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ON INTERACTIVE SOLUTION FOR TWO POINT FUZZY BOUNDARY VALUE PROBLEM

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Abstract. In this manuscript, the eigenvalues and eigenfunctions of the two-point fuzzy boundary value problem (FBVP) are analyzed under the concept of interactivity between the fuzzy numbers found in the boundary conditions. A fuzzy solution is provided for this problem via sup-J extension, which can be seen as a generalization of Zadeh's extension principle. Finally, an example is presented in order to compare the given features.

1. Introduction

In this paper, the FBVP is considered

$$\widehat{u}'' + \lambda\widehat{u} = 0, \quad t \in [a, b] \tag{1.1}$$

which satisfies the conditions

$$\widehat{a}_1\widehat{u}\,(a) -^h \widehat{a}_2\widehat{u}'\,(a) = 0 \tag{1.2}$$

$$\widehat{b}_1\widehat{u}\,(b) -^h \widehat{b}_2\widehat{u}'\,(b) = 0 \tag{1.3}$$

where $\widehat{a}_1, \widehat{a}_2, \widehat{b}_1, \widehat{b}_2$ non-negative triangular fuzzy numbers, $\lambda > 0$, at least one of the numbers $\widehat{a}_1$ and $\widehat{a}_2$ and at least one of the numbers $\widehat{b}_1$ and $\widehat{b}_2$ are nonzero and $-^h$ is Hukuhara difference.

Fuzzy differential equation (FDE) is utilized to model problems in science and engineering.In most of the problems there are uncertain structural parameters. Instead, many researchers have modeled these uncertain structural parameters as fuzzy numbers in this area [4, 10]. This occurs a fuzzy boundary value problem with fuzzy boundary conditions.

The studies of two-point FBVP have been made with the Hukuhara derivative [11, 14] and generalized Hukuhara derivative [6, 15, 22, 27–29]. But in some cases

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the fuzzy solutions with Hukuhara derivative suffer from certain disadvantages since the diameter of the solutions is unbounded as time increases [13, 14] and the fuzzy solutions with generalized Hukuhara derivative have some not interval solutions which are associated with the existence of switch points [20]. Moreover, Gasilov et al. argued that the solutions obtained with the method of Khastan and Nieto [15] are difficult to interpret because the solutions of the four different problems may not reflect the nature of the studied phenomenon [9].

Another approach to solving FBVP has been proposed, including the Zadeh's extension principle [1, 17]. For a boundary value problem, the associated crisp problem is solved and in the solution, the fuzzy boundary value is substituted instead of the real constant. Then the arithmetic operations are regarded as operations on fuzzy numbers [16].

Recently, several authors have used the concept of interactivity to study fuzzy differential equations (FDEs) [5, 25]. The relation of interactivity between two fuzzy numbers arises in the presence of a joint possibility distribution J for them. In this case, the solution is obtained in terms of the $sup-J$ extension principle of the solution of an associated classical BVP. Moreover, this proposed approach always produces a proper fuzzy solution, in contrast to other methods presented in the literature [14, 15, 23]. This means that its $\alpha-$ cuts are proper intervals. Moreover, the fuzzy solution obtained by this approach always has a smaller or equal to the solution via Zadeh's extension [12, 17].

This paper analyses FBVP with fuzzy boundary values given by interactive fuzzy values. The fuzzy solution is obtained using the $sup-J$ extension principle [5]. In order to illustrate the utility of this $sup-J$ proposal, the solution of a second order FBVP is presented.

2. Preliminaries

2.1. Solution for a crisp boundary value problem. Let the fuzzy problem (1.1-1.3) be considered as a crisp problem.

Then we shall make use of solutions of (1.1) defined by initial conditions instead of boundary conditions in a manner similar to Titchmarsh's method [24].

Lemma 2.1. *([24]) For any $\lambda > 0$ the equation*

$$u'' + \lambda u = 0, \quad t \in [a, b]$$

has a unique solution $u = u(t, \lambda)$ satisfying the initial conditions

$$u(a) = a_2, \qquad u'(a) = a_1 \qquad (\text{ or } u(b) = b_2, \quad u'(b) = b_1).$$

For each $t \in [a, b]$, $u(t, \lambda)$ is an entire function of λ

Two solutions $\Phi_\lambda(t)$ and $\Psi_\lambda(t)$ of the equation (1.1) are defined as follows. Let $\Phi_\lambda(t) = \Phi(t, \lambda)$ be the solution of equation (1.1) on $[a, b]$, which satisfies the initial conditions

$$\begin{pmatrix} u(a) \\ u'(a) \end{pmatrix} = \begin{pmatrix} a_2 \\ a_1 \end{pmatrix} \tag{2.1}$$

and $\Psi_\lambda(t) = \Psi(t, \lambda)$ be the solution of equation (1.1) on $[a, b]$, which satisfies the initial conditions

$$\begin{pmatrix} u(b) \\ u'(b) \end{pmatrix} = \begin{pmatrix} b_2 \\ b_1 \end{pmatrix}. \tag{2.2}$$

Let us consider the following linear and homogeneous differential equation with (2.1) and (2.2) initial conditions, where $a_1, a_2, b_1, b_2 \in \mathbb{R}$, given by

$$\begin{cases} \Phi'' + \lambda\Phi = 0 \\ \Phi(a) = a_2, \; \Phi'(a) = a_1 \end{cases} \tag{2.3}$$

and

$$\begin{cases} \Psi'' + \lambda\Psi = 0 \\ \Psi(b) = b_2, \; \Psi'(b) = b_1. \end{cases} \tag{2.4}$$

First, let's search for the solution of the problem in (2.3) with the help of the algorithm created by Sanchez et al. [19]. Then a solution is found for the problem (2.4) by doing similar operations.

Firstly, the general solution of (2.3) is given

$$\Phi_\lambda(t) = C_1\Phi_1(t) + C_2\Phi_2(t), \tag{2.5}$$

where Φ_1, Φ_2 are linearly independent solutions of the homogeneous differential equation which is given as in (2.3).

