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# Innovative Decision-Making with VFP-Soft Sets: A Comparative Analysis

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**Abstract:** This study delves into fundamental set properties within the framework of virtual fuzzy parameterized (VFP-)soft set theory. It provides a comprehensive examination of these properties, offering essential insights and considerations. The study also simplifies the definition of VFP-soft sets to streamline data analysis, making it more accessible and less complex. Furthermore, the paper explores the integration of two distinct approaches for parameter weighting in VFP-soft sets. Notably, the research introduces a novel decision-making algorithm grounded in VFP-soft sets and conducts a comparative analysis to evaluate its effectiveness. This work contributes to the field by enhancing the understanding of VFP-soft sets and their applications, while also providing a practical decision-making tool for real-world scenarios.

**Key words:** Fuzzy et, soft set, VFP-soft set, algorithm, decision-making.

**VFP-Esnek Kümelerle Yenilikçi Karar Verme: Karşılaştırmalı Bir Analiz**

**Öz:** Bu çalışma sanal bulanık parametreli (VFP-)yumuşak küme teorisi çerçevesinde temel küme özelliklerini incelemektedir. Bu özelliklerin kapsamlı bir incelemesini sağlayarak temel içgörüleri ve değerlendirmeleri sunar. Çalışma aynı zamanda veri analizini kolaylaştırmak için VFP-soft kümelerinin tanımını basitleştirerek onu daha erişilebilir ve daha az karmaşık hale getiriyor. Ayrıca makale, VFP-soft kümelerinde parametre ağırlıklandırmaya yönelik iki farklı yaklaşımın entegrasyonunu araştırıyor. Özellikle araştırma, VFP-soft kümelerine dayanan yeni bir karar verme algoritması sunmakta ve bunun etkinliğini değerlendirmek için karşılaştırmalı bir analiz gerçekleştirmektedir. Bu çalışma, VFP-soft kümelerinin ve uygulamalarının anlaşılmasını geliştirerek alana katkıda bulunurken, aynı zamanda gerçek dünya senaryoları için pratik bir karar verme aracı da sağlıyor.

**Anahtar kelimeler:** Bulanık küme, esnek küme, VFP-esnek küme, algoritma, karar verme.

## 1. Introduction

Uncertainty is an important feature that must be addressed during data analysis to increase the robustness of the results. However, parsing the uncertainty of the data is generally not that easy. Therefore, many mathematical approaches based on the analysis of specific data may be insufficient to capture this component. Many theories have been introduced to deal with the uncertainty in the data. One of these theories is the fuzzy set (FS) theory, introduced to the literature by Zadeh [1] in 1965. In the following years, another important mathematical model of the effort to cope with uncertainty, the rough set (RS) theory [2] was proposed. However, FS and RS theories are difficult to apply objectively to uncertainty problems. Molodtsov [3], who thinks that the reason for this difficulty is due to the lack of a parameterization tool, proposed the soft set (SS) theory in 1999. Then, Maji et al. [4] defined the basic operations of SSs in order to make a detailed theoretical study on SSs. In addition, in the following years, Maji et al. [5] defined the concept of fuzzy soft sets (FSS) and gave an application based on FSS theory for a decision-making problem in [6]. In addition, Çağman and Enginoğlu [7] have worked on soft decision-making problems and Çağman et al. proposed an application of SS theory for a decision-making problem in [8]. On the other hand, Chen et al. [9] discussed the parameterization reduction and applications of SS. Moreover, an adjustable approach for FSS based on decision-making has been given by Feng et al. [10]. Later, Çağman et al. presented a new perspective on FSSs in their studies [11]. The combination of SS and FSS theories enabled the development of different algorithms for solving uncertainty problems [12-21].

In the following years, Çağman [22] defined the concept of fuzzy parameterized (FP-)soft set (FPSS) in order to achieve near ideal results for uncertainty problems encountered in almost every field. However, FPSSs were insufficient to express this situation when the uncertainty consisted of more than one stage. In order to overcome this deficiency, virtual fuzzy parameterized (VFP)-soft sets (VFPSSs) were proposed by Dalkılıç and Demirtaş [23] in 2021. This set theory [23] consists of lower and upper approximate functions in expressing more than one stage, so three FP-soft sets can be evaluated. This novel set theory incorporated lower and upper approximate

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functions, facilitating evaluation across multiple stages of uncertainty, effectively surpassing the limitations of FPSSs. Motivated by the need for a comprehensive approach to uncertainty management, this study revisits VFP-soft sets, aiming to streamline their representation and enhance their utility in decision-making contexts. The primary motivations and contributions of this paper are outlined as follows:

- **Simplification of Complex Structures:** We propose a streamlined representation of VFP-soft sets to tackle complex data, enhancing accessibility and usability.

- **Exploration of Incomplete Operations:** We examine incomplete basic set operations on VFP-soft sets, contributing to the ongoing development of the theory and expanding its applicability.

- **Development of a New Decision-Making Algorithm:** We present a novel decision-making algorithm leveraging fuzzy decision sets. Notably, this algorithm integrates the importance weights of parameters, accommodating two distinct approaches within VFP-soft sets. The most important feature of this algorithm is that it also considers the importance weights of the parameters for VFP-soft sets that focus on two different approaches.

Through this study, we aim to provide a comprehensive overview of VFP-soft sets, elucidating their potential in addressing uncertainty challenges. A short representation is expressed for this analyzed set theory and some new properties and remarks were included. Additionally, we introduce a new algorithm tailored to optimize decision-making processes within this framework, offering a comparative analysis with existing methodologies [23]. By addressing these key points, we endeavor to enhance the understanding and applicability of VFP-soft sets, ultimately contributing to more effective uncertainty management across diverse domains.

## 2. Preliminaries

In this section, some definitions and results are reminded. Detailed explanations related to VFPSSs can be found in [23].

Throughout this paper, let $U = \{u_1, u_2, \dots\}$ be a universe set, $P = \{p_1, p_2, \dots\}$ be a set of parameters and $X$ be a FS over $P$. In this case, the lower virtual parameter set and the upper virtual parameter set are expressed as $\underline{P} = \left\{p_1^{\underline{\alpha_1}}, p_2^{\underline{\alpha_2}}, \dots\right\}$ and $\overline{P} = \left\{p_1^{\overline{\alpha_1}}, p_2^{\overline{\alpha_2}}, \dots\right\}$, respectively. Also, let $2^U$ denote the power set of $U$ and $\emptyset \neq A \subseteq P$.

