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Some new results on quasi-ordered residuated systems

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Abstract. Quasi-ordered residuated system is a commutative residuated integral monoid ordered under a quasi-order was introduced in 2018 by Bonzio and Chajda as a generalization of commutative residuated lattices and hoop-algebras. This paper introduces the concept of atoms in these systems and analyzes its properties. Additionally, two extensions of the system $\mathfrak{A}$ to the system $\mathfrak{A} \cup \{w\}$ were designed so that the element w is an atom in $\mathfrak{A} \cup \{w\}$.

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1. Introduction

The algebraic structure recognizable under the name quasi-ordered residuated system (QRS, by short) was introduced in 2018 by Bonzio and Chajda ([1]) as a generalization of hoop-algebras (in the sense of [3]) and commutative residuated lattices (in the sense of [7]). Quasi-ordered residuated system $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1, \preccurlyeq \rangle$ is an integral monoid ordered by a quasi-order $\preccurlyeq$ at which the residuum $' \rightarrow '$ is associated with internal binary operation $' \cdot '$ in A by a special relationship

$$(\forall x, y, z \in A)(x \cdot y \preccurlyeq z \Longleftrightarrow x \preccurlyeq y \rightarrow z).$$

The results of the study of the internal structure of the QRS as well as its substructures were announced by by the author of this article in several of his reports (see, for example [11,12]). One of the specifics by which this algebraic structure differs from the commutative residuated lattice is that its residuum part $(A, \rightarrow, 1)$ is a BE-algebra with some additional features. Besides that, this algebraic structure, in the general case, does not satisfy the condition

$$(\forall x, y \in A)(x \cdot (x \rightarrow y) = y \cdot (y \rightarrow x)),$$

which is one of the hoop-algebras axioms.

The concept of atoms, as a specific phenomenon in many logical algebras, such as, for example, BCK/BCI/BCC-algebras, has been studied by several researchers (see, for example [4–6,8,10,14]). Atomic elements in residual lattices are also studied (see, for example, [9,15]).

This paper is a report on the properties of atoms in quasi-ordered residuated systems. However, due to the specificity of the quasi-order relation in QRSs, the method of defining the concept of atom used in the residual lattice and the mentioned logical algebra is not expedient for defining the concept of atom in QRSs. The definition of this concept in QRSs, which is used here, to be expedient must be specific. The paper is organized as follows: In the Preliminaries section, the necessary data and propositions on quasi-ordered residuated system for the comfonious monitoring of exposure in Section 3. Section 3 is central part of this report. The concept of atoms in the quasi-ordered residuated system was introduced. Some of the important features of this notion were registered. For example, the set $L(A)$ of all atoms in a quasi-ordered residuated system $\mathfrak{A}$ if not empty, is an anti-chain. However, this subset need not

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be a sub-semigroup of the monoid $(A, \cdot, 1)$. Second, a criterion was found for determining whether an element $w \in A$ is an atom in $\mathfrak{A}$ or not. There are also several examples are included that illustrate the characteristics of this phenomenon in a quasi-ordered residuated system. Additionally, in the second subsection, two extensions of the system $\mathfrak{A}$ to the system $\mathfrak{A} \cup \{w\}$ were designed, so that the element w is an atom in $\mathfrak{A} \cup \{w\}$. The second of them is created so that $L(A \cup \{w\}) = L(A) \cup \{w\}$.

2. Preliminaries

It should be emphasized here that the formulas in this text are written in a standard way, as is usual in mathematical logic, with the standard use of labels for logical functions. Thus, the labels $\wedge$, $\vee$, $\Longrightarrow$, $\neg$, and so on, are labels for the logical functions of conjunction, disjunction, implication, negation, and so on. Brackets in formulas are used in the standard way, too. All formulas appearing in this paper are closed by some quantifier. If one of the formulas is open, then the variables that appear in it should be seen as free variables. In addition to the previous one, the sign $=:$, in the use of $A =: B$, should be understood in the sense that the letter A is the abbreviation for the formula B.

Recall that a *quasi-order relation* $' \preccurlyeq '$ on a set A is a binary relation which is reflexive and transitive.

