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An Exact Multiplicity Result for Singular Subcritical Elliptic Problems

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Abstract

For a bounded and smooth enough domain Ω in $\mathbb{R}^n$, with $n \geq 2$, we consider the problem $-\Delta u = au^{-\beta} + \lambda h(.,u)$ in Ω, $u = 0$ on $\partial\Omega$, $u > 0$ in Ω, where $\lambda > 0$, $0 < \beta < 3$, $a \in L^\infty(\Omega)$, $ess\,inf\,(a) > 0$, and with $h = h(x,s) \in C\left(\overline{\Omega} \times [0,\infty)\right)$ positive on $\Omega \times (0,\infty)$ and such that, for any $x \in \Omega$, $h(x,.)$ is strictly convex on $(0,\infty)$, nondecreasing, belongs to $C^2(0,\infty)$, and satisfies, for some $p \in \left(1, \frac{n+2}{n-2}\right)$, that $\lim_{s\to\infty} \frac{h_s(x,s)}{s^p} = 0$ and $\lim_{s\to\infty} \frac{h(x,s)}{s^p} = k(x)$, in both limits uniformly respect to $x \in \overline{\Omega}$, and with $k \in C\left(\overline{\Omega}\right)$ such that $\min_{\overline{\Omega}} k > 0$. Under these assumptions it is known the existence of $\Sigma > 0$ such that for $\lambda = 0$ and $\lambda = \Sigma$ the above problem has exactly a weak solution, whereas for $\lambda \in (0,\Sigma)$ it has at least two weak solutions, and no weak solutions exist if $\lambda > \Sigma$. For such a Σ we prove that for $\lambda \in (0,\Sigma)$ the considered problem has it has exactly two weak solutions.

1. Introduction

Let $n \geq 2$, and let Ω be a C^2 bounded domain in $\mathbb{R}^n$, let $a : \Omega \to \mathbb{R}$, and let $h : \overline{\Omega} \times [0,\infty) \to \mathbb{R}$. For $\lambda \geq 0$ and $\beta > 0$, consider the problem:

$$\begin{cases} -\Delta u = au^{-\beta} + \lambda h(.,u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega \\ u > 0 \text{ in } \Omega. \end{cases} \tag{1.1}$$

Singular problems like the above appear in many applications to physical and chemical process (cf. [1], [2], [3] and their references). After the pioneers works [4] [1], [3], [5], [6], [7], [2] and [8], singular elliptic problems have received a lot of interest in the literature, and many articles concern them. Let us recall some of these works:

The case when $h = 0$ in (1.1) was studied, under different hypothesis on the function a, in [5], [9], [10], and [11]. In particular, [11] gives, when a is regular enough, accurate asymptotic estimates near the boundary for the solutions. [12] studied (1.1) when $h = 0$ and a is a Radon's measure. Also, [2] studied problem (1.1) when $a = -1$, but with $h(.,u)$ replaced by a suitable positive function $h \in L^1(\Omega)$.

[8] considered the problem $-\Delta u = au^{-\beta} + h(.,\lambda u)$ in Ω, $u = 0$ on $\partial\Omega$, $u > 0$ in Ω, and proved that if $\beta > 0$, $a \in C^1\left(\overline{\Omega}\right)$, $a > 0$ in $\overline{\Omega}$, $h \in C^1\left(\overline{\Omega} \times [0,\infty)\right)$ and if, for some positive constant c, $h(x,s) > c(1+s)$ for all $(x,s) \in \overline{\Omega} \times [0,\infty)$, then there exists $\lambda^* > 0$ such that the studied problem has a positive classical solution $u \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$ for any $\lambda \in [0,\lambda^*)$, and has no positive classical solution if $\lambda > \lambda^*$.

[13] addressed the equation $-\Delta u = au^{-\beta} + \lambda u^p$ in Ω, $u = 0$ on $\partial\Omega$, $u > 0$ in Ω, and obtained existence and nonexistence theorems when a is a regular enough function, with indefinite sign, $0 < \beta < 1$, $0 < p < 1$ and $\lambda \geq 0$.

