3.32}$$

and

$$\nabla^\perp e_4 = \nabla^\perp e_5 = 0$$

where c is a non-zero constant and f, f_2 are some smooth functions such that $e_2(f) = e_3(f) = e_1(f_2) = 0$.

Next, by using Proposition 3.2, we have the following theorem.

Theorem 3.2. *Let $\psi : (M^3, g) \hookrightarrow \mathbb{E}^5_1$ be a biconservative PNMCV immersion with two distinct principal curvatures in the direction of the mean curvature vector. Then, H is space-like vector if and only if ψ is one of the following immersions.*

Case I. An isometric immersion ψ_1 which has an orthonormal frame field $\{e_1, e_2, e_3; e_4, e_5\}$ such that

$$A_{e_4} = \begin{pmatrix} \frac{-3f}{2} & 0 & 0 \\ 0 & \frac{9f}{4} & 0 \\ 0 & 0 & \frac{9f}{4} \end{pmatrix}, \quad A_{e_5} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & f_2 f^{3/5} & 0 \\ 0 & 0 & -f_2 f^{3/5} \end{pmatrix}, \tag{3.33}$$

and

$$\begin{aligned} &\omega_{12}(e_1) = \omega_{13}(e_1) = \omega_{12}(e_3) = \omega_{13}(e_2) = \omega_{23}(e_1) = 0, \\ &\omega_{12}(e_2) = \omega_{13}(e_3) = \frac{-3}{5}\frac{e_1(f)}{f}, \\ &\omega_{23}(e_2) = \frac{1}{2}\frac{e_3(f_2)}{f_2}, \quad \omega_{23}(e_3) = -\frac{1}{2}\frac{e_2(f_2)}{f_2} \end{aligned} \tag{3.34}$$

for some smooth functions f and f_2 such that $e_2(f) = e_3(f) = e_1(f_2) = 0$ and f does not vanish.

Case II. An isometric immersion ψ_2 which has an orthonormal frame field $\{e_1, e_2, e_3; e_4, e_5\}$ such that

$$A_{e_4} = \begin{pmatrix} \frac{-3f}{2} & 0 & 0 \\ 0 & \frac{9f}{4} & 0 \\ 0 & 0 & \frac{9f}{4} \end{pmatrix}, \quad A_{e_5} = \begin{pmatrix} 2cf^{9/5} & 0 & 0 \\ 0 & -cf^{9/5} & 0 \\ 0 & 0 & -cf^{9/5} \end{pmatrix}, \tag{3.35}$$

and

$$\begin{aligned} &\omega_{12}(e_1) = \omega_{13}(e_1) = \omega_{12}(e_3) = \omega_{13}(e_2) = \omega_{23}(e_2) = \omega_{23}(e_3) = 0, \\ &\omega_{12}(e_2) = \omega_{13}(e_3) = \frac{e_1(f)}{f} \end{aligned} \tag{3.36}$$

for a smooth non-vanishing function f satisfying $e_2(f) = e_3(f) = 0$.

Proof. The proof of theorem can be obtained by using similar way as given in Theorem 3.1. □
Finally, we obtain the following theorem.

Theorem 3.3. *Let $\psi : (M^3, g) \hookrightarrow \mathbb{E}^5_1$ be a biconservative PNMCV immersion with two distinct principal curvatures in the direction of the mean curvature vector field. Then, the vector field e_1 and the functions f, f_2, c appearing in Theorem 3.2 can be computed intrinsically.*

Proof. We suppose that M^3 admits the isometric immersion ψ_1 given in Case I of Theorem 3.2. By making similar calculations given in the Section 3.1, we obtain the eigenvalues of Ric as

$$\lambda_1 = -\frac{27f^2}{4}, \lambda_2 = \lambda_3 = \frac{27}{16}f^2 + f_2^2 f^{6/5} \tag{3.37}$$

which implies that we can find f^2 and f_2^2 in terms of eigenvalues of Ric and

$$e_1 = \frac{\nabla\lambda_1}{\|\nabla\lambda_1\|}.$$

Moreover, we suppose that M^3 admits the isometric immersion ψ_2 given in Case II of Theorem 3.2. Similarly, we find the eigenvalues of Ric as

$$\lambda_1 = \frac{-27f^2}{4} + 4c^2 f^{18/5}, \lambda_2 = \lambda_3 = \frac{27f^2}{16} + c^2 f^{18/5}$$

which implies that f^2 and c^2 can be calculated in terms of λ_1, λ_2. This completes the proof. □

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Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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