Definition 1 ([1], Definition 2.1)**.** *A residuated relational system is a structure $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1, R\rangle$, where $\langle A, \cdot, \rightarrow, 1\rangle$ is an algebra of type $\langle 2, 2, 0\rangle$ and R is a binary relation on A and satisfying the following properties:*
(1) $(A, \cdot, 1)$ *is a commutative monoid;*
(2) $(\forall x \in A)((x, 1) \in R)$*;*
(3) $(\forall x, y, z \in A)((x \cdot y, z) \in R \Longleftrightarrow (x, y \rightarrow z) \in R)$.
We will refer to the operation $\cdot$ as multiplication, to $\rightarrow$ as its residuum and to condition (3) as residuation. A quasi-ordered residuated system (QRS, in short) is a residuated relational system $\langle A, \cdot, \rightarrow, 1, \preccurlyeq\rangle$, where $\preccurlyeq$ is a quasi-order relation in the monoid $(A, \cdot)$. We denote this axiomatic system by ***QRS***.

The following proposition shows the basic properties of quasi-ordered residuated systems.

Proposition 1 ([1], Proposition 3.1)**.** *Let $\mathfrak{A}$ be a quasi-ordered residuated system. Then*
(4) *The operation '$\cdot$' preserves the pre-order in both positions;*

$$(\forall x, y, z \in A)(x \preccurlyeq y \Longrightarrow (x \cdot z \preccurlyeq y \cdot z \wedge z \cdot x \preccurlyeq z \cdot y));$$

(5) $(\forall x, y, z \in A)(x \preccurlyeq y \Longrightarrow (y \rightarrow z \preccurlyeq x \rightarrow z \wedge z \rightarrow x \preccurlyeq z \rightarrow y))$;
(6) $(\forall y, z \in A)(x \cdot (y \rightarrow z) \preccurlyeq y \rightarrow x \cdot z)$;
(7) $(\forall x, y, z \in A)(x \cdot y \rightarrow z \preccurlyeq x \rightarrow (y \rightarrow z))$;
(8) $(\forall x, y, z \in A)(x \rightarrow (y \rightarrow z) \preccurlyeq x \cdot y \rightarrow z)$;
(9) $(\forall x, y, z \in A)(x \rightarrow (y \rightarrow z) \preccurlyeq y \rightarrow (x \rightarrow z))$;
(10) $(\forall x, yz \in A)((x \rightarrow y) \cdot (y \rightarrow z) \preccurlyeq x \rightarrow z)$;
(11) $(\forall x, y \in A)((x \cdot y \preccurlyeq x) \wedge (x \cdot y \preccurlyeq y))$;
(12) $(\forall x, y, z \in A)(x \rightarrow y \preccurlyeq (y \rightarrow z) \rightarrow (x \rightarrow z))$;
(13) $(\forall x, y, z \in A)(y \rightarrow z \preccurlyeq (x \rightarrow y) \rightarrow (x \rightarrow z))$.

It is generally known that a quasi-order relation $\preccurlyeq$ on a set A generates an equivalence $\equiv_{\preccurlyeq} := \preccurlyeq \cap \preccurlyeq^{-1}$ on A. Due to properties (4) and (5), this equality relation is compatible with the operations in $\mathfrak{A}$. Thus, $\equiv_{\preccurlyeq}$ is a congruence on $\mathfrak{A}$. In what follows, we will sometime write these relations with $\equiv$ if there is no misunderstanding. In connection with the previous one, the quotient structure $\mathfrak{A}/\equiv =: \langle A/\equiv, *, \rightrightarrows, [1]_{\equiv}\rangle$ is a QRS, where the operations $*$ and $\rightrightarrows$ are determined as follows

$$(\forall x, y \in A)(([x]_{\equiv} * [y]_{\equiv} =: [x \cdot y]_{\equiv}) \wedge ([x]_{\equiv} \rightrightarrows [y]_{\equiv} =: [x \rightarrow y]_{\equiv})).$$

In the light of the previous note, it is easy to see that the following applies:
(7) and (8) give:

$$(\forall x, y, z \in A)(x \cdot y \rightarrow z \equiv_{\preccurlyeq} x \rightarrow (y \rightarrow z)).$$

Due to the universality of formula (9), we have:
(14) $(\forall x, y, z \in A)(x \rightarrow (y \rightarrow z) \equiv_{\preccurlyeq} y \rightarrow (x \rightarrow z))$.

In addition to the previous one it is easy to prove that

$$(\forall x, y \in A)(x \preccurlyeq y \Longleftrightarrow x \rightarrow y \equiv_{\preccurlyeq} 1).$$

Indeed, for any $x, y \in A$ the following holds $x \preccurlyeq y \iff 1 \preccurlyeq x \to y \preccurlyeq 1$, relying on (3) and (2).