[10] studied existence, nonexistence, uniqueness and stability issues for weak solutions of the problem $-\Delta u = p(x)u^{-\beta}$ in Ω, $u = 0$ on $\partial\Omega$, $u > 0$ in Ω, when $\beta > 0$ and $p(x)$ behaves like $d_\Omega^{-\gamma}(x)$ as $x \to \partial\Omega$, with $d_\Omega(x) := dist\,(x,\partial\Omega)$ and $0 < \gamma < 2$.

[14] investigates equations with singular nonlinearities that involve two bifurcation parameters.

[15] gives existence and nonexistence theorems for equations of the form $-\Delta u = g(x,u) + \lambda f(x,u,|\nabla u|)$ in Ω, $u = 0$ on $\partial\Omega$, $u > 0$ in Ω with $g(x,s)$ singular at $s = 0$ and also at $x \in \partial\Omega$, and where $f(x,u,|\nabla u|)$ involves a power of $|\nabla u|$.

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[16] studied the problem

$$-\Delta u=a(x)g(u)+\lambda h(u) \text{ in } \Omega,\ u=0 \text{ on } \partial\Omega,\ u>0 \text{ in } \Omega, \tag{1.2}$$

where $h(s)$ is nondecreasing, positive on $(0,\infty)$, and such that $s^{-1}h(s)$ is nonincreasing; and with g satisfying $\lim_{s\to0^+}g(s)=\infty$ but in such a way that, for some $\alpha\in(0,1)$ and $\varepsilon>0$, $s^{\alpha}g(s)$ is bounded on $(0,\varepsilon)$. There it was introduced the space $E:=\left\{v\in C^2(\Omega)\cap C^{1,1-\alpha}\left(\overline{\Omega}\right):\Delta v\in L^1(\Omega)\right\}$ and, among other results, it was proved that if g and h are regular enough on $(0,\infty)$ and $[0,\infty)$ respectively, and if a is regular enough on $\overline{\Omega}$, then:
i) if $\lim_{s\to\infty}s^{-1}h(s)=0$, problem (1.2) has a solution in E for any $\lambda\geq0$.
ii) If $\lim_{s\to\infty}s^{-1}h(s)>0$ and $\lambda\geq\frac{\lambda_1}{\lim_{s\to\infty}s^{-1}h(s)}$ (where λ_1 is the principal eigenvalue for $-\Delta$ in Ω with Dirichlet boundary condition) then (1.2) has no solutions u in E.
iii) If $\lim_{s\to\infty}s^{-1}h(s)>0$ and $\min_{\overline{\Omega}}a>0$ then (1.2) has a unique weak solution in E for any λ such that $0\leq\lambda<\frac{\lambda_1}{\lim_{s\to\infty}s^{-1}h(s)}$.
[17] studied semilinear elliptic problems with singular nonlocal Neumann boundary conditions, obtaining existence and uniqueness (up to a constant) results.
In [18] existence results were obtained for a one dimensional problem involving the fractional $p-$Laplacian with multipoint boundary conditions.
Concerning multiplicity results [19] studied, for $\beta>0$ and $1<p\leq2$, the problem $-\Delta_p u=g(u)+\lambda h(u)$ in Ω, $u=0$ on $\partial\Omega$, $u>0$ in Ω on a smooth, bounded and strictly convex domain in $\mathbb{R}^n$, , and under suitabe conditions on g and h, there was proved that for some $\varepsilon>0$ if $0<\lambda<\varepsilon$ then there exist at least two weak solutions.
[20] addressed existence and multiplicity issues for positive weak solutions of a family of (p,q)-Laplacian systems on an open, bounded, and regular enough domain in $\mathbb{R}^n$. Under suitable assumptions on the problem's data, there it was proved the existence of at least two (weak) positive solutions of the system.
[21] proved that if $B:\overline{\Omega}\to M_n(\mathbb{R})$ satisfies the standard symmetry, ellipticity, and regularity conditions, and if $0<\beta<1<p<\frac{n+2}{n-2}$ then, for λ positive and small enough, the problem $-\operatorname{div}(B(x)\nabla u)=u^{-\beta}+\lambda u^p$ in Ω, $u=0$ on $\partial\Omega$ has two positive weak solutions in $H_0^1(\Omega)$.
[22] addressed the problem $-\Delta_p u=\lambda u^{-\beta}+u^q$ in Ω, $u=0$ on $\partial\Omega$, $u>0$ in Ω under the assumptions that $0<\beta<1$, $1<p<\infty$, $q<\infty$ and $p-1<q\leq p^*-1$, with p^* given by $p^*:=\frac{np}{n-p}$ if $p<n$, $p^*=Q$ with $Q>p$ if $p=n$, and $p^*=\infty$ if $p>n$. With these assumptions [22] proved that, for some $\lambda^*\in(0,\infty)$, the problem has a weak solution if $\lambda=\lambda^*$, has no weak solution if $\lambda>\lambda^*$, and has at least two weak solutions if $\lambda\in(0,\lambda^*)$.
[23] studied problems of the form