**Definition 1.** [1] A FS $X$ over $U$ is a set defined by $\mu_X: U \to [0,1]$. $\mu_X$ is called the membership function of $X$, and the value $\mu_X(u)$ is called the grade of membership of $u \in U$. Thus, as given in Equation (1), a FS $X$ over $U$ can be represented as follows:

$$X = \{(\mu_X(u)/u): u \in U, \mu_X(u) \in [0,1]\}. \tag{1}$$

**Definition 2.** [3] A pair $(F, P)$ is called a SS over $U$, where $F$ is a mapping given by $F: P \to 2^U$. In other words, a SS over $U$ is a parameterized family of subsets of $U$ for $p \in P$, $F(p)$ may be considered as the set of $p$-approximate elements of $(F, P)$.

**Definition 3.** [22] A FPSS $\Phi_X$ on $U$ is defined by the set of ordered pairs, given in Equation (2)

$$\Phi_X = \left\{\left(\frac{\mu_X(p)}{p}, \varphi_X(p)\right): p \in P, \mu_X(p) \in [0,1]\right\}, \tag{2}$$

where the function $\varphi_X: P \to 2^U$ is called approximate function such that $\varphi_X(p) = \emptyset$ if $\mu_X(p) = 0$, and the function $\mu_X: U \to [0,1]$ is called membership function of FPSS $\Phi_X$.

State that the set of all FPSSs over $U$ will be denoted by $FPS(U)$.

**Definition 4.** [22] Let $\Phi_X \in FPS(U)$. Then a fuzzy decision set of $\Phi_X$, denoted by $\Phi_X^d$, is defined by Equation (3).

$$\Phi_X^d = \left\{\mu_{\Phi_X^d}(u)/u : u \in U\right\} \tag{3}$$

which is a FS over $U$, its membership function $\mu_{\Phi_X^d}$ is defined by $\mu_{\Phi_X^d}: U \to [0,1]$, given in Equation (4)

$$\mu_{\Phi_X^d}(u) = \frac{1}{|supp(X)|}\sum_{p\in supp(X)} \mu_X(p)\chi_{\varphi_X(p)}(u) \tag{4}$$

where $supp(X)$ is the support set of $X$, $\varphi_X(p)$ is the crisp subset determined by the parameter $p$ and

$$\chi_{\varphi_X(p)}(u) = \begin{cases} 1, & u \in \varphi_X(p) \\ 0, & u \notin \varphi_X(p) \end{cases}. \tag{5}$$

as given in Equation (5).

**Definition 5.** [23] Let $\underline{X}$, $X$, $\overline{X}$ be a FS over $\underline{P}$, $P$, $\overline{P}$, respectively. Thus, in Equations (6), (7), (8) and (9), a VFPSS $\Psi_X$ over $U$ is defined as follows:

$$\Psi_X = \underline{\Upsilon_X} \cup \Upsilon_X \cup \overline{\Upsilon_X} \tag{6}$$

such that

$$\underline{\Upsilon_X} = \left\{\left(\frac{\mu_X(p)-\underline{\alpha}}{p}, \underline{\psi_X}(p^{\underline{\alpha}})\right) : p^{\underline{\alpha}} \in \underline{P}, p \in P, \mu_X(p) \in [0,1], 0 \le \underline{\alpha} < \mu_X(p)\right\}, \tag{7}$$

$$\Upsilon_X = \left\{\left(\frac{\mu_X(p)}{p}, \psi_X(p)\right) : p \in P, \mu_X(p) \in [0,1]\right\}, \tag{8}$$

$$\overline{\Upsilon_X} = \left\{\left(\frac{\mu_X(p)+\overline{\alpha}}{p}, \overline{\psi_X}(p^{\overline{\alpha}})\right) : p \in P, p^{\overline{\alpha}} \in \overline{P}, \mu_X(p) \in [0,1], 0 \le \overline{\alpha} \le 1-\mu_X(p)\right\}, \tag{9}$$

where the functions $\underline{\psi_X}: \underline{P} \to 2^U$, $\psi_X: P \to 2^U$, $\overline{\psi_X}: \overline{P} \to 2^U$ are called lower approximate function, approximate function, upper approximate function, respectively, and the functions $\mu_X: P \to [0,1]$ is called membership function of the set $X$. Here $\psi_X(p) = \emptyset$ if $\mu_X(p) = 0$. Moreover, $\underline{\psi_X}(p^{\underline{\alpha}}) = \emptyset$ if $\mu_X(p) - \underline{\alpha} = 0$ and $\overline{\psi_X}(p^{\overline{\alpha}}) = \emptyset$ if $\mu_X(p) + \overline{\alpha} = 0$.

Obviously, each ordinary SSs can be written as VFPSSs.

From now on, $VFPS(U)$ denotes the family of all VFPSSs over $U$ with $P$ as the set of parameters.

**Example 1.** Let $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$ be an universe set, $P = \{p_1, p_2, p_3, p_4\}$ be the set of parameters and $X = \{0.45/p_2, 0.6/p_4\}$ be a FS over $P$. If $\underline{X} = \{0.32/p_2, 0.24/p_4\}$, $\overline{X} = \{0.8/p_2, 0.92/p_4\}$, and

$$\begin{aligned} \underline{\psi_X}(p_2^{0.13}) &= \{u_2, u_4, u_5, u_6, u_7\}, & \underline{\psi_X}(p_4^{0.36}) &= \{u_1, u_2, u_3, u_5, u_6\}, \\ \psi_X(p_2) &= \{u_2, u_5, u_6, u_7\}, & \psi_X(p_4) &= \{u_2, u_3, u_5, u_6\}, \\ \overline{\psi_X}(p_2^{0.35}) &= \{u_5, u_6, u_7\}, & \overline{\psi_X}(p_4^{0.32}) &= \{u_3, u_5, u_6\}, \end{aligned}$$

then the VFPSS $\Psi_X$ is written by

$$\Psi_X = \begin{Bmatrix} (0.32/p_2, \{u_2, u_4, u_5, u_6, u_7\}), (0.24/p_4, \{u_1, u_2, u_3, u_5, u_6\}) \\ (0.45/p_2, \{u_2, u_5, u_6, u_7\}), (0.6/p_4, \{u_2, u_3, u_5, u_6\}) \\ (0.8/p_2, \{u_5, u_6, u_7\}), (0.92/p_4, \{u_3, u_5, u_6\}) \end{Bmatrix}$$

where

$$\underline{\Upsilon_X} = \{(0.32/p_2, \{u_2, u_4, u_5, u_6, u_7\}), (0.24/p_4, \{u_1, u_2, u_3, u_5, u_6\})\},$$

$$\Upsilon_X = \{(0.45/p_2, \{u_2, u_5, u_6, u_7\}), (0.6/p_4, \{u_2, u_3, u_5, u_6\})\}$$

and

$$\overline{\Upsilon_X} = \{(0.8/p_2, \{u_5, u_6, u_7\}), (0.92/p_4, \{u_3, u_5, u_6\})\}.$$