Definition 2. *By a hoop ([3]) we mean an algebra* $(H, \cdot, \to, 1)$ *in which* $(H, \cdot, 1)$ *is a commutative semigroup with the identity and the following assertions are valid:*

(H1) $(\forall x \in H)(x \to x = 1)$,
(H2) $(\forall x, y \in H)(x \cdot (x \to y) = y \cdot (y \to x))$ *and*
(H3) $(\forall x, y, z \in A)(x \cdot y \to z = x \to (y \to z))$.

A relation $\leqslant$ on hoop $(A, \cdot, 1)$ is defined ([3], pp. 62) by

$$(\forall x, y \in A)(x \leqslant y \iff x \to y = 1).$$

The relation $\leqslant$ is a partial order on A compatible with the operation in the hoop $(A, \cdot, 1)$ in accordance with Proposition 2.2 in [3] (see [2], also). It is easy to see that every hoop is a (quasi-)ordered residuated system and vice versa does not have to be because, in the general case, the formula (H2) is not a valid formula in the **QRS** axiom system.

Example 1. *For a commutative monoid* A, *let* $\mathfrak{P}(A)$ *be denote the powerset of* A *ordered by set inclusion and '$\cdot$' the usual multiplication of subsets of* A. *Then* $\langle \mathfrak{P}(A), \cdot, \to, A, \subseteq \rangle$ *is a quasi-ordered residuated system in which the residuum is given by* $(\forall X, Y \in \mathfrak{P}(A))(Y \to X =: \{z \in A : Yz \subseteq X\})$. □

Examples 2, 3 and 4, included in this section, have an important application to Section 3 as well.

Example 2. *Let* $A = \{1, 2, 3, 4\}$ *and operations '$\cdot$' and '$\to$' defined on* A *as follows:*

$\cdot$	1	a	b	c	d
1	1	a	b	c	d
a	a	a	a	a	a
b	b	a	b	b	b
c	c	a	b	c	b
d	d	a	b	b	d

and

$\to$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	1	1
b	1	a	1	1	1
c	1	a	d	1	d
d	1	a	c	c	1

Then $\mathfrak{A} =: \langle A, \cdot, \to, 1 \rangle$ *is a quasi-ordered residuated systems where the relation '$\preccurlyeq$' is defined as follows* $\preccurlyeq =: \{(1,1), (a,1), (a,a), (a,b), (a,c), (a,d), (b,b), (b,c), (b,d), (b,1), (c,c), (c,1), (d,d), (d,1)\}$. □

Example 3. *Let* $A = \{1, a, b, c, d, e\}$ *and operations '$\cdot$' and '$\to$' defined on* A *as follows:*

$\cdot$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	a	a	a	a	a	a
b	b	a	b	a	b	a
c	c	a	a	a	a	c
d	d	a	b	a	b	c
e	e	a	a	c	c	e

and

$\to$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	1	1	1	1	1	1
b	1	e	1	e	1	e
c	1	b	b	1	1	1
d	1	a	b	e	1	e
e	1	b	b	d	d	1

Then $\mathfrak{A} =: \langle A, \cdot, \to, 1 \rangle$ *is a quasi-ordered residuated systems where the relation '$\preccurlyeq$' is defined as follows* $\preccurlyeq =: \{(1,1), (a,1), (b,1), (c,1), (d,1), (e,1), (a,a), (a,b), (a,c), (a,d), (a,e), (b,b), (b,d), (c,c), (c,d), (c,e), (d,d), (e,e)\}$. *By direct verification it can be proved that* $\mathfrak{A}$ *is a quasi-ordered residuated system.* □

Definition 3 ([11], Definition 3.1). *For a non-empty subset* F *of a quasi-ordered residuated system* $\mathfrak{A}$ *we say that it is a* filter *of* $\mathfrak{A}$ *if it satisfies conditions*

(F2) $(\forall u, v \in A)((u \in F \wedge u \preccurlyeq v) \implies v \in F)$, *and*
(F3) $(\forall u, v \in A)((u \in F \wedge u \to v \in F) \implies v \in F)$.

If the non-empty subset F of a quasi-ordered system $\mathfrak{A}$ satisfies the condition (F2), then it also satisfies the conditions:

(F0) $1 \in F$ and
(F1) $(\forall u, v \in A)((u \cdot v \in F \implies (u \in F \wedge v \in F))$,

as it is shown ([11], Proposition 3.4 and Proposition 3.2).