The scalar coefficients C_1 and C_2 are determined from the initial values a_2 and $-a_1$:

$$C_1 = \frac{\Phi_2(a)(a_1) + \Phi_2'(a)a_2}{\Phi_1(a)\Phi_2'(a) - \Phi_2(a)\Phi_1'(a)} \text{ and } C_2 = -\frac{\Phi_1(a)(a_1) + \Phi_1'(a)a_2}{\Phi_1(a)\Phi_2'(a) - \Phi_2(a)\Phi_1'(a)}.$$

Thus, from (2.5), the general solution of (2.3) is given by

$$\Phi_\lambda(t) = a_1 m_1(t) + a_2 m_2(t), \tag{2.6}$$

where $m_1(t)$ and $m_2(t)$ are defined for $\Phi_1(a)\Phi_2'(a) - \Phi_2(a)\Phi_1'(a) \neq 0$ as follows [9]:

$$m_1(t) = \frac{\Phi_2(a)\Phi_1(t) - \Phi_1(a)\Phi_2(t)}{\Phi_1(a)\Phi_2'(a) - \Phi_2(a)\Phi_1'(a)}, \text{ and } m_2(t) = \frac{\Phi_2'(a)\Phi_1(t) - \Phi_1'(a)\Phi_2(t)}{\Phi_1(a)\Phi_2'(a) - \Phi_2(a)\Phi_1'(a)}. \tag{2.7}$$

Similarly, the general solution of (2.4) is given by

$$\Psi_\lambda(t) = b_1 m_3(t) + b_2 m_4(t), \tag{2.8}$$

where $m_3(t)$ and $m_4(t)$ are defined for $\Phi_1(b)\Phi_2'(b) - \Phi_2(b)\Phi_1'(b) \neq 0$ as follows

$$m_3(t) = \frac{\Phi_2(b)\Phi_1(t) - \Phi_1(b)\Phi_2(t)}{\Phi_1(b)\Phi_2'(b) - \Phi_2(b)\Phi_1'(b)}, \text{ and } m_4(t) = \frac{\Phi_2'(b)\Phi_1(t) - \Phi_1'(b)\Phi_2(t)}{\Phi_1(b)\Phi_2'(b) - \Phi_2(b)\Phi_1'(b)}. \tag{2.9}$$

Then this solutions $\Phi_\lambda(t)$ and $\Psi_\lambda(t)$ are put in the Wronskians function

$$w(\lambda) = W_\lambda(\Phi, \Psi; t) = \Phi_\lambda(t)\Psi_\lambda'(t) - \Phi_\lambda'(t)\Psi_\lambda(t) \tag{2.10}$$

which are independent of $t \in [a, b]$. For each fixed t these functions and derivatives are entire in λ [24].

Lemma 2.2. *([24] If $\lambda = \lambda_0$ is an eigenvalue, then $\Phi(t, \lambda_0)$ and $\Psi(t, \lambda_0)$ are linearly dependent and eigenfunctions corresponding to this eigenvalue .*

Theorem 2.3. *([24]) The eigenvalues of the problem (1.1-1.3) are the zeros of the function $w(\lambda)$.*

In Section 3, equations (2.6) and (2.8) will be used to define the fuzzy solution of the second order two point boundary values problem with fuzzy boundary values.

Before the approach applied to solve an FBV problem is introduced, it is necessary first to review some concepts of fuzzy sets theory.

2.2. **Basic concepts of fuzzy sets.**

Definition 2.4. ([18]) Let E be a universal set. A fuzzy subset $\widehat{A}$ of E is given by its membership function $\mu_{\widehat{A}} : E \rightarrow [0,1]$, where $\mu_{\widehat{A}}(t)$ represents the degree to which $t \in E$ belongs to $\widehat{A}$. We denote the class of the fuzzy subsets of E by the sembol $F(E)$.

Definition 2.5. ([16]) The $\alpha-$ cut of a fuzzy set $\widehat{A} \subseteq E$denoted by $\left[\widehat{A}\right]^{\alpha}$, is defined as $\left[\widehat{A}\right]^{\alpha} = \left\{x \in E : \widehat{A}(t) \geq \alpha\right\}$, $\forall \alpha \in (0,1]$. If E is also topological space, then the $0-$ cut is defined as the closure of the support of $\widehat{A}$,that is, $\left[\widehat{A}\right]^{0} = \overline{\left\{x \in E : \widehat{A}(t) > 0\right\}}$. The $1-$ cut of a fuzzy subset $\widehat{A}$ is also called as core of $\widehat{A}$ and denoted by $\left[\widehat{A}\right]^{1} = core\left(\widehat{A}\right)$

Definition 2.6. ([21]) A fuzzy subset $\widehat{u}$ on $\mathbb{R}$ is called a fuzzy real number (fuzzy interval), whose $\alpha-$ cut set is denoted by $[\widehat{u}]^{\alpha}$, i.e., $[\widehat{u}]^{\alpha} = \{x : \widehat{u}(t) \geq 0\}$, if it satisfies two axioms:

i. There exists $r \in \mathbb{R}$ such that $\widehat{u}(r) = 1$,

ii. For all $0 < \alpha \leq 1$, there exist real numbers $-\infty < u_{\alpha}^{-} \leq u_{\alpha}^{+} < +\infty$ such that $[\widehat{u}]^{\alpha}$ is equal to the closed interval $[u_{\alpha}^{-}, u_{\alpha}^{+}]$.

The set of all fuzzy real numbers (fuzzy intervals) is denoted by $\mathbb{R}_F$. $F_K(\mathbb{R})$, the family of fuzzy sets of $\mathbb{R}$ whose $\alpha-$ cuts are nonempty compact convex subsets of $\mathbb{R}$. If $\widehat{u} \in \mathbb{R}_F$ and $\widehat{u}(t) = 0$ whenever $t < 0$, then $\widehat{u}$ is called a non-negative fuzzy real number and $\mathbb{R}_F^{+}$ denotes the set of all non-negative fuzzy real numbers. For all $\widehat{u} \in \mathbb{R}_F^{+}$ and each $\alpha \in (0,1]$, real number u_{α}^{-} is positive.