**Definition 6.** [23] Let $\Psi_X \in VFPS(U)$.

i. If $\underline{\psi_X}(p^{\underline{\alpha}}) = \psi_X(p) = \overline{\psi_X}(p^{\overline{\alpha}}) = \emptyset$ for all $p^{\underline{\alpha}} \in \underline{P}$, $p \in P$, $p^{\overline{\alpha}} \in \overline{P}$, then VFPSS $\Psi_X$ is called an $X$-empty VFPSS, denoted by $\Psi_{\emptyset_X}$. If $X = \emptyset$, then $\Psi_X$ is called an empty VFPSS, denoted by $\Psi_\emptyset$.

ii. If $\underline{X}$, $X$, $\overline{X}$ is a crisp subset of $\underline{P}$, $P$, $\overline{P}$, respectively, and $\underline{\psi_X}(p^{\underline{\alpha}}) = \psi_X(p) = \overline{\psi_X}(p^{\overline{\alpha}}) = U$ for all $p^{\underline{\alpha}} \in \underline{P}$, $p \in P$, $p^{\overline{\alpha}} \in \overline{P}$, then VFPSS $\Psi_X$ is called an $X$-universal VFPSS, denoted by $\Psi_{\tilde{X}}$. If $X = P$, then the $X$-universal VFPSS is called universal VFPSS, denoted by $\Psi_{\tilde{P}}$.

**Definition 7.** [23] Let $\Psi_X, \Psi_Y \in VFPS(U)$. Then, $\Psi_X$ is a VFP-soft subset of $\Psi_Y$, denoted by $\Psi_X \tilde{\sqsubseteq} \Psi_Y$, if

i. $\mu_X(p) - \underline{\alpha} \le \mu_Y(p) - \underline{\beta}$ and $\underline{\psi_X}(p^{\underline{\alpha}}) \subseteq \underline{\psi_Y}(p^{\underline{\beta}})$ for all $p^{\underline{\alpha}}, p^{\underline{\beta}} \in \underline{P}$,
ii. $\mu_X(p) \le \mu_Y(p)$ and $\psi_X(p) \subseteq \psi_Y(p)$ for all $p \in P$,
iii. $\mu_X(p) + \overline{\alpha} \le \mu_Y(p) + \overline{\beta}$ and $\overline{\psi_X}(p^{\overline{\alpha}}) \subseteq \overline{\psi_Y}\left(p^{\overline{\beta}}\right)$ for all $p^{\overline{\alpha}}, p^{\overline{\beta}} \in \overline{P}$.

Also, $\Psi_X$ is a VFP-soft equal to $\Psi_Y$, denoted by $\Psi_X = \Psi_Y$, if

i. $\mu_X(p) - \underline{\alpha} = \mu_Y(p) - \underline{\beta}$ and $\underline{\psi_X}(p^{\underline{\alpha}}) = \underline{\psi_Y}(p^{\underline{\beta}})$ for all $p^{\underline{\alpha}}, p^{\underline{\beta}} \in \underline{P}$,
ii. $\mu_X(p) = \mu_Y(p)$ and $\psi_X(p) = \psi_Y(p)$ for all $p \in P$,
iii. $\mu_X(p) + \overline{\alpha} = \mu_Y(p) + \overline{\beta}$ and $\overline{\psi_X}(p^{\overline{\alpha}}) = \overline{\psi_Y}\left(p^{\overline{\beta}}\right)$ for all $p^{\overline{\alpha}}, p^{\overline{\beta}} \in \overline{P}$.

**Proposition 1.** [23] Let $\Psi_X \in VFPS(U)$. $s\left(\overline{\psi_X}(p^{\overline{\alpha}})\right) \subseteq s(\psi_Y(p)) \subseteq s\left(\underline{\psi_X}(p^{\underline{\alpha}})\right)$ is valid for all $p^{\underline{\alpha}} \in \underline{P}, p \in P, p^{\overline{\alpha}} \in \overline{P}$.

**Definition 8.** [23] Let $\Psi_X \in VFPS(U)$. Then, complement $\Psi_X$, denoted by $\Psi_X^c$, is a VFPSS defined by the approximate and membership functions as

i. $\mu_{X^c}(p) - \underline{\tilde{\alpha}} = 1 - \left(\mu_X(p) - \underline{\alpha}\right)$ and $\underline{\psi_{X^c}}\left(p^{\underline{\tilde{\alpha}}}\right) = U / \underline{\psi_X}(p^{\underline{\alpha}})$ for all $p^{\underline{\alpha}}, p^{\underline{\tilde{\alpha}}} \in \underline{P}$,
ii. $\mu_{X^c}(p) = 1 - \mu_X(p)$ and $\psi_{X^c}(p) = U / \psi_X(p)$ for all $p \in P$,
iii. $\mu_{X^c}(p) + \overline{\tilde{\alpha}} = 1 - (\mu_X(p) + \overline{\alpha})$ and $\overline{\psi_{X^c}}\left(p^{\overline{\tilde{\alpha}}}\right) = U / \overline{\psi_X}(p^{\overline{\alpha}})$ for all $p^{\overline{\alpha}}, p^{\overline{\tilde{\alpha}}} \in \overline{P}$.