Remark 1. *It is easy to see that the determination of filters in quasi-ordered residuated systems differs from the determination of filters either in residuated lattices or hoop-algebras.*

Example 4. *Let $A = \langle -\infty, 1] \subset \mathbb{R}$ (the real numbers field). If we define '$\cdot$' and '$\rightarrow$' as follows, $(\forall u, v \in A)(u \cdot v =: min\{u, v\})$ and $u \rightarrow v =: 1$ if $u \leqslant v$ and $u \rightarrow v =: v$ if $v < u$ for all $u, v \in A$, then $\mathfrak{A} =: \langle A, \cdot, \rightarrow, 1, < \rangle$ is a quasi-ordered residuated system. All filters in $\mathfrak{A}$ are in the form of $\langle x, 1]$, for $x \in \langle -\infty, 1]$.* □

Example 5. *Let $A = \{1, a, b, c, d\}$ as in Example 2. The subsets $F_0 = \{1\}$, $F_1 = \{1, d\}$, and $F_2 = \{1, b, c, d\}$ are filters in $\mathfrak{A}$. Subsets $\{1, a\}$, $\{1, b\}$, $\{1, c\}$, $\{1, a, b\}$, $\{1, a, c\}$, $\{1, a, d\}$, $\{1, b, c\}$, $\{1, b, d\}$, $\{1, c, d\}$, $\{1, a, b, c\}$, $\{1, a, b, d\}$ are not filters in $\mathfrak{A}$.* □

Example 6. *Let $A = \{1, a, b, c, d, e\}$ as in Example 3. Subsets $F_0 = \{1\}$, $F_1 = \{1, d\}$, $F_2 = \{1, e\}$, $F_3 = \{1, b, d\}$, $F_4 = \{1, c, d, e\}$ are non-trivial filters in $\mathfrak{A}$. Subsets $\{1, a\}$, $\{1, b\}$, $\{1, c\}$, $\{1, a, b\}$, $\{1, a, c\}$, $\{1, a, d\}$, $\{1, a, e\}$, $\{1, b, c\}$, $\{1, b, e\}$, $\{1, c, d\}$, $\{1, c, e\}$, $\{1, d, e\}$, $\{1, a, b, c\}$, $\{1, a, b, d\}$, $\{1, a, b, e\}$, $\{1, b, c, d\}$, $\{1, b, c, e\}$ are not filters in $\mathfrak{A}$.* □

3. The Main Results

This section is the central part of this report. In the first subsection, the concept of a quasi-ordered residuated system is introduced and its basic properties are registered. Also, a criterion was found that enables recognition of whether an element is an atom in a quasi-ordered residuated system or not. Several examples are given that illustrate this phenomenon and its characteristics. In the second subsection, it was shown that every quasi-ordered residuated system can be embedded in a quasi-ordered residuated system that has at least one atom.

3.1. **Concept of atoms in quasi-ordered residuated system.** First, we will determine the concept of atoms in a quasi-ordered residuated system.

Definition 4. *Let $\mathfrak{A}$ be a quasi-ordered residuated system. An element $(1 \neq)a \in A$ is an* atom *in $\mathfrak{A}$ if*

(At) $(\forall x \in A)(a \preccurlyeq x \implies (x \equiv_{\preccurlyeq} a \vee x \equiv_{\preccurlyeq} 1))$

holds. The set of all atoms in $\mathfrak{A}$ is denoted by $L(A)$.

It can immediately be concluded that:

Theorem 1. *Elements of $L(A)$ are not comparable.*

Proof. Let $a, b \in L(A)$ be such that $a \neq b$. If we assume that $a \preccurlyeq b$, we would have $b \equiv_{\preccurlyeq} a$ or $b \equiv_{\preccurlyeq} 1$ because a is an atom in $\mathfrak{A}$. Since none of the obtained options is possible, we conclude that the elements a and b are not comparable. □

The following proposition gives a criterion for recognizing atoms in a quasi-ordered residuated system.