Definition 2.7. ([7]) An arbitrary fuzzy number $\widehat{u}$ in the parametric form is represented by an ordered pair of functions $[u_{\alpha}^{-}, u_{\alpha}^{+}]$, $0 \leq \alpha \leq 1$, which satisfy the following requirements

i. u_{α}^{-} is bounded non-decreasing left continuous function on $(0,1]$ and right- continuous for $\alpha = 0$,

ii. u_{α}^{+} is bounded non- increasing left continuous function on $(0,1]$ and right-continuous for $\alpha = 0$,

iii. $u_{\alpha}^{-} \leq u_{\alpha}^{+}$, $0 < \alpha \leq 1$.

Definition 2.8. ([2,10]) A fuzzy number $\widehat{A}$ is said to be triangular if the parametric representation of its $\alpha-$ cut is of the form for all $\alpha \in [0,1]$

$\left[\widehat{A}\right]^{\alpha} = [(m - a_{\alpha}^{-})\,\alpha + a_{\alpha}^{-}, (m - a_{\alpha}^{+})\,\alpha + a_{\alpha}^{+}]$, where $\left[\widehat{A}\right]^{0} = [a_{\alpha}^{-}, a_{\alpha}^{+}]$ and *core* $\left(\widehat{A}\right) = m\dot{\text{A}}$ triangular fuzzy number is denoted by the triple $(a_{\alpha}^{-}; m; a_{\alpha}^{+})$.

Zadeh's extension principle is a mathematical method to extend classical functions to deal with fuzzy sets as input arguments [26]. For multiple fuzzy variables as arguments, Zadeh's extension principle is defined as follows.

Definition 2.9. ([1]) Let $f : X_1 \times X_2 \to Z$ a classical function and let $\widehat{A}_i \in F(X_i)$, for $i = 1, 2$. The Zadeh's extension $\widehat{f}$ of f, applied to $(\widehat{A}_1, \widehat{A}_2)$, is the fuzzy set $\widehat{f}$ $(\widehat{A}_1, \widehat{A}_2)$ of Z, whose membership function is defined by

$$\widehat{f}(\widehat{A}_1, \widehat{A}_2)(z) = \begin{cases} \underset{(x_1,x_2)\in f^{-1}(z)}{sup} \; min\{\widehat{A}_1(x_1), \widehat{A}_2(x_2)\}, & \text{if } f^{-1}(z) \neq \varnothing, \\ 0 & \text{, if } f^{-1}(z) = \varnothing \end{cases}$$

where $f^{-1}(z) = \{(x_1, x_2) \in X_1 \times X_2 : f(x_1, x_2) = z\}$.

We can apply Zadeh's extension principle to define the standard arithmetic for fuzzy numbers [26] . Let $[\widehat{u}]^{\alpha} = [u_{\alpha}^{-}, u_{\alpha}^{+}]$ and $[\widehat{v}]^{\alpha} = [v_{\alpha}^{-}, v_{\alpha}^{+}]$. For all $\alpha \in [0, 1]$ and $\lambda \in \mathbb{R}$, we have

$$\begin{aligned} [\widehat{u} \oplus \widehat{v}]^{\alpha} &= [\widehat{u}]^{\alpha} + [\widehat{v}]^{\alpha} = \{x + y : x \in [\widehat{u}]^{\alpha}, y \in [\widehat{v}]^{\alpha}\}, \\ [\lambda \odot \widehat{u}]^{\alpha} &= \lambda \odot [\widehat{u}]^{\alpha} = \{\lambda x : x \in [\widehat{u}]^{\alpha}\}. \end{aligned}$$

Theorem 2.10. *([2]) Let X and Y be topological spaces, $f : X \to Y$ be a continuous function and $\widehat{A}$ a fuzzy subset of $X\dot{S}o$ for all $\alpha \in [0, 1]$, we have*

$$[\hat{f}(\widehat{A})]^{\alpha} = f\left(\left[\widehat{A}\right]^{\alpha}\right).$$

As a consequence of Theorem 2.10, it is obtained that $\hat{f}(\widehat{A})$ is a fuzzy number whenever the function $f : X \to Y$ be a continuous function and $\widehat{A}$ is a fuzzy number.

The concept of interactivity between fuzzy numbers is based on the notion of joint possibility distributions [5]. More precisely, a fuzzy subset J of $\mathbb{R}^n$ is called a joint possibility distribution of $\widehat{A}_1, ..., \widehat{A}_n \in \mathbb{R}_F$ if

$$\widehat{A}_i(x_i) = \underset{x_j\in\mathbb{R}, j\neq i}{sup} J\left(x_1, ..., x_n\right),$$

for all $x_i \in \mathbb{R}$ and for all $i = 1, ..., n$. Moreover, the fuzzy numbers $\widehat{A}_1, ..., \widehat{A}_n$ are said to be non-interactive if their joint possibility distribution is given by and for all $i = 1, ..., n$. Moreover, the fuzzy numbers $\widehat{A}_1, ..., \widehat{A}_n$ are said to be non-interactive if their joint possibility distribution is given by

$$J_{\hat{}}\left(x_1, ..., x_n\right) = min\{\widehat{A}_1(x_1), ..., \widehat{A}_n(x_n)\}, \forall (x_1, ..., x_n) \in \mathbb{R} \tag{2.11}$$

Otherwise, the fuzzy numbers $\widehat{A}_1, ..., \widehat{A}_n$ are said to be interactive . Next, the notion of $sup - J$ extension principle proposed by Carlsson et al. is presented [5].