**Definition 9.** [23] Let $\Psi_X, \Psi_Y \in VFPS(U)$. Then, union $\Psi_X$ and $\Psi_Y$, denoted by $\Psi_X \tilde{\sqcup} \Psi_Y$, is defined by

i. $\mu_{X \cup Y}(p) - \underline{\gamma} = max\left\{\mu_X(p) - \underline{\alpha}, \mu_Y(p) - \underline{\beta}\right\}$ and $\underline{\psi_{X \cup Y}}(p^{\underline{\gamma}}) = \underline{\psi_X}(p^{\underline{\alpha}}) \cup \underline{\psi_Y}(p^{\underline{\beta}})$ for all $p^{\underline{\alpha}}, p^{\underline{\beta}}, p^{\underline{\gamma}} \in \underline{P}$,
ii. $\mu_{X \cup Y}(p) = max\{\mu_X(p), \mu_Y(p)\}$ and $\psi_{X \cup Y}(p) = \psi_X(p) \cup \psi_Y(p)$ for all $p \in P$,

iii. $\mu_{X\cup Y}(p)+\overline{\gamma}=max\{\mu_X(p)+\overline{\alpha},\mu_Y(p)+\overline{\beta}\}$ and $\overline{\psi_{X\cup Y}}(p^{\overline{\gamma}})=\overline{\psi_X}(p^{\overline{\alpha}})\cup\overline{\psi_Y}\left(p^{\overline{\beta}}\right)$ for all $p^{\overline{\alpha}},p^{\overline{\beta}},p^{\overline{\gamma}}\in\overline{P}$.

**Definition 10.** [23] Let $\Psi_X,\Psi_Y\in VFPS(U)$. Then, intersection $\Psi_X$ and $\Psi_Y$, denoted by $\Psi_X\,\widetilde{\sqcap}\,\Psi_Y$, is defined by

i. $\mu_{X\cap Y}(p)-\underline{\gamma}=min\left\{\mu_X(p)-\underline{\alpha},\mu_Y(p)-\underline{\beta}\right\}$ and $\underline{\psi_{X\cap Y}}(p^{\underline{\gamma}})=\underline{\psi_X}(p^{\underline{\alpha}})\cap\underline{\psi_Y}(p^{\underline{\beta}})$ for all $p^{\underline{\alpha}},p^{\underline{\beta}},p^{\underline{\gamma}}\in\underline{P}$,
ii. $\mu_{X\cap Y}(p)=min\{\mu_X(p),\mu_Y(p)\}$ and $\psi_{X\cap Y}(p)=\psi_X(p)\cap\psi_Y(p)$ for all $p\in P$,
iii. $\mu_{X\cap Y}(p)+\overline{\gamma}=min\{\mu_X(p)+\overline{\alpha},\mu_Y(p)+\overline{\beta}\}$ and $\overline{\psi_{X\cap Y}}(p^{\overline{\gamma}})=\overline{\psi_X}(p^{\overline{\alpha}})\cap\overline{\psi_Y}\left(p^{\overline{\beta}}\right)$ for all $p^{\overline{\alpha}},p^{\overline{\beta}},p^{\overline{\gamma}}\in\overline{P}$.

## 3. On VFP-Soft Sets

In this section; in order to avoid complex representations in data analysis, a simpler representation for VFP-soft sets is proposed. Moreover, some new properties of VFP-soft sets are studied. In addition, a fuzzy decision set is given for the decision-making algorithm built in the next section.

**Definition 11.** Let $\Psi_X\in VFPS(U)$. The presentation of

$$\Psi_X=\left\{\begin{matrix}\left(\frac{\mu_X(p)-\underline{\alpha}}{p},\underline{\psi_X}(p^{\underline{\alpha}})\right)\cup\\ \left(\frac{\mu_X(p)}{p},\psi_X(p)\right)\cup\\ \left(\frac{\mu_X(p)+\overline{\alpha}}{p},\overline{\psi_X}(p^{\overline{\alpha}})\right)\end{matrix}:\begin{matrix}p^{\underline{\alpha}}\in\underline{P} & 0\le\underline{\alpha}<\mu_X(p)\\ p\in P, & \mu_X(p)\in[0,1]\\ p^{\overline{\alpha}}\in\overline{P} & 0\le\overline{\alpha}\le1-\mu_X(p)\end{matrix}\right\}\tag{10}$$

is said to be a short representation of VFPSS $\Psi_X$ as given in Equation (10). Here, the internal structure of the set expressed in (10) denotes the union of the sets (7), (8) and (9). It is the same as the definition of VFP-soft set given in Definition 5, but this representation is shorter and clearer. In this way, it is aimed to express indefinite data sets more easily.

**Remark 1.** $\Psi_X\,\widetilde{\sqsubseteq}\,\Psi_Y$ does not imply that every element of $\Psi_X$ is an element of $\Psi_Y$. For example, let's consider the VFPSSs $\Psi_Y$ given below and $\Psi_X$ of Example 1,

$$\Psi_X=\left\{\begin{matrix}(0.32/p_2,\{u_2,u_4,u_5,u_6,u_7\}),(0.24/p_4,\{u_1,u_2,u_3,u_5,u_6\})\\ (0.45/p_2,\{u_2,u_5,u_6,u_7\}),(0.6/p_4,\{u_2,u_3,u_5,u_6\})\\ (0.8/p_2,\{u_5,u_6,u_7\}),(0.92/p_4,\{u_3,u_5,u_6\})\end{matrix}\right\},$$

$$\Psi_Y=\left\{\begin{matrix}(0.5/p_2,\{u_1,u_2,u_4,u_5,u_6,u_7\}),(0.4/p_4,\{u_1,u_2,u_3,u_5,u_6,u_7\})\\ (0.65/p_2,\{u_2,u_5,u_6,u_7\}),(0.7/p_4,\{u_1,u_2,u_3,u_5,u_6\})\\ (0.9/p_2,\{u_5,u_6,u_7\}),(0.95/p_4,\{u_2,u_3,u_5,u_6\})\end{matrix}\right\}.$$

Then, for $p_4$

i. $0.24=\mu_X(p_4)-0.36\le\mu_Y(p_4)-0.3=0.4$ and $\underline{\psi_X}(p_4^{0.36})\subseteq\underline{\psi_Y}(p_4^{0.3})$ for $p_4^{0.36},p_4^{0.3}\in\underline{P}$,
ii. $0.6=\mu_X(p_4)\le\mu_Y(p_4)=0.7$ and $\psi_X(p)\subseteq\psi_Y(p)$ for all $p\in P$,
iii. $0.92=\mu_X(p_4)+0.32\le\mu_Y(p_4)+0.25=0.95$ and $\overline{\psi_X}(p_4^{0.32})\subseteq\overline{\psi_Y}(p_4^{0.25})$ for $p_4^{0.32},p_4^{0.25}\in\overline{P}$.

It can be shown similarly for $p_2$. Therefore, $\Psi_X \tilde{\sqsubseteq} \Psi_Y$. It is clear that $(0.24/p_4, \{u_1, u_2, u_3, u_5, u_6\}) \in \Psi_X$ but $(0.24/p_4, \{u_1, u_2, u_3, u_5, u_6\}) \notin \Psi_Y$.