$$\begin{cases} -\Delta u=\lambda\left(u^{-\delta}+u^q+\rho(u)\right) \text{ in } \Omega,\\ u=0 \text{ on } \partial\Omega,\\ u>0 \text{ in } \Omega, \end{cases} \tag{1.3}$$

where Ω is a bounded and regular enough domain in $\mathbb{R}^n$ with $n\geq3$, $\lambda>0$, $\delta>0$, $0<q\leq2^*-1$ where $2^*-1=\frac{n+2}{n-1}$ and $\rho\in C^1([0,\infty))$ satisfies:
a) $\rho(0)=\rho'(0)=0$, $\rho(t)+t^q\geq0$, if $q<2^*-1$;
b) There exists $\beta<2^*-2$ such that $\lim_{t\to\infty}t^{-\beta}\rho^-(t)=0$ and $\lim_{t\to\infty}t^{--2^*+1}\rho^+(t)=0$ if $q=2^*-1$.
Under these assumptions there it was proved, for λ positive and small enough, the existence of at least two positive solutions of (1.3).
[24] studied problems of the form

$$\begin{cases} -\Delta u=\lambda\left(u^{-\delta}+h(u)e^{u^{\alpha}}\right) \text{ in } \Omega,\\ u=0 \text{ on } \partial\Omega,\\ u>0 \text{ in } \Omega, \end{cases} \tag{1.4}$$

where $\lambda>0$, $0<\delta<1$, $1\leq\alpha<2$, and $h\in C^2[0,\infty)$ satisfies $h(0)=0$, $s\to s^{-\delta}+h(s)e^{s^{\alpha}}$ is convex, and for any $\varepsilon>0$, $\lim_{s\to\infty}h(s)e^{-\varepsilon s^{\alpha}}=0$ and $\lim_{s\to\infty}h(s)e^{\varepsilon s^{\alpha}}=\infty$. Under these asumptions there were proved several existence, multiplicity, and bifurcation results for problem (1.4).
[25] studied the problem $-\Delta_N u=\lambda f(.,u)$ in Ω, $u=0$ on $\partial\Omega$, $u>0$ in Ω, where Ω is a bounded and regular domain in R^N, Δ_N is the $N-$Laplacian on Ω, and where $f(x,s)$ is a regular enough function which may be singular at $s=0$ and with exponential growth as $s\to\infty$. Under suitable additional assumptions on f, there it was proved the existence of $\Sigma>0$ such that: for $0<\lambda<\Sigma$ the problem has at least two solutions, one solution if $\lambda=\Sigma$, and no solutions when $\lambda>\Sigma$.
We mention also that the Nehari manifold method, adapted to the presence of singular nonlinearities through the study of the associated fibering functions, were used to establish multiplicity results for degenerated elliptic singular nonlinear problems involving either the pLaplacian or the weighted $p-q$ Laplacian in [26], [27], and [28]. For additional works concerning singular elliptic problems see e.g., [29], [30], [31], [32], [33], [34], [35], [36], [37], [38]; and for a systematic treatment of the subject of singular problems, we refer the readers to the research books [39] and [40] and their references.
Our aim in this work is to prove an exact multiplicity result for weak solutions of problem (1.1). By a weak sollution we mean, as usual, the given by following:

Definition 1.1. *If $\rho : \Omega \to \mathbb{R}$ is a a measurable function such that $\rho\varphi \in L^1(\Omega)$ for any $\varphi \in H_0^1(\Omega)$, and if u is a function defined on Ω we say that u is a weak solution of the problem*

$$\begin{cases} -\Delta u = \rho \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

if, and only if, $u \in H_0^1(\Omega)$ and $\int_\Omega \langle \nabla u, \nabla \varphi \rangle = \int_\Omega \rho\varphi$ for all $\varphi \in H_0^1(\Omega)$.
Also, for $u \in H^1(\Omega)$ and ρ as above, we will write $-\Delta u \geq \rho$ in Ω (respectively $-\Delta u \leq \rho$ in Ω) to mean that $\int_\Omega \langle \nabla u, \nabla \varphi \rangle \geq \int_\Omega \rho\varphi$ (resp. $\int_\Omega \langle \nabla u, \nabla \varphi \rangle \leq \int_\Omega \rho\varphi$) for any nonnegative $\varphi \in H_0^1(\Omega)$.

Since our results depend largely on those of [35], [36], and [37], let us to briefly review them in the next three remarks:

Remark 1.2. *In [35] and [36], it was considered, for $\beta \in (0,3)$, the problem*

$$\begin{cases} -\Delta u = au^{-\beta} + f(\lambda, ., u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \\ u > 0 \text{ in } \Omega, \end{cases} \tag{1.5}$$

with (1.5) understood in weak sense.
Under suitable assumptions on a and f, ([35] Theorem 1.1) states that there exists $\Sigma > 0$ such that problem (1.5) has (at least) a weak solution $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$, if and only if, $\lambda \in [0,\Sigma]$
Let us mention also that ([35] Theorems 1.2) says that, for λ positive and small enough, there exist at least two weak solutions in $H_0^1(\Omega) \cap L^\infty(\Omega)$. In addition, ([35] Theorem 1.1) says also that any solution u in $H_0^1(\Omega) \cap L^\infty(\Omega)$ of (1.5) belongs to $C(\overline{\Omega})$.
In [36] all the hypothesis of [35] were assumed, plus an additional one, and in ([36] Theorem 1.2) it was proved that, for Σ as in [35] and $\lambda \in [0,\Sigma]$, problem (1.5) has a solution $u_\lambda \in H_0^1(\Omega) \cap C(\overline{\Omega})$ which is minimal in the sense that $u_\lambda \leq v$ for all weak solution $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$ of (1.5). Additionally, ([36] Theorem 1.2) says that $\lambda \to u_\lambda$ is strictly increasing from $[0,\Sigma]$ into $C(\overline{\Omega})$; and ([36] Theorem 1.3) asserts that, for each $\lambda \in (0,\Sigma)$, problem (1.5) has at least two weak solutions $u \in H_0^1(\Omega) \cap C(\overline{\Omega})$.

Remark 1.3. *Problem (1.5) was again considered in [37], where, with further hypothesis added, in ([37], Theorem 1.3) it was proved that the map $\lambda \to u_\lambda$, defined for $\lambda \in [0,\Sigma]$, with Σ and u_λ as in Remark 1.2, is continuous from $[0,\Sigma]$ into $C(\overline{\Omega})$, and belongs to $C^1((0,\Sigma), C(\overline{\Omega}))$.*

Remark 1.4. *Also, again for Σ as in Remark 1.2, ([37], Lemma 5.7) states, for each $\lambda \in [0,\Sigma]$, the existence of a solution $v_\lambda \in H_0^1(\Omega) \cap L^\infty(\Omega)$ of problem (1.5) which is maximal respect to the partial order $\leq$, that is: v_λ has the property that if $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is a weak solution of (1.5) and $u \geq v_\lambda$ a.e. in Ω, then $u = v_\lambda$. We mention also that ([37], Theorem 1.4) states that, for $\lambda = \Sigma$, there exists a unique solution in $H_0^1(\Omega) \cap L^\infty(\Omega)$ of problem (1.5) (and so, in particular, $u_\Sigma = v_\Sigma$).*

We assume, from now on and without anymore mention, the following conditions H1)-H6) (with the convention that $\frac{n+2}{n-2} = \infty$ if $n = 2$):