Definition 2.11. ([5]) Let $\widehat{A}_1, ..., \widehat{A}_n \in \mathbb{R}_F$ and $f : \mathbb{R}_n \to \mathbb{R}$. Given a joint possibility distribution J of $\widehat{A}_1, ..., \widehat{A}_n$, the $sup - J$ extension of f at $(\widehat{A}_1, ..., \widehat{A}_n)$ is the fuzzy set $\widehat{f}(J) := f_J(A_1, ..., A_n)$ of $\mathbb{R}$ whose membership function is given by

$$\widehat{f}(J)(z) = \underset{f(x_1,...,x_n)=z}{sup} J(x_1, ..., x_n), \forall z \in \mathbb{R}$$

for all $z \in \mathbb{R}$, where $f^{-1}(z) = \{(x_1, ..., x_n) : f((x_1, ..., x_n)) = z\}$.

Remark 2.12. If $\widehat{A}_1, ..., \widehat{A}_n \in \mathbb{R}_F$ are non-interactive that is, if the corresponding joint possibility distribution J is defined as in (2.11), then the $sup - J$ extension principle corresponds to the Zadeh's extension principle of a function $f : \mathbb{R}^n \to \mathbb{R}$ at $(\widehat{A}_1, ..., \widehat{A}_n) \in \mathbb{R}_F^n$. In this case, the symbol $\widehat{f}(\widehat{A}_1, ..., \widehat{A}_n)$ is used simply instead of $f_J(\widehat{A}_1, ..., \widehat{A}_n)$ to denote the Zadeh's extension of f at $(\widehat{A}_1, ..., \widehat{A}_n)$.

The next corollary is an immediate consequence of Theorem 2.10

Corollary 2.13. ([5]) Let $\widehat{A}_1, ..., \widehat{A}_n \in \mathbb{R}_F$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function. If J is a joint possibility distribution of the fuzzy numbers $\widehat{A}_1, ..., \widehat{A}_n$, then we have

$$\left[f_J(\widehat{A}_1, ..., \widehat{A}_n)\right]_\alpha = f([J]_\alpha)$$

for all $\alpha \in [0, 1]$.

The usual arithmetic operations of addition, subtraction, multiplication, and division for fuzzy numbers are defined Definition 2.14. Other forms of arithmetic operations between fuzzy numbers can be established using the notion of $sup - J$ extension principle. Next, an arithmetic defined for the class of the linearly correlated (or completely correlated) fuzzy numbers is presented [1].

Definition 2.14. ([3,5]) Two fuzzy numbers A and B are linearly correlated if there exists $q, r \in R$, $q \neq 0$, such that $[B]^\alpha = q[A]^\alpha + r$ for each $\alpha \in [0, 1]$ or, equivalently, if A and B are interactive with respect to J_L given cutwise by

$$[J_L]_\alpha = \{(xq + r : x \in [A]^\alpha\}.$$

In this case, we may simply write $B = qA + r$ is written.

If A and B are linearly interactive fuzzy numbers $[B]^\alpha = q[A]^\alpha + r$, with $[A]^\alpha = [a_\alpha^-, a_\alpha^-]$ and $[B]^\alpha = [b_\alpha^-, b_\alpha^+]$, then the addition $(+_L)$ subtraction $(-_L)$ are given bye

$$[B +_L A]^\alpha = (q + 1)[A]^\alpha + r = \begin{cases} [b_\alpha^- + a_\alpha^-, b_\alpha^+ + a_\alpha^+] & \text{if} \quad q > 0, \\ [b_\alpha^+ + a_\alpha^-, b_\alpha^- + a_\alpha^+] & \text{if} \quad -1 \leq q < 0, \\ [b_\alpha^- + a_\alpha^+, b_\alpha^+ + a_\alpha^-] & \text{if} \quad q < -1, \end{cases} \tag{2.12}$$

$$[B -_L A]^\alpha = (q - 1)[A]^\alpha + r = \begin{cases} [b_\alpha^- - a_\alpha^-, b_\alpha^+ - a_\alpha^+] & \text{if} \quad q \geq 1, \\ [b_\alpha^+ - a_\alpha^+, b_\alpha^- - a_\alpha^-] & \text{if} \quad 0 \leq q < 1, \\ [b_\alpha^- - a_\alpha^+, b_\alpha^+ - a_\alpha^-] & \text{if} \quad q < 0, \end{cases} \tag{2.13}$$

for all $\alpha \in [0; 1]$ [3].

Definition 2.15. ([23]) Let $\widehat{u} \in E$ and for $\alpha \in [0,1]$, $[\widehat{u}]^\alpha = [u_\alpha^-, u_\alpha^+]$. Then $-^h[\widehat{u}]^\alpha$ is defined as follows:

$$-^h[\widehat{u}]^\alpha = -^h \left[u_\alpha^-, u_\alpha^+\right] = 0 -^h \left[u_\alpha^-, u_\alpha^+\right] = \left[-u_\alpha^-, -u_\alpha^+\right].$$

3. Solution Method of the FBVP

In this section we concern with the fuzzy initial value problems obtained by replacing the initial values a_1, a_2 and b_1, b_2 with fuzzy numbers $\widehat{a}_1$, $\widehat{a}_2$ and $\widehat{b}_1$, $\widehat{b}_2$ in Equations (2.3) and (2.4). More precisely, let us consider the following FIVPs:

$$\begin{cases} \Phi'' + \lambda\Phi = 0 \\ \Phi(a) = \widehat{a}_2, \quad \Phi'(a) = \widehat{a}_1 \end{cases} \tag{3.1}$$

and

$$\begin{cases} \Psi'' + \lambda\Psi = 0 \\ \Psi(b) = \widehat{b}_2, \quad \Psi'(b) = \widehat{b}_1 \end{cases} \tag{3.2}$$

where $\widehat{a}_1 = (a_{10}, a_1, a_{11})$, $\widehat{a}_2 = (a_{20}, a_2, a_{21})$, $\widehat{b}_1 = (b_{10}, b_1, b_{11})$,$\widehat{b}_2 = (b_{20}, b_2, b_{21}) \in \mathbb{R}_F$, λ is crisp number and $\lambda = p^2$, $p > 0$.