**Proposition 2.** Let $\Psi_X, \Psi_Y \in VFPS(U)$. Then,

i. $\Psi_\emptyset \tilde{\sqsubseteq} \Psi_X \tilde{\sqsubseteq} \Psi_X \tilde{\sqsubseteq} \Psi_{\tilde{P}}$,
ii. If $\Psi_X \tilde{\sqsubseteq} \Psi_Y$ and $\Psi_Y \tilde{\sqsubseteq} \Psi_Z$, then $\Psi_X \tilde{\sqsubseteq} \Psi_Z$,
iii. If $\Psi_X = \Psi_Y$ and $\Psi_Y = \Psi_Z$, then $\Psi_X = \Psi_Z$,
iv. If $\Psi_X \tilde{\sqsubseteq} \Psi_Y$ and $\Psi_Y \tilde{\sqsubseteq} \Psi_X$, then $\Psi_X = \Psi_Y$.

Proof. They can be proved easily by using the approximate and membership functions of the VFPSSs.

**Remark 2.** Let $\Psi_X \in VFPS(U)$. If $\Psi_\emptyset \neq \Psi_X \neq \Psi_{\tilde{P}}$, then $\Psi_X \tilde{\sqcup} \Psi_X^c \neq \Psi_{\tilde{P}}$ and $\Psi_X \tilde{\sqcap} \Psi_X^c \neq \Psi_\emptyset$.

**Proposition 3.** Let $\Psi_X, \Psi_Y \in VFPS(U)$. Then De Morgan's laws are valid

i. $(\Psi_X \tilde{\sqcup} \Psi_Y)^c = \Psi_X^c \tilde{\sqcap} \Psi_Y^c$,
ii. $(\Psi_X \tilde{\sqcap} \Psi_Y)^c = \Psi_X^c \tilde{\sqcup} \Psi_Y^c$.

Proof. For all $p^{\underline{\alpha}} \in \underline{P}$, $p \in P$, $p^{\overline{\alpha}} \in \overline{P}$;

i. $\mu_{(X\cup Y)^c}(p) - \underline{\tilde{\gamma}} = 1 - \left(\mu_{X\cup Y}(p) - \underline{\gamma}\right) = 1 - max\left\{\mu_X(p) - \underline{\alpha}, \mu_Y(p) - \underline{\beta}\right\} = min\left\{1 - \left(\mu_X(p) - \underline{\alpha}\right), 1 - \left(\mu_Y(p) - \underline{\beta}\right)\right\} = min\left\{\mu_{X^c}(p) - \underline{\tilde{\alpha}}, \mu_{Y^c}(p) - \underline{\tilde{\beta}}\right\} = \mu_{X^c\cap Y^c}(p) - \underline{\tilde{\gamma}}$
ii. $\mu_{(X\cup Y)^c}(p) = 1 - \mu_{X\cup Y}(p) = 1 - max\{\mu_X(p), \mu_Y(p)\} = min\{1 - \mu_X(p), 1 - \mu_Y(p)\} = min\{\mu_{X^c}(p), \mu_{Y^c}(p)\} = \mu_{X^c\cap Y^c}(p)$
iii. $\mu_{(X\cup Y)^c}(p) + \overline{\tilde{\gamma}} = 1 - (\mu_{X\cup Y}(p) + \overline{\gamma}) = 1 - max\{\mu_X(p) + \overline{\alpha}, \mu_Y(p) + \overline{\beta}\} = min\{1 - (\mu_X(p) + \overline{\alpha}), 1 - \left(\mu_Y(p) + \overline{\beta}\right)\} = min\left\{\mu_{X^c}(p) + \overline{\tilde{\alpha}}, \mu_{Y^c}(p) + \overline{\tilde{\beta}}\right\} = \mu_{X^c\cap Y^c}(p) + \overline{\tilde{\gamma}}$

and

i. $\underline{\psi_{(X\cup Y)^c}}(p^{\underline{\tilde{\gamma}}}) = U \setminus \underline{\psi_{X\cup Y}}(p^{\underline{\gamma}}) = U \setminus \left(\underline{\psi_X}(p^{\underline{\alpha}}) \cup \underline{\psi_Y}(p^{\underline{\beta}})\right) = \left(U \setminus \underline{\psi_X}(p^{\underline{\alpha}})\right) \cap \left(U \setminus \underline{\psi_Y}(p^{\underline{\beta}})\right) = \underline{\psi_{X^c}}(p^{\underline{\tilde{\alpha}}}) \cap \underline{\psi_{Y^c}}\left(p^{\underline{\tilde{\beta}}}\right) = \underline{\psi_{X^c\cap Y^c}}(p^{\underline{\tilde{\gamma}}})$
ii. $\psi_{(X\cup Y)^c}(p) = U \setminus \psi_{X\cup Y}(p) = U \setminus \left(\psi_X(p) \cup \psi_Y(p)\right) = \left(U \setminus \psi_X(p)\right) \cap \left(U \setminus \psi_Y(p)\right) = \psi_{X^c}(p) \cap \psi_{Y^c}(p) = \psi_{X^c\cap Y^c}(p)$
iii. $\overline{\psi_{(X\cup Y)^c}}\left(p^{\overline{\tilde{\gamma}}}\right) = U \setminus \overline{\psi_{X\cup Y}}(p^{\overline{\gamma}}) = U \setminus \left(\overline{\psi_X}(p^{\overline{\alpha}}) \cup \overline{\psi_Y}\left(p^{\overline{\beta}}\right)\right) = \left(U \setminus \overline{\psi_X}(p^{\overline{\alpha}})\right) \cap \left(U \setminus \overline{\psi_Y}\left(p^{\overline{\beta}}\right)\right) = \overline{\psi_{X^c}}\left(p^{\overline{\tilde{\alpha}}}\right) \cap \overline{\psi_{Y^c}}\left(p^{\overline{\tilde{\beta}}}\right) = \overline{\psi_{X^c\cap Y^c}}\left(p^{\overline{\tilde{\gamma}}}\right)$

Likewise, the proof of (ii) can be made similarly.
**Proposition 4.** Let $\Psi_X \in VFPS(U)$. Then,

i. $(\Psi_X^c)^c = \Psi_X$,
ii. $\Psi_\emptyset^c = \Psi_{\tilde{P}}$.

Proof. Straightforward.