Proposition 2. *Let $\mathfrak{A}$ be a quasi-ordered residuated system and $a \in A$ such that $1 \neq a$. Then a is an atom in $\mathfrak{A}$ if the set $\{1, a\}$ is a filter in $\mathfrak{A}$.*

Proof. Let the subset $\{1, a\}$ be filter in $\mathfrak{A}$. Then holds

$$(\forall x \in A)((a \in \{1, a\} \wedge a \preccurlyeq x) \implies x \in \{1, a\})$$

according (F2). This means $x = 1$ or $x = a$. □

Formula (At) can be written in the form

$$(\forall x \in A)(a \rightarrow x \equiv_{\preccurlyeq} 1 \implies (x \equiv_{\preccurlyeq} a \vee x \equiv_{\preccurlyeq} 1)).$$

In this case, the proof of the previous proposition is demonstrated by referring to (F3) instead of (F2).

Example 7. *Let $A = \{1, a, b, c, d\}$ as in Examples 2 and 5. Then $\mathfrak{A} =: \langle A, \cdot, \rightarrow, 1 \rangle$ is a quasi-ordered residuated system. Since the subsets $\{1, c\}$ and $\{1, d\}$ are filters in $\mathfrak{A}$, then, by Proposition 2, elements c and d are atoms in $\mathfrak{A}$.* □

Example 8. *Let $A = \{1, a, b, c, d, e\}$ is as in Example 3 and example 6. Then $\mathfrak{A} =: \langle A, \cdot, \rightarrow, 1 \rangle$ is a quasi-ordered residuated system. Subsets $\{1, d\}$ and $\{1, e\}$ are filters in $\mathfrak{A}$ (see Example 6). Therefore the elements d and e are atoms in $\mathfrak{A}$.* □

Remark 2. *An additional explanation of the concept of atoms in a quasi-ordered residuated system can be given by using the concept of 'covering'. We say that z cover x if and only if $x \preccurlyeq z$ and there does not exists $y \in A$ such that $x \preccurlyeq y \preccurlyeq z$ and $x \neq y$. Thus, an atom in the quasi-ordered residuated system $\mathfrak{A} =: \langle A, \cdot, \rightarrow, 1\rangle$ is an element in A which is covered by the neutral 1.*

As the following example shows, not every quasi-ordered residuated system has to have atoms:

Example 9. *Let $\mathfrak{A}$ be as in Example 4. For an element a to be an atom in $\mathfrak{A}$, it must be $(\forall x \in A)(a < x \Longrightarrow (x = a \vee x = 1))$ which is impossible, because A is not a discrete set. On the contrary, for $a < 1$, there are infinitely many elements x such that $a < x < 1$.* □

Remark 3. *In our effort to prove that the converse of Proposition 2 is valid, we encountered the following problem: Let $\mathfrak{A} =: \langle A, \cdot, \rightarrow, 1\rangle$ be a quasi-ordered residuated system and let a be an atom in $\mathfrak{A}$. Then the set $F = \{1, a\}$ is a filter in $\mathfrak{A}$. We need to prove that the set F satisfies the conditions (F2) and (F3):*

(F2)*: Let $x, y \in A$ be arbitrary elements such that $x \in F = \{1, a\}$ and $x \preccurlyeq y$. This means $x \equiv_{\preccurlyeq} a$ or $x \equiv_{\preccurlyeq} 1$. In the first case, we have that $a \preccurlyeq y$ implies $y \equiv_{\preccurlyeq} a$ or $y \equiv_{\preccurlyeq} 1$ because a is an atom in $\mathfrak{A}$. Thus $y \in F$. In the second case, we have $1 \equiv_{\preccurlyeq} x \preccurlyeq y$ so $y \equiv_{\preccurlyeq} 1 \in F$ as well. This shows that the set F satisfies the condition (F2).*

(F3)*: Let $x, y \in A$ be arbitrary elements such that $x \in F = \{1, a\}$ and $x \rightarrow y \in F$. Two options are possible:*

(i) *$x = 1$ and $1 \rightarrow y \in \{1, a\}$. If $1 \rightarrow y = 1$, then $1 \preccurlyeq y$ and, therefore, $y \equiv_{\preccurlyeq} 1 \in F$. If $a = 1 \rightarrow y$, then $a \preccurlyeq y$. From here it follows $y \equiv_{\preccurlyeq} a \in F$ or $y \equiv_{\preccurlyeq} 1 \in F$ because a is an atom in $\mathfrak{A}$.*