We present two different methods such as sup-J and Zadeh extension principle to solve the FIVPs (3.1) and (3.2).

Let $\Phi(.; a_1, a_2)$ and $\Psi(.; b_1, b_2)$ be the deterministic solution of the associated IVPs of equations (3.1) and (3.2), given in (2.6) and (2.8), where a_1, a_2, b_1, b_2 are the initial conditions. Let's first consider solution Φ and then similarly we get solution Ψ. Let U be an open subset of $\mathbb{R}^2$ such that $\left([\widehat{a}_1]^0 \times [\widehat{a}_2]^0\right) \subset U$. For each t, let be the operator $S_t : U \to \mathbb{R}$,given by

$$S_t(\Phi_0) = \Phi(t, \Phi_0)$$

and $J = J_L$ be a joint possibility distribution of $\widehat{a}_1, \widehat{a}_2 \in \mathbb{R}_F$. The fuzzy solution of (3.1) via $sup - J$ extension principle is given by

$$\widehat{\Phi}_J(t) = S_t(\widehat{a}_1, \widehat{a}_2).$$

If S_t is a continuous function, then by Corollary 2.13 , we have ([19]):

$$\begin{aligned} \widehat{\Phi}_J(t) &= [(S_t)_J(\widehat{a}_1, \widehat{a}_2)]^\alpha = (S_t)([J]^\alpha) \\ &= \left\{S_t(z, qz + r) : z \in [\widehat{a}_2]^\alpha = \left[(a_2)_\alpha^-, (a_2)_\alpha^+\right]\right\} \\ &= m_1(t) z + m_2(t)(qz + r) : z \in [\widehat{a}_2]^\alpha \\ &= (m_1(t) + q m_2(t)) \left[(a_2)_\alpha^-, (a_2)_\alpha^+\right] + r m_2(t) \end{aligned} \tag{3.3}$$

for all $\alpha \in [0,1]$.

If the initial conditions are non interactive fuzzy numbers, we can use Zadeh's extension principle to obtain a solution given by

$$\begin{aligned} \widehat{\Phi}(t) &= \left[\widehat{S}_t(\widehat{a}_1, \widehat{a}_2)\right]^\alpha = S_t([\widehat{a}_1]^\alpha \times [\widehat{a}_2]^\alpha) \\ &= \left\{S_t(\widehat{a}_1, \widehat{a}_2) : z \in [\widehat{a}_2]^\alpha = \left[(a_2)_\alpha^-, (a_2)_\alpha^+\right]\right\} \end{aligned}$$

for all $t \in [t_0, T]$. If S_t is a continuous function, then by Corollary 2.13, we have:

$$\begin{aligned} \widehat{\Phi}(t) &= \left[\widehat{S}_t\left(\widehat{a}_1, \widehat{a}_2\right)\right] \\ &= \left\{S_t\left(z, qz+r\right) : a_1 \in \left[\widehat{a}_1\right]^{\alpha} = \left[\left(a_1\right)_{\alpha}^{-}, \left(a_1\right)_{\alpha}^{+}\right], a_2 \in \left[\widehat{a}_2\right]^{\alpha} = \left[\left(a_2\right)_{\alpha}^{-}, \left(a_2\right)_{\alpha}^{+}\right]\right\} \\ &= m_1\left(t\right)\left[\left(a_2\right)_{\alpha}^{-}, \left(a_2\right)_{\alpha}^{+}\right] + m_2\left(t\right)\left[\left(a_1\right)_{\alpha}^{-}, \left(a_1\right)_{\alpha}^{+}\right] \end{aligned} \tag{3.4}$$

for all $\alpha \in [0,1]$.

Theorem 3.1. *([8]) Let $\widehat{\Phi}\left(t\right)$ and $\widehat{\Phi}_J\left(t\right)$ be the Zadeh and linear linear interactive solutions to the FIVP, respectively. Thus, $\widehat{\Phi}_J\left(t\right) \subseteq \widehat{\Phi}\left(t\right)$ for all $t \in \mathbb{R}$.*

Similarly, we get $\widehat{\Psi}\left(t\right)$ and $\widehat{\Psi}_J\left(t\right)$ be the Zadeh and linear interactive solutions to the FIVP, respectively.

The above theorem reveals that the linear interactive solution is contained in Zadeh's fuzzy solution. In fact, this result holds for every joint possibility distribution J, such that $J \subseteq J_{\wedge}$ [8].

Since λ is crisp (non-fuzzy) we substitute classical cases of the obtained fuzzy solutions $\widehat{\Phi}_{\lambda}\left(t\right) = \widehat{\Phi}\left(t, \lambda\right)$ and $\widehat{\Psi}_{\lambda}\left(t\right) = \widehat{\Psi}\left(t, \lambda\right)$ in (2.10). So we get the Wronskian function as follows

$$w(\lambda) = W_{\lambda}\left(\Phi, \Psi; t\right) = \Phi_{\lambda}\left(t\right)\Psi_{\lambda}'\left(t\right) - \Phi_{\lambda}'\left(t\right)\Psi_{\lambda}\left(t\right). \tag{3.5}$$

Definition 3.2. ([11]) Let $\left[\widehat{u}\left(t, \lambda\right)\right]^{\alpha} = \left[u_{\alpha}^{-}\left(t, \lambda\right), u_{\alpha}^{+}\left(t, \lambda\right)\right]$ be a solution of the fuzzy differential equation 1.1 where $\alpha \in [0,1]$. If the fuzzy differential equation 1.1 has the nontrivial solutions such that $u_{\alpha}^{-}\left(t, \lambda\right) \neq 0$ and $u_{\alpha}^{+}\left(t, \lambda\right) \neq 0$, then the $\lambda = \lambda_0$ is eigenvalue of (1.1)

Theorem 3.3. *([11]) The roots of equations (3.5) coincide with the eigenvalues of the fuzzy boundary value problem (1.1-1.3).*

The next section presents an example of FBVP with interactive and non-interactive boundary values.