**Proposition 5.** Let $\Psi_X, \Psi_Y, \Psi_Z \in VFPS(U)$. Then,

i. $\Psi_X \tilde{\cup} (\Psi_Y \tilde{\cap} \Psi_Z) = (\Psi_X \tilde{\cup} \Psi_Y) \tilde{\cap} (\Psi_X \tilde{\cup} \Psi_Z)$,
ii. $\Psi_X \tilde{\cap} (\Psi_Y \tilde{\cup} \Psi_Z) = (\Psi_X \tilde{\cap} \Psi_Y) \tilde{\cup} (\Psi_X \tilde{\cap} \Psi_Z)$.

Proof. For all $p^{\underline{\alpha}} \in \underline{P}$, $p \in P$, $p^{\overline{\alpha}} \in \overline{P}$

i. $\mu_{X\cup(Y\cap Z)}(p) - \underline{\alpha_{1,2,3}} = max\left\{\mu_X(p) - \underline{\alpha_1}, \mu_{Y\cap Z}(p) - \underline{\alpha_{2,3}}\right\} = max\left\{\mu_X(p) - \underline{\alpha_1}, min\left\{\mu_Y(p) - \underline{\alpha_2}, \mu_Z(p) - \underline{\alpha_3}\right\}\right\} = min\left\{max\left\{\mu_X(p) - \underline{\alpha_1}, \mu_Y(p) - \underline{\alpha_2}\right\}, max\left\{\mu_X(p) - \underline{\alpha_1}, \mu_Z(p) - \underline{\alpha_3}\right\}\right\} = min\left\{\mu_{X\cup Y}(p) - \underline{\alpha_{1,2}}, \mu_{X\cup Z}(p) - \underline{\alpha_{1,3}}\right\} = \mu_{(X\cup Y)\cap(X\cup Z)}(p) - \underline{\alpha_{1,2,3}}$
ii. $\mu_{X\cup(Y\cap Z)}(p) = max\{\mu_X(p), \mu_{Y\cap Z}(p)\} = max\{\mu_X(p), min\{\mu_Y(p), \mu_Z(p)\}\} = min\{max\{\mu_X(p), \mu_Y(p)\}, max\{\mu_X(p), \mu_Z(p)\}\} = min\{\mu_{X\cup Y}(p), \mu_{X\cup Z}(p)\} = \mu_{(X\cup Y)\cap(X\cup Z)}(p)$
iii. $\mu_{X\cup(Y\cap Z)}(p) + \overline{\alpha_{1,2,3}} = max\{\mu_X(p) + \overline{\alpha_1}, \mu_{Y\cap Z}(p) + \overline{\alpha_{2,3}}\} = max\{\mu_X(p) + \overline{\alpha_1}, min\{\mu_Y(p) + \overline{\alpha_2}, \mu_Z(p) + \overline{\alpha_3}\}\} = min\{max\{\mu_X(p) + \overline{\alpha_1}, \mu_Y(p) + \overline{\alpha_2}\}, max\{\mu_X(p) + \overline{\alpha_1}, \mu_Z(p) + \overline{\alpha_3}\}\} = min\{\mu_{X\cup Y}(p) + \overline{\alpha_{1,2}}, \mu_{X\cup Z}(p) + \overline{\alpha_{1,3}}\} = \mu_{(X\cup Y)\cap(X\cup Z)}(p) + \overline{\alpha_{1,2,3}}$

and

i. $\underline{\psi_{X\cup(Y\cap Z)}}(p^{\underline{\alpha_{1,2,3}}}) = \underline{\psi_X}(p^{\underline{\alpha_1}}) \cup \underline{\psi_{Y\cap Z}}(p^{\underline{\alpha_{2,3}}}) = \underline{\psi_X}(p^{\underline{\alpha_1}}) \cup \left(\underline{\psi_Y}(p^{\underline{\alpha_2}}) \cap \underline{\psi_Z}(p^{\underline{\alpha_3}})\right) = \left(\underline{\psi_X}(p^{\underline{\alpha_1}}) \cup \underline{\psi_Y}(p^{\underline{\alpha_2}})\right) \cap \left(\underline{\psi_X}(p^{\underline{\alpha_1}}) \cup \underline{\psi_Z}(p^{\underline{\alpha_3}})\right) = \underline{\psi_{X\cup Y}}(p^{\underline{\alpha_{1,2}}}) \cap \underline{\psi_{X\cup Z}}(p^{\underline{\alpha_{1,3}}}) = \underline{\psi_{(X\cup Y)\cap(X\cup Z)}}(p^{\underline{\alpha_{1,2,3}}})$
ii. $\psi_{X\cup(Y\cap Z)}(p) = \psi_X(p) \cup \psi_{Y\cap Z}(p) = \psi_X(p) \cup \left(\psi_Y(p) \cap \psi_Z(p)\right) = \left(\psi_X(p) \cup \psi_Y(p)\right) \cap \left(\psi_X(p) \cup \psi_Z(p)\right) = \psi_{X\cup Y}(p) \cap \psi_{X\cup Z}(p) = \psi_{(X\cup Y)\cap(X\cup Z)}(p)$
iii. $\overline{\psi_{X\cup(Y\cap Z)}}(p^{\overline{\alpha_{1,2,3}}}) = \overline{\psi_X}(p^{\overline{\alpha_1}}) \cup \overline{\psi_{Y\cap Z}}(p^{\overline{\alpha_{2,3}}}) = \overline{\psi_X}(p^{\overline{\alpha_1}}) \cup \left(\overline{\psi_Y}(p^{\overline{\alpha_2}}) \cap \overline{\psi_Z}(p^{\overline{\alpha_3}})\right) = \left(\overline{\psi_X}(p^{\overline{\alpha_1}}) \cup \overline{\psi_Y}(p^{\overline{\alpha_2}})\right) \cap \left(\overline{\psi_X}(p^{\overline{\alpha_1}}) \cup \overline{\psi_Z}(p^{\overline{\alpha_3}})\right) = \overline{\psi_{X\cup Y}}(p^{\overline{\alpha_{1,2}}}) \cap \overline{\psi_{X\cup Z}}(p^{\overline{\alpha_{1,3}}}) = \overline{\psi_{(X\cup Y)\cap(X\cup Z)}}(p^{\overline{\alpha_{1,2,3}}})$

Likewise, the proof of (ii) can be made in a similar way.