H1) $\beta \in (0,3)$.
H2) $a \in L^\infty(\Omega)$ and $ess\,inf\,(a) > 0$.
H3) $h \in C^2(\overline{\Omega} \times [0,\infty))$ and there exists $p \in \left(1, \frac{n+2}{n-2}\right)$ such that $\lim_{s\to\infty} \frac{h(x,s)}{s^p} = k(x)$ uniformly on $x \in \overline{\Omega}$, with $k \in C(\overline{\Omega})$ such that $\min_{\overline{\Omega}} k > 0$.
H4) For all $x \in \Omega$, the function $s \to h(x,s)$ is positive, nondecreasing, strictly convex, and belongs to $C^2(0,\infty)$.
H5) $h_s > 0$ in $\overline{\Omega} \times (0,\infty)$, and $\lim_{s\to\infty} \frac{h_s(x,s)}{s^p} = 0$ uniformly on $x \in \overline{\Omega}$, where h_s denotes the partial derivative of h respect of s.
H6) There exists $q \in [1,\infty)$ and a nonnegative and nonidentically zero function $b \in L^\infty(\Omega)$, such that $h(.,s) \geq bs^q$ *a.e.* in Ω, for any $s \geq 0$.

It is immediate to check that, if β, a, and h, satisfy H1)-H6) and if $f : [0,\infty) \times \overline{\Omega} \times [0,\infty) \to \mathbb{R}$ is defined by $f(\lambda, ., s) := \lambda h(., s)$, then β, a, and f satisfy all the conditions required in [37] (and so also all the conditions imposed in [35] and [36] hold), thus all the results in [35], [36], and [37] hold for problem (1.1).

Remark 1.5. *We fix, from now on, Σ as given by Remark 1.2, but taking there $\lambda h(., s)$ instead of $f(\lambda, ., s)$, and for $\lambda \in [0,\Sigma]$, u_λ and v_λ will denote the functions provided by Remarks 1.2 and 1.4, again now with $\lambda h(., s)$ instead of $f(\lambda, ., s)$.*

Our aim in this work is to prove the following

Theorem 1.6. *Let Ω be a C^2 and bounded domain in $\mathbb{R}^n$, $n \geq 2$, and assume the conditions H1)-H6). Let Σ be as in Remark 1.5. Then for $\lambda \in (0,\Sigma)$ problem (1.1) has exactly two weak solutions.*

.
Let us briefly outline the structure of the article. In Section 2 we recall some results of [11] concerning existence, uniqueness, and asymptotic properties near the boundary, for classical solutions of problems of the form $-\Delta u = a^*(x)u^{-\beta}$ in Ω, $u=0$ on $\partial\Omega$, $u>0$ in Ω. Again in Section 2, Lemma 2.10 improves, under the assumptions H1)-H6), the regularity results of [35], [36], and [37]. In fact, it proves that any weak solution of (1.1) belongs to $C^1(\Omega)\cap C(\overline{\Omega})$.
The main objective in Section 3 is to prove that the function v_λ provided by Remark 1.4 is a maximal solution of (1.1), in the sense that $w \le v_\lambda$ for each weak solution of (1.1). After some preliminary lemmas, this is done in Lemma 3.6 by using a sub-supersolution argument. This property of v_λ plays a crucial role in the proof of Theorem 1.6
Section 4 concerns certain principal eigenvalue problems with singular potential needed for the proof of Theorem 1.6.
In Section 5 we prove Theorem 1.6 by a contradiction argument. To do it, we suppose that for some $\lambda\in(0,\Sigma)$ there exists a weak solution w of (1.1) such that $w\ne u_\lambda$ and $w\ne v_\lambda$. We rewrite (1.1) as $S(\lambda,u)=0$, where