4. Example

Consider the two point fuzzy boundary value problem

$$\widehat{u}'' + \lambda\widehat{u} = 0 \tag{4.1}$$

$$\widehat{2}u\left(0\right) + \widehat{1}u'\left(0\right) = 0 \tag{4.2}$$

$$\widehat{4}u\left(1\right) + \widehat{3}u'\left(1\right) = 0 \tag{4.3}$$

where $\widehat{1} = (0,1,2)$, $\widehat{2} = (1,2,3)$, $\widehat{3} = (2,3,4)$, $\widehat{4} = (3,4,5)$and $\lambda = p^2$, $p > 0$.

From (4.1-4.3) problem and Definition 2.15, we get two FIVPs involving a crisp differential equation (4.1) with fuzzy initial values as follows:

$$\Phi'' + p^2\Phi = 0, \quad \Phi\left(0\right) = \widehat{1}, \quad \Phi'\left(0\right) = -^{h}\widehat{2} \tag{4.4}$$

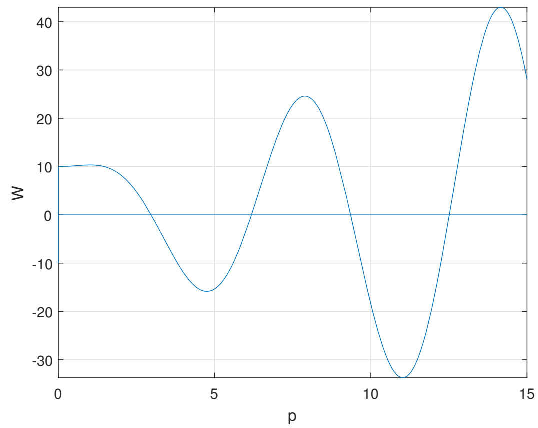

FIGURE 1. The function $W(\lambda) = \left(3p + \frac{8}{p}\right)\sin(p) + (2)\cos(p)$

and

$$\Psi'' + p^2\Psi = 0, \quad \Psi(1) = \widehat{3}, \quad \Psi'(1) = -^h\widehat{4} \tag{4.5}$$

We shall define two solutions $\widehat{\Phi}_\lambda(t)$ and $\widehat{\Psi}_\lambda(t)$ of the equations (4.4) and (4.5). The linearly independent classical solution the homogeneous ODE of (4.4) are given

$$\Phi_1(x) = \cos(pt), \text{ and } \Phi_2(x) = \sin(pt).$$

Thus, we have from 2.7

$$m_1(t) = -\frac{\sin(pt)}{p} \text{ and } m_2(t) = \cos(pt)$$

First the solution obtained from the Zadeh's extension principle (3.4) is provided, which is given by

$$\begin{aligned} \left[\widehat{\Phi}(t,\lambda)\right]^\alpha &= m_1(t)\,[\alpha, 2-\alpha] + m_2(t)\,[\alpha+1, 3-\alpha] \\ &= -\frac{1}{p}\sin(pt)\,[\alpha, 2-\alpha] + \cos(pt)\,[\alpha+1, 3-\alpha] \end{aligned} \tag{4.6}$$

for all $\alpha \in [0,1]$ and $t \in [0, 3.5]$.

Analogically $\widehat{\Psi}(t,\lambda)$ is obtained as follows

$$\begin{aligned} \left[\widehat{\Psi}(t,\lambda)\right]^\alpha &= m_1(t)\,[\alpha+2, 4-\alpha] + m_2(t)\,[\alpha+3, 5-\alpha] \\ &= -\frac{1}{p}\sin(pt-p)\,[\alpha+2, 4-\alpha] + \cos(pt-p)\,[\alpha+3, 5-\alpha] \end{aligned} \tag{4.7}$$

for all $\alpha \in [0,1]$ and $t \in [0, 3.5]$.

These $\widehat{\Phi}(t,\lambda)$ and $\widehat{\Psi}(t,\lambda)$ have unique solution [24]. Then putting the classical cases of (4.6) and (4.7) in equation (3.5), Wronskian function is obtained as

$$w(p)=\left(3p+\frac{8}{p}\right)\sin(p)+(2)\cos(p)\,. \tag{4.8}$$

From Theorem 3.3 , the eigenvalues of the fuzzy problem (4.1)-(4.3) are zeros the functions $w(\lambda)$ in (4.8).

If the values satisfying the equation (4.8) compute with Matlab Program, then eigenvalues of fuzzy problem are obtained in Table 1. as follows:

	p_n	λ_n
$n=1$	2.9709	8.8262
$n=2$	6.1827	38.2257
$n=3$	9.3557	87.5291
$n=4$	12.514	156.6
$n\approx$	$n\pi$	$(n\pi)^2$

Table 1. Eigenvalues of the fuzzy problem

The first five eigenvalues are found numerically and then the approximation of the remaining eigenvalues will be used. From Figure 1 It can be seen that the graphs intersect at infinitely many point $p_n\approx n\pi \quad (n=1,2,3...)$, where the error in this approximation approaches zero as $n\to\infty$. Given this estimate, Matlab program can be used to compute p_n more accurately.

From the equations (4.6) and (4.7)

$$\left[\widehat{\Phi}(t,\lambda_n)\right]^{\alpha}=-\frac{1}{p_n}\sin(p_nt)\,[\alpha,2-\alpha]+\cos(p_nt)\,[\alpha+1,3-\alpha] \tag{4.9}$$

and

$$\left[\widehat{\Psi}(t,\lambda_n)\right]^{\alpha}=-\frac{1}{p_n}\sin(p_nt-p_n)\,[\alpha+2,4-\alpha]+\cos(p_nt-p_n)\,[\alpha+3,5-\alpha] \tag{4.10}$$

are eigenfunctions associated with $\lambda_n=(p_n)^2$.

In particular, $p_1=2.9709$ is selected in Table 1 and substituted this value respectively in (4.9) and (4.10).