**Proposition 6.** Let $\Psi_X, \Psi_Y, \Psi_Z \in VFPS(U)$. Then,

i. $\Psi_X \tilde{\cup} \Psi_X = \Psi_X$ and $\Psi_X \tilde{\cap} \Psi_X = \Psi_X$,
ii. $\Psi_X \tilde{\cup} \Psi_\emptyset = \Psi_X$ and $\Psi_X \tilde{\cap} \Psi_\emptyset = \Psi_\emptyset$,
iii. $\Psi_X \tilde{\cup} \Psi_{\tilde{P}} = \Psi_{\tilde{P}}$ and $\Psi_X \tilde{\cap} \Psi_{\tilde{P}} = \Psi_X$,
iv. $\Psi_X \tilde{\cup} \Psi_Y = \Psi_Y \tilde{\cup} \Psi_X$ and $\Psi_X \tilde{\cap} \Psi_Y = \Psi_Y \tilde{\cap} \Psi_X$,
v. $(\Psi_X \tilde{\cup} \Psi_Y) \tilde{\cup} \Psi_Z = \Psi_X \tilde{\cup} (\Psi_Y \tilde{\cup} \Psi_Z)$ and $(\Psi_X \tilde{\cap} \Psi_Y) \tilde{\cap} \Psi_Z = \Psi_X \tilde{\cap} (\Psi_Y \tilde{\cap} \Psi_Z)$.

Proof. The proofs can be easily obtained from Definition 8 and 9.

**Definition 12.** Let $\Psi_X \in VFPS(U)$. Then a fuzzy decision set of $\Psi_X$, denoted by $\Psi_X^d$, is defined by Equations (11), (12), (13), (14) and (15)

$$\Psi_X^d = \left\{\mu_{\Psi_X^d}(u)/u: u \in U\right\} \tag{11}$$

which is a FS over $U$ and $\mu_{\Psi_X^d}$ is defined by $\mu_{\Psi_X^d}: U \to [0,1]$, (as shown in Equation (12), (13), (14) and (15))

$$\mu_{\Psi_X^d}(u) = \frac{1}{3}\left(\sum_{p^{\underline{\alpha}} \in \underline{X}} \frac{(\mu_X(p) - \underline{\alpha})\, \chi_{\underline{\psi_X}(p^{\underline{\alpha}})}(u)}{\underline{w}|\underline{X}|} + \sum_{p \in X} \frac{\mu_X(p)\, \chi_{\psi_X(p)}(u)}{w|X|} + \sum_{p^{\overline{\alpha}} \in \overline{X}} \frac{(\mu_X(p) + \overline{\alpha})\, \chi_{\overline{\psi_X}(p^{\overline{\alpha}})}(u)}{\overline{w}|\overline{X}|}\right) \tag{12}$$

where $|\underline{X}|$, $|X|$, $|\overline{X}|$ are the cardinality of $\underline{X}$, $X$, $\overline{X}$ and

$$\chi_{\underline{\psi_X}(p^{\underline{\alpha}})}(u) = \begin{cases} 1, & u \in \underline{\psi_X}(p^{\underline{\alpha}}) \\ 0, & u \notin \underline{\psi_X}(p^{\underline{\alpha}}) \end{cases} \tag{13}$$

$$\chi_{\psi_X(p)}(u) = \begin{cases} 1, & u \in \psi_X(p) \\ 0, & u \notin \psi_X(p) \end{cases} \tag{14}$$

$$\chi_{\overline{\psi_X}(p^{\overline{\alpha}})}(u) = \begin{cases} 1, & u \in \overline{\psi_X}(p^{\overline{\alpha}}) \\ 0, & u \notin \overline{\psi_X}(p^{\overline{\alpha}}) \end{cases} \tag{15}$$

Here, $\underline{w}, w, \overline{w}$ represent the total weight of objects with membership degrees $\mu_X(p) - \underline{\alpha}$, the total weight of objects with membership degrees $\mu_X(p)$, the total weight of objects with membership degrees $\mu_X(p) + \overline{\alpha}$, respectively, for all $p \in P$ and $\underline{w}, w, \overline{w} \in \mathbb{R}^+$.

**Remark 3.** As the membership degree of a parameter in Proposition 1 decreases, the number of objects that provide that parameter increases. For this reason, when we increase the membership degree, it is clear that there should be a relationship $\overline{w} < w < \underline{w}$ between the total weights in order to highlight the objects that still provide that parameter. So the choice of weights is not random. In this case, if there is a contrary situation between the total weights, it becomes difficult to highlight the best objects that meet the desired parameters.

**4. Numerical Example**

In this section, a decision-making algorithm is proposed for the selection of the best choice. Moreover, the given algorithm has been analyzed comparatively with an algorithm proposed for VFPSS.

Firstly; let's construct a decision-making method over a fuzzy decision set of $\Psi_X$ by the following algorithm;

**Algorithm.** The algorithm for the selection of the best choice is given as:

**Step 1:** Express the uncertainty encountered with the help of a VFPSS $\Psi_X$.
**Step 2:** Compute the fuzzy decision set $\Psi_X^d$.
**Step 3:** Find $r$, for which $\mu_{\Psi_X^d}(u_r) = max\left\{\mu_{\Psi_X^d}(u) : u \in U\right\}$.

**Remark 4.** If $r$ has more than one value then any one of them may be chose.

Now, let's consider the uncertainty problem given below to analyze the algorithm.

**Example 2.** Suppose a school wants to choose the students that best suit its parameters. For this, the school has posted an announcement. According to the announcement, a three-stage exam will be held for candidate students. Participation conditions for these exams are stated as follows:

(A) All student candidates who apply for all three-stage exams held by the school will be able to participate.
(B) The first exam is less decisive than the second exam and is less decisive in the second exam than the third exam.

(C) Increasing decisive in an exam means that the score to be obtained from this exam is higher.

Assume that the set of candidate students applying for admission to the school under these conditions is $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10}\}$ and the set of parameters the school requires from students is $P =$

$\{p_1 = self - confident, p_2 = successful, \}$. Moreover, the difficulty levels of the exams to be made by the school administration to evaluate candidate students are determined as follows:

For each parameter, the difficulty level of the first exam is "0.28, 0.38", the second exam's difficulty level is "0.45, 0.57" and finally, the third exam's difficulty level is "0.68, 0.75", respectively. The difficulty levels of the exams expressed here construct FSs of parameter sets $\underline{P}$, $P$, $\overline{P}$ and are $\underline{X} = \{0.28/p_1, 0.38/p_2\}$, $X = \{0.45/p_1, 0.57/p_2\}$, $\overline{X} = \{0.68/p_1, 0.75/p_2\}$ respectively.