(ii) *$x = a$ and $a \rightarrow y \in \{1, a\}$. If $1 = a \rightarrow y$, then $a \preccurlyeq y$, so $y \equiv_{\preccurlyeq} 1 \in F$ or $y \equiv_{\preccurlyeq} a \in F$ because a is an atom in $\mathfrak{A}$. Let now, us assume, that $a = a \rightarrow y$. Then $a = a \rightarrow y \preccurlyeq 1$. From here we conclude that it is not $a \preccurlyeq y$. Indeed, if there were $a \preccurlyeq y$, according to (5) we would have $1 \equiv_{\preccurlyeq} a \rightarrow a \preccurlyeq a \rightarrow y = a$, which is impossible. So it must be $\neg(a \preccurlyeq y)$. The obtained conclusion does not allow us to demonstrate the implication $a \in F \wedge a \rightarrow y \in F \Longrightarrow y \in F$.*

The following theorem says something more about the set $L(A)$ of all atoms for a given quasi-ordered residuated system $\mathfrak{A} = \langle A, \cdot, \rightarrow\rangle$.

Theorem 2. *Let $\mathfrak{A}$ be a quasi-ordered residuated system and $a, b \in L(A)$. Then:*

(a) *If $a \neq b$, then $a \rightarrow b = b$ and $b \rightarrow a = a$.*

(b) *$(\forall x \in A)((a \rightarrow x) \rightarrow x = a \vee (a \rightarrow x) \rightarrow x = 1)$.*

(c) *$(\forall x \in A)(x \rightarrow a = a \vee x \rightarrow a = 1)$.*

Proof. (a) Since $a \cdot b \preccurlyeq a$ according to (11), we conclude that $a \preccurlyeq b \rightarrow a$. From here it follows $b \rightarrow a = a$ or $b \rightarrow a = 1$ because a is an atom in A. Since the second option is not possible according to Theorem 1, we have $b \rightarrow a = a$.

The statement $a \rightarrow b = b$ can be proved analogously to the previous proof.

(b) Let A be a quasi-ordered residuated system, $a \in L(A)$ and $x \in A$ be an arbitrary element. From the valid formula $(a \rightarrow x) \rightarrow (a \rightarrow x) = 1$, it follows $a \rightarrow ((a \rightarrow x) \rightarrow x) = 1$ according to (14). This means $a \preccurlyeq (a \rightarrow x) \rightarrow x$. Since a is an atom in $\mathfrak{A}$, from here we get $a = (a \rightarrow x) \rightarrow x$ or $(a \rightarrow x) \rightarrow x = 1$. The second option gives $a \rightarrow x \preccurlyeq x$.

(c) For the elements a and $x \in A$, we have $a \cdot x \preccurlyeq a$ according to (11). Then $a \preccurlyeq x \rightarrow a$ by (3). Thus $x \rightarrow a = a$ or $x \rightarrow a = 1$ since a is an atom in $\mathfrak{A}$. The second option means $x \preccurlyeq a$. □

Example 10. *Let $A = \{1, a, b, c, d, e\}$ as in Examples 3, 6 and 8. Elements d and e are atoms in the quasi-ordered residuated system $\mathfrak{A}$. $d \rightarrow e = e$ and $e \rightarrow d = d$ hold for them, which illustrates the statement (a) in the previous theorem.*

To illustrate statement (b), we calculate:
$(d \rightarrow a) \rightarrow a = a \rightarrow a = 1$ *and* $d \rightarrow a = a \preccurlyeq a$,
$(d \rightarrow b) \rightarrow b = b \rightarrow b = 1$ *and* $d \rightarrow b = b \preccurlyeq b$,
$(d \rightarrow c) \rightarrow c = e \rightarrow c = d$ *and* $d \rightarrow c = e$ *and* $\neg(e \preccurlyeq c)$,
$(d \rightarrow e) \rightarrow e = e \rightarrow e = 1$ *and* $d \rightarrow e = e \preccurlyeq e$.

In this example, the following calculation illustrates statement (c):
$a \rightarrow d = 1$ *and* $a \preccurlyeq d$,

$b \to d = 1$ and $b \preccurlyeq d$,
$c \to d = 1$ and $c \preccurlyeq d$,
$e \to d = d$ and $\neg(e \preccurlyeq d)$. □

Remark 4. *Let $\mathfrak{A}$ be a quasi-ordered residuated system. For any $a \in A$ we define a subset $V(a)$ of A as follows $V(a) = \{x \in A : x \preccurlyeq a\}$. Note that $V(a)$ is non-empty, because $a \preccurlyeq a$ gives $a \in V(a)$. If $a \in L(A)$, then the set $V(a)$ is called a branch of $\mathfrak{A}$. A characteristic of the concept of branches $V(a) \cap V(b) = \emptyset$ in some logical algebras such as, for example, BCI-algebra ([6], Proposition 3.15) and weak BCC-algebra ([5], Corollary 3.18) in the case of quasi-ordered residuated systems is not present, as the following example shows. In Example 3, for atoms $d, e \in L(A)$ we have $V(d) = \{a, b, c, d\}$ and $V(e) = \{a, c, e\}$, so, therefore, is $V(d) \cap V(e) = \{a, c\} \neq \emptyset$.*