First the solutions obtained from the Zadeh's extension principle 3.4 are provided ,which are given by

$$\begin{aligned}\left[\widehat{\Phi}(t,p_1)\right]^{\alpha} &= m_1(t,p_1)\,[\alpha,2-\alpha]+m_2(t,p_1)\,[\alpha+1,3-\alpha]\\ &= -\frac{1}{2.9709}\sin(2.9709t)\,[\alpha,2-\alpha]+\cos(2.9709t)\,[\alpha+1,3-\alpha]\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}\left[\widehat{\Psi}(t,p_1)\right]^{\alpha} &= m_1(t,p_1)\,[\alpha+2,4-\alpha]+m_2(t,p_1)\,[\alpha+3,5-\alpha]\\ &= -\frac{1}{2.9709}\sin(2.9709t-2.9709)\,[\alpha+2,4-\alpha]\\ &\quad+\cos(2.9709t-2.9709)\,[\alpha+3,5-\alpha]\end{aligned} \tag{4.12}$$

for all $\alpha\in[0,1]$ and for all $t\in[0,3.5]$.

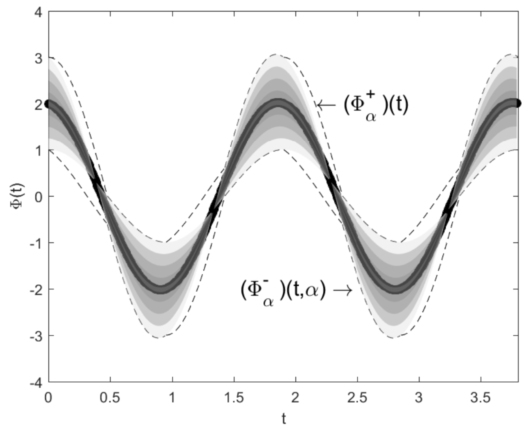

FIGURE 2. The gray-scale lines varying from white to black represent the $\alpha-$cuts of the fuzzy solution (4.11) via $sup - J$ extension principle, where their endpoints for varying from 0 to 1. Black dashed lines represent the 0-cut of the fuzzy solution (4.13) via Zadeh extension principle of $\widehat{\Phi}$

Then, it is assumed that $\widehat{1}$, $\widehat{2}$ and $\widehat{3}$, $\widehat{4}$ are linearly interactive, then there exists (q, r) such that $B = qA + r$ with ($q = 1$, $r = 1$ for $\widehat{1}, \widehat{2}$ and $\widehat{3}, \widehat{4}$ linearly interactive numbers). In this case the solutions $\widehat{\Phi}_j$ and $\widehat{\Psi}_j$ obtained from the $sup - j$ extension principle by means of 3.3 whose $\alpha-$cut is given by

$$\begin{aligned} \left[\widehat{\Phi}_j(t,p_1)\right]^\alpha &= (m_1(t,p_1) + qm_2(t,p_1))[\alpha, 2-\alpha] + rm_2(t,p_1) \\ &= \left(-\frac{1}{2.9709}\sin(2.9709t) + q\cos(2.9709t)\right)[\alpha, 2-\alpha] \\ &\quad +r\cos(2.9709t) \end{aligned} \tag{4.13}$$

and

$$\begin{aligned} \left[\widehat{\Psi}_j(t,p_1)\right]^\alpha &= (m_1(t,p_1) + qm_2(t,p_1))[\alpha, 2-\alpha] + rm_2(t,p_1) \\ &= -\frac{1}{2.9709}\sin(2.9709t - 2.9709)[\alpha+2, 4-\alpha] \\ &\quad +\cos(2.9709t - 2.9709)[\alpha+3, 5-\alpha] \end{aligned} \tag{4.14}$$

for all $\alpha \in [0, 1]$ and for all $t \in [0, 3.5]$. Fig. 2 and Fig. 3 illustrate the fuzzy solutions (4.13) and (4.14) of this FBVP for the cases where the boundary values are interactive as well as non-interactive.

Note that the solution via sup-J extension principle is contained in the solution via Zadeh's extension principle, corroborating the statement provided in [12].

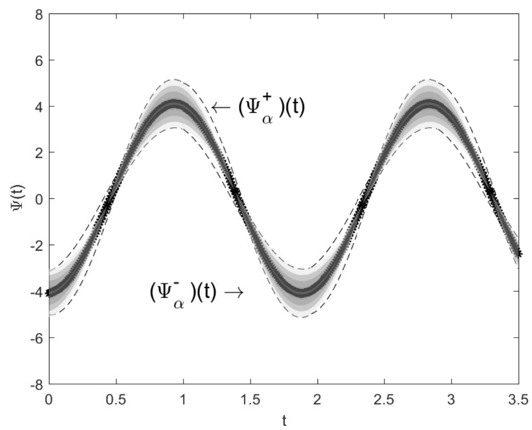

FIGURE 3. The gray-scale lines varying from white to black represent the $\alpha-$cuts of the fuzzy solution (4.12) via $sup-J$ extension principle, where their endpoints for varying from 0 to 1. Black dashed lines represent the 0-cut of the fuzzy solution (4.14) via Zadeh's extension principle of $\widehat{\Psi}$

5. Conclusion

This manuscript studies linear ordinary differential equations with two point boundary values given by interactive fuzzy numbers. The solution is obtained by means of the $sup-J$ and Zadeh's extension principle from the deterministic solutions of the associated BVP. The boundary values are non-interactive fuzzy numbers, then the fuzzy solution is given via Zadeh's extension principle.

We study linear two point FBVP with boundary values given by interactive and non-interactive fuzzy numbers. We show that the interactive fuzzy solution is contained in the non-interactive fuzzy solution by Fig.2 and Fig. 3. So it can be concluded that the fuzzy interactive solution with uncertain boundary conditions (with a membership degree given by their α-cuts) that is closer to the classical deterministic solution.

The approach via H-derivative or gH-derivative for two-point FBVP is equivalent to the study some systems of classical differential equations, which can result in an additional study of switching points. In contrast to this approach, the fuzzy solutions obtained by means of the extension principle are always well defined and do not require the analysis of the existence of switching points. Moreover from Zadeh's extension principle, the sign of the solution is not considered itself and the signs of its first and second derivatives.