**Remark 5.** It should be noted that; changing the difficulty levels for each parameter in the exams can directly affect the success of candidate students in the exams. In order to obtain more detailed information about this stated situation, studies [23-25] can be examined.

**Step 1:** During the evaluation of candidate students, the data obtained from the school administration are assumed as follows:

$$\underline{\psi_X}(p_1^{0.17}) = \{u_1, u_2, u_3, u_4, u_5, u_7, u_9, u_{10}\}, \qquad \underline{\psi_X}(p_2^{0.19}) = \{u_1, u_4, u_5, u_6, u_8, u_{10}\},$$
$$\psi_X(p_1) = \{u_1, u_2, u_3, u_5, u_7, u_{10}\}, \qquad \psi_X(p_2) = \{u_4, u_5, u_6, u_8\},$$
$$\overline{\psi_X}(p_1^{0.23}) = \{u_7, u_{10}\}, \qquad \overline{\psi_X}(p_2^{0.18}) = \{u_8\},$$

These data can be expressed as a whole with the help of a VFPSS $\Psi_X$,

$$\Psi_X = \left\{ \begin{matrix} (0.28/p_1, \{u_1, u_2, u_3, u_4, u_5, u_7, u_9, u_{10}\}), (0.38/p_2, \{u_1, u_4, u_5, u_6, u_8, u_{10}\}) \\ (0.45/p_1, \{u_1, u_2, u_3, u_5, u_7, u_{10}\}), (0.57/p_2, \{u_4, u_5, u_6, u_8\}) \\ (0.68/p_1, \{u_7, u_{10}\}), (0.75/p_2, \{u_8\}) \end{matrix} \right\}.$$

**Step 2:** The fuzzy decision set of $\Psi_X$ can be found as, (for $\underline{w} = 3, w = 2, \overline{w} = 1$)

$$\Psi_X^d = \left\{ \begin{matrix} 0.074/u_1, 0.053/u_2, 0.053/u_3, 0.084/u_4, 0.121/u_5 \\ 0.068/u_6, 0.166/u_7, 0.193/u_8, 0.015/u_9, 0.1875/u_{10} \end{matrix} \right\}.$$

For example; considering the $u_2$,

$$\mu_{\Psi_X^d}(u_2) = \frac{1}{3}\left(\frac{(0.28 * 1) + (0.38 * 0)}{3 * 2} + \frac{(0.45 * 1) + (0.57 * 0)}{2 * 2} + \frac{(0.68 * 0) + (0.75 * 0)}{1 * 2}\right) = 0.053$$

Here, attention should be paid to the values selected in accordance with the condition $\overline{\mathrm{w}} < w < \underline{\mathrm{w}}$.

**Step 3:** We conclude from the values of $u$ that $\mu_{\Psi_X^d}(u_8) = max\left\{\mu_{\Psi_X^d}(u) : u \in U\right\} = 0.193$ and hence $r = 8$. Thus $u_8$ is the optimal choice candidate and so $u_8$ is the most suitable student candidate for the desired parameters.

**A comparison:** Only one algorithm has been proposed for VFPSS theory, since it is very new. If we apply the algorithm suggested by Dalkılıç and Demirtaş [4] for the example given above, the results obtained are as follows:

**Table 1.** Comparison of algorithms for VFPSSs.

| Algorithm | $u_1$ | $u_2$ | $u_3$ | $u_4$ | $u_5$ | $u_6$ | $u_7$ | $u_8$ | $u_9$ | $u_{10}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| [4] | 1.11 | 0.73 | 0.73 | 1.23 | 1.68 | 0.95 | 1.41 | 1.7 | 0.28 | 1.79 |
| Suggested | 0.074 | 0.053 | 0.053 | 0.084 | 0.121 | 0.068 | 0.166 | 0.193 | 0.015 | 0.1875 |

According to the scores obtained in Table 1, the ranking among the candidate students is as follows:

For the algorithm [4], $u_{10} > u_8 > u_5 > u_7 > u_4 > u_1 > u_6 > u_2 = u_3 > u_9$.
For the suggested algorithm, $u_8 > u_{10} > u_7 > u_9 > u_5 > u_4 > u_1 > u_6 > u_2 = u_3$.

When the results are examined, it is striking that there is a serious difference. First of all, it should be noted that an error such as the best student $u_{10}$ is avoided. There are two important reasons for the difference between the algorithms given for the VFPSS,

i. Their algorithm is very complex, but the algorithm proposed in this study is so simple. Moreover, their algorithm has more variables and constraints than the algorithm proposed in this study.

ii. In the algorithm proposed by Dalkılıç and Demirtaş [4], the scores obtained from each approximate function were evaluated equally. Therefore, students who passed the more difficult exams were also subjected to an equal score. However, the algorithm proposed in this study eliminated this problem.

For these reasons, it is recommended to use the algorithm given in this study in expressing any uncertainty and in obtaining of the decision-making process in a more ideal way.

## 5. Conclusion

In conclusion, this study has significantly advanced the field of set theory, particularly within the context of VFP-soft sets, a novel mathematical tool tailored to address uncertainties in data analysis. The research's importance lies in its dual objective of enhancing existing theory while offering practical solutions to complex data analysis challenges. Firstly, the study simplifies the representation of intricate data, making analysis more accessible. By exploring fundamental set operations within the framework of VFP-soft sets, the research contributes to strengthening the theoretical foundation of this mathematical model. A crucial aspect of this research is the examination of parameter importance weights in VFP-soft sets, encompassing three different types of fuzzy parameterized soft sets. This investigation sheds light on nuanced approaches to parameter weighting, a fundamental aspect of the theory, thereby enhancing its applicability. Furthermore, the development of a novel decision-making algorithm based on VFP-soft sets represents a significant practical outcome of this research. This algorithm, derived from theoretical enhancements and insights, holds promise for addressing real-world uncertainty problems effectively. However, it's important to acknowledge certain limitations and challenges associated with the proposed methodology. While VFP-soft sets offer a versatile approach, their implementation may require specialized expertise, and computational complexity could pose challenges for large-scale datasets. Additionally, the effectiveness of the proposed algorithm may vary depending on the specific characteristics of the data and the context of the decision problem. In light of these considerations, future research should focus on refining the methodology to overcome these limitations and explore its applicability across diverse domains. By addressing these challenges and leveraging the advantages of VFP-soft sets, further advancements in data analysis and decision-making can be achieved.

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