It seems that this tool in the case of quasi-ordered residuated systems is not of any use in studying the phenomenon of atoms in this algebraic structure. For the sake of illustration, in the Example 3, for atoms $d, e \in L(A)$ we have $e \cdot d = d \cdot e = c \notin L(A)$. Therefore, $L(A)$ is not a subsemigroup in A.

The converse of statement (b) in the Theorem 2 is valid:

Theorem 3. *Let $\mathfrak{A} =: \langle A, \cdot, \to \rangle$ be a quasi-ordered residuated system. If the element $a \in A$ satisfies the condition* (b)*, then a is an atom in $\mathfrak{A}$.*

Proof. Let $x \in A$ be such that $a \preccurlyeq x$. This means $a \to x = 1$. Then $(a \to x) \to x = 1 \to x = x$. If $(a \to x) \to x = a$, then $a = x$. If $(a \to x) \to x = 1$, we have $x = 1$. This proves that a is an atom in $\mathfrak{A}$. □

3.2. **Two types of extensions of quasi-ordered residuated systems.** Let $\mathfrak{A} =: \langle A, \cdot, \to, 1, \preccurlyeq \rangle$ be a quasi-ordered residuated system and $w \notin A$. We can extend the system $\mathfrak{A} =: \langle A, \cdot, \to, 1, \preccurlyeq \rangle$ to the system $\mathfrak{B} =: \langle A \cup \{w\}, *, \rightsquigarrow, 1, \preccurlyeq \rangle$ so that the element w is an atom in the system $\mathfrak{B}$.

Here we demonstrate two such extensions.

Theorem 4. *Let $\mathfrak{A} =: \langle A, \cdot, \to, 1 \rangle$ be a quasi-ordered residuated system and $w \notin A$. We can extend the system $\langle A, \cdot, \to, 1 \rangle$ to the system $\mathfrak{B} =: \langle A \cup \{w\}, *, \rightsquigarrow, 1 \rangle$ so that the element w is an atom in the system $\mathfrak{B}$.*

Proof. System $\mathfrak{B}$ can be created in the following way:

$$x * y = \begin{cases} x \cdot y & \text{for } x \in A \ \wedge \ y \in A, \\ x & \text{for } x \in A \ \wedge \ y = w, \\ y & \text{for } x = w \ \wedge \ y \in A, \\ w & \text{for } x = w \ \wedge \ y = w, \end{cases}$$

and

$$x \rightsquigarrow y = \begin{cases} x \to y & \text{for } x \in A \ \wedge \ y \in A, \\ 1 & \text{for } x \in A \ \wedge \ y = w, \\ y & \text{for } x = w \ \wedge \ y \in A, \\ 1 & \text{for } x = w \ \wedge \ y = w. \end{cases}$$

For elements $x, y \in A$ we have
$w * (x \cdot y) = x \cdot y$ and $(w * x) \cdot y = x \cdot y$ and
$w * (w * x) = w * x = x$ and $(w * w) * x = w * x = x$.
So, the set $B = A \cup \{w\}$ is a commutative monoid. Second, $w * x \preccurlyeq w \iff w \preccurlyeq x \rightsquigarrow a = 1$, $w * x \preccurlyeq x \iff x \preccurlyeq w \rightsquigarrow w = 1$ and $w * w = w \preccurlyeq w \iff w \preccurlyeq w \rightsquigarrow w = 1$. Therefore, $\mathfrak{B}$ is a quasi-ordered residuated system. It is immediately clear that w is an atom in $\mathfrak{B}$, because if $w \preccurlyeq x$ holds, then $1 = w \rightsquigarrow x = x$. □

The following example illustrates the first extension of a quasi-ordered residuated system.