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References

[1] L.C. Barros, R.C. Bassanezi and P.A. Tonelli, On the continuity of the Zadeh's extension, Seventh IFSA World Congress, Prague, pp. 22-26 (1997).
[2] L.C. Barros, R.C. Bassanezi and W.A. Lodwick, A first Course in Fuzzy Logic, Fuzzy Dynamical Systems, and Biomathematics: Theory and applications, Springer Cham., London, (2017).
[3] L.C. Barros, F.S. Pedro, Fuzzy differential equations with interactive derivative, Fuzzy Sets and Systems, Vol. 309, pp. 64–80 (2017).
[4] B. Bede and L. Stefanini, Generalized differentiability of fuzzy-valued functions, Fuzzy Sets and Systems, Vol. 230, pp. 119–141 (2013).
[5] C. Carlsson, R. Fuller and P. Majlender, Additions of Completely Correlated Fuzzy Numbers, IEEE International Conference on Fuzzy Systems, (2004).
[6] T. Ceylan and N. Altınısık, Eigenvalue problem with fuzzy coefficients of boundary conditions, Scholars Journal of Physics, Mathematics and Statistics, Vol. 5, N. 2, pp. 187–193 (2018).
[7] P. Diamond and P. Kloeden, Metric spaces of fuzzy sets World Scientific, World Scientific, Singapore, (1994).
[8] E. Esmi, D.E. Sanchez, V.F. Wasques and L.C. Barros, Solutions of higher order linear fuzzy differential equations with interactive fuzzy values, Fuzzy Sets and Systems, Vol. 419, N. 1, pp. 122–140 (2021).
[9] N. Gasilov, S.E. Amrahov and A.G. Fatullayev, Linear differential equations with fuzzy boundary values, CoRR, pp. 696–700 (2011).
[10] L.T. Gomes, L.C. Barros and B. Bede, Fuzzy Differential Equations in Various Approaches,Springer Cham, London, pp. 120 (2015).

[11] H.G. Çitil and N. Altınışık, On the eigenvalues and the eigenfunctions of the Sturm-Liouville fuzzy boundary value problem, J. Math. Comput. Sci., Vol. 7, N. 4, pp. 786–805 (2017).

[12] D.S Ibanez, E. Esmi and L.C. Barros, Linear Ordinary Differential Equations with Linearly Correlated Boundary Values, Proceedings of 2018 IEEE international conference on fuzzy systems (FUZZ-IEEE), (2018).

[13] A. Kandel and W.J. Byatt, Fuzzy differential equations, Proceedings of the International Conference on Cybernetics and Society, (1978).

[14] O. Kaleva , Fuzzy differential equations, Fuzzy sets and systems, Vol. 24, N. 3, pp. 301–317 (1987).

[15] A. Khastan and J.J. Nieto, A boundary value problem for second order fuzzy differential equations, Fuzzy sets and systems, Vol. 72, N. 9-11, pp. 3583–3593 (2010).

[16] G. Klir and B. Yuan, Fuzzy Sets and Fuzzy Logic,Prentice Hall Inc., Upper Saddle River, (1995).

[17] M.T. Mizukoshi, L.C. Barros, Y. Chalco-Cano, H. Román-Flores and R.C. Bassanezi, Fuzzy differential equations and the extension principle, Information Sciences, Vol. 177, pp. 3627–3635 (2007).

[18] M. L. Puri and D. A. Ralescu , Differentials of fuzzy functions, Journal of Math. Analysis and App., Vol. 91, N. 2, pp. 552–558 (1983).

[19] D.E. Sanchez, V.F. Wasques, E. Esmi and L.C. Barros, Solution to the Bessel differential equation with interactive fuzzy boundary conditions, Computational and Applied Mathematics, Vol. 4, N. 1, pp. 1–12 (2022).

[20] L. Stefanini and B. Bede, Generalized Hukuhara differentiability of interval-valued functions and interval differential equations, Nonlinear Analysis: Theory, Methods and Applications, Vol. 71, N. 3-4, pp. 1311–1328 (2009).

[21] I. Sadeqi, M. Moradlou and M. Salehi, On approximate cauchy equation in Felbin's type fuzzy normed linear spaces, Iranian Journal of Fuzzy Systems, Vol. 10, N. 3, pp. 51–63 (2013).

[22] T. Allahviranloo and K. Khalilpour, A numerical method for two-point fuzzy boundary value problems, World Applied Sciences Journal, Vol. 13, N. 10, pp. 2137–2147 (2011).

[23] K. Sabzi, T. Allahviranloo and S. Abbasbandy, A fuzzy generalized power series method under generalized Hukuhara differentiability for solving fuzzy Legendre differential equation, Soft Computing, Vol. 24, pp. 8763–8779 (2020).

[24] E. C. Titchmarsh, Eigenfunction expansions associated with second-order differential equations I, 2nd edn., Oxford University Press, London, (1962).

[25] V.F. Wasques, E. Esmi, L.C. Barros and P. Sussner, Numerical Solutions for Bidimensional Initial Value Problem with Interactive Fuzzy Numbers, Fuzzy Information Processing, Springer, Cham., (2018).

[26] L.A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning-III, Information Sciences, Vol. 8, N. 3, pp. 199–249 (1975).

[27] H. Gültekin, N. Altınışık, On Solution of Two point Fuzzy Boundary Value Peoblem, The Bulletin of Society for Mathematical Services and Standards, Vol. 11, pp 31-39, (2014).

[28] A. Armand, Z. Gouyandeh, Solving two-point fuzzy boundary value problem using variational iteration method, Communications on Advanced Com Science with Applications, Vol. 2013, pp. 1–10 (2013).

[29] M. H. Suhhiem, R. I. Khwayyit, Approximate Polynomial Solution for Two-Point Fuzzy Boundary Value Problems, Vol. 7 No. 1, pp. 64-79 (2024)

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