$$S(\lambda,u) := u-(-\Delta)^{-1}\left(au^{-\beta}+\lambda h(.,u)\right),$$

and where $(-\Delta)^{-1}$ denotes the solution operator for the problem $-\Delta u = h$ in $\Omega, u=0$ on $\partial\Omega$.
From [37] we know that $S:(0,\infty)\times U_\beta\to Y_\beta$ is a continuously Frechet differentiable operator, where Y_β and U_β are, respectively, a suitable Banach's space and a suitable nonempty open subset of Y_β, with U_β such that any weak solution u of (1.1) belongs to U_β (for the definitions Y_β and U_β, see Definition 2.8 in Section 2).
In Remark 5.2 we observe that, as in [37], if $w\le v_\lambda$ and $w\ne v_\lambda$, then $r_{\lambda,w}>1$, where $r_{\lambda,w}$ denotes the principal eigenvalue of the operator $-\Delta+\beta aw^{-\beta-1}$ in Ω, with weight function $\lambda h_s(.,w)$, and with homogeneous Dirichlet boundary condition. (notice that the potential $\beta aw^{-\beta-1}$ is singular at $\partial\Omega$).
We observe also in Remark 5.2, Lemma 5.3, and Lemma 5.4 that the condition $r_{\lambda,w}>1$ allows, as in [37], the use of the implicit function theorem to obtain, for some $\varepsilon>0$, a local continuously differentiable branch $\xi:(\lambda-\varepsilon,\lambda+\varepsilon)\to U_\beta$ such that $S(\sigma,\xi(\sigma))=0$ for all $\sigma\in(\lambda-\varepsilon,\lambda+\varepsilon)$ and $\xi(\lambda)=w$. Then we show that ξ can be extended to a continuously differentiable branch $\Theta:(0,\lambda+\varepsilon)\to U_\beta$ such that $S(\sigma,\Theta(\sigma))=0$ for any $\sigma\in(0,\lambda+\varepsilon)$ and $\lim_{\sigma\to0^+}\Theta=u_0$ with convergence in Y_β, where u_0 is the unique weak solution of (1.1) for $\lambda=0$.
Next we repeat the same process, but starting with u_λ instead of w, to obtain, for some $\varepsilon'>0$, a continuously differentiable branch, $\Phi:(0,\lambda+\varepsilon')\to Y_\beta$ such that $S(\sigma,\Phi(\sigma))=0$ for $\sigma\in(0,\lambda+\varepsilon')$ and $\lim_{\sigma\to0^+}\Phi=u_0$ with convergence in Y_β. Our final step within the proof of Theorem 1.6 will be to obtain, for $\sigma\in(0,\lambda)$, an estimate of the norm $\|\Phi(\sigma)-\Theta(\sigma)\|_{H_0^1(\Omega)}$ which, by taking the limit as $\sigma\to0^+$, will give a contradiction.

2. Preliminaries

Let us introduce some notations we will use: δ_Ω will denote the function defined on $\overline{\Omega}$ by

$$\delta_\Omega(x) := dist(x,\partial\Omega). \tag{2.1}$$

and $(-\Delta)^{-1}$ will denote the inverse of the bijection $-\Delta: H_0^1(\Omega)\to H^{-1}(\Omega)$.
If ξ is a measurable functon defined on Ω we will write $\xi\in H^{-1}(\Omega)$ to mean that the map $\phi\to\int_\Omega\xi\phi$ belongs to $H^{-1}(\Omega)$
If f and g, are two functions defined *a.e.* in Ω, we will write $f\approx g$ to mean that, for some positive constants c_1 and c_2, $c_1 f\le g\le c_2 f$ in Ω, and we will write $f\lessapprox g$ (respectively $f\gtrapprox g$) to mean that for some positive constant c, $f\le cg$ in Ω (resp. $f\ge cg$ in Ω).
If f and g are functions defined *a.e.* in Ω, and if no confusion arises, we will write $f=g$ in Ω, $f\le g$ in Ω and $f\ge g$ in Ω to mean that $f=g$ *a.e.* in Ω, $f\le g$ *a.e.* in Ω and $f\ge g$ *a.e.* in Ω respectively.
We will need the following elementary comparison lemma for singular equations:

Lemma 2.1. *i) Let $\beta>0$, and for $i=1,2$, let $u_i\in H_0^1(\Omega)$, and let $a_i\in L^\infty(\Omega)$ be such that $ess\,inf\,(a_i)>0$. If $a_2\ge a_1$ and if u_1 and u_2 satisfy, in weak sense,*

$$\begin{cases} -\Delta u_1 \le a_1 u_1^{-\beta} \text{ in } \Omega, \\ u_1 = 0 \text{ on } \partial\Omega \\ u_1>0 \text{ in } \Omega \end{cases} \quad \text{and} \quad \begin{cases} -\Delta u_2 \ge a_2 u_2^{-\beta} \text{ in } \Omega, \\ u_2 = 0 \text{ on } \partial\Omega \\ u_2>0 \text{ in } \Omega, \end