Example 11. *Let $A = \{1, a, b, c, d\}$ as in Example 2. Then $\mathfrak{A} =: \langle A, \cdot, \to, 1 \rangle$ is a quasi-ordered residuated system. Here is $L(A) = \{c, d\}$. Let us put $B = A \cup \{w\} = \{1, a, b, c, d, w\}$ and define the operations in B as follows*

$*$	1	a	b	c	d	w
1	1	a	b	c	d	w
a	1	a	a	a	a	a
b	1	a	b	b	b	b
c	1	a	b	c	b	c
d	1	a	b	b	d	d
w	w	a	b	c	d	w

and

$\rightsquigarrow$	1	a	b	c	d	w
1	1	a	b	c	d	w
a	1	1	1	1	1	1
b	1	a	1	1	1	1
c	1	a	d	1	d	1
d	1	a	c	c	1	1
w	1	a	b	c	d	1

Then $\mathfrak{B} =: \langle A \cup \{w\}, *, \rightsquigarrow, 1\rangle$ *is a quasi-ordered residuated system. It is obvious that* w *is a single atom in the system* $\mathfrak{B}$*. Hence* $L(B) = \{w\}$*.*

Theorem 5. *The extension of a quasi-ordered residuated system* $\langle A, \cdot, \longrightarrow, 1, \preccurlyeq\rangle$ *to system* $\mathfrak{B} =: \langle A \cup \{w\}, *, \rightsquigarrow, 1, \preccurlyeq\rangle$ *can also be realized so that the set* $L(A)$ *of all atoms of the system* $\mathfrak{A}$ *is expanded by one element, that is* $L(B) = L(A) \cup \{w\}$*.*

Proof. Let us take $w \notin A$. Let's form the set $B = A \cup \{w\}$ and design operations on B in the following way:

$$x * y \;=\; \begin{cases} x \cdot y & \text{for } \; x \in A \;\wedge\; y \in A, \\ x & \text{for } \; x \in A \setminus L(A) \;\wedge\; y = w, \\ y & \text{for } \; x = w \;\wedge\; y \in A \setminus L(A), \\ w & \text{for } \; x = w \;\wedge\; y = w, \\ max\{z \in A : z \preccurlyeq x \;\wedge\; z \preccurlyeq w\} & \text{for } \; x \in L(A) \;\wedge\; y = w \end{cases}$$

and

$$x \rightsquigarrow y \;=\; \begin{cases} x \longrightarrow y & \text{for } \; x \in A \;\wedge\; y \in A, \\ 1 & \text{for } \; x \in A \setminus L(A) \;\wedge\; y = w, \\ y & \text{for } \; x = w \;\wedge\; y \in A \setminus L(A), \\ 1 & \text{for } \; x = w \;\wedge\; y = w, \\ w & \text{for } \; x \in L(A) \;\wedge\; y = w \\ y & \text{for } \; x = w \;\wedge\; y \in L(A). \end{cases}$$

It can be shown that $\mathfrak{B}$ is a quasi-ordered residuated system by patient and careful calculation. It is easy to conclude that a is an atom in $\mathfrak{B}$. Indeed, from $w \preccurlyeq x$ it follows $w \rightsquigarrow x = 1$ and $(w \rightsquigarrow x) \rightsquigarrow x = 1$, from which it follows that w is an atom in $\mathfrak{B}$ according to Theorem 3. ☐

The following example illustrates another way of extending the quasi-ordered residuated system mentioned above. The extension, described in the following example, is significantly different from the previous one.

Example 12. *Let* $A = \{1, a, b, c, d\}$ *as in Example 2. Elements* c *and* d *are atoms in this quasi-ordered residuated system:* $L(A) = \{c, d\}$*. Let us put* $B = A \cup \{w\}$ *and define the operations in* B *as follows*

$*$	1	a	b	c	d	w
1	1	a	b	c	d	w
a	1	a	a	a	a	a
b	1	a	b	b	b	b
c	1	a	b	c	b	b
d	1	a	b	b	d	b
w	w	a	b	c	d	w

and

$\rightsquigarrow$	1	a	b	c	d	w
1	1	a	b	c	d	w
a	1	1	1	1	1	1
b	1	a	1	1	1	1
c	1	a	d	1	d	w
d	1	a	c	c	1	w
w	1	a	b	c	d	1

From the second table, which determined the $\rightsquigarrow$ *operation, it can be seen that* $a \preccurlyeq c < 1 \wedge b \preccurlyeq c < 1$*,* $a \preccurlyeq d < 1 \wedge b \preccurlyeq d < 1$*, and* $a \preccurlyeq w < 1 \wedge b \preccurlyeq w < 1$*. So,* $L(B) = \{c, d, w\}$*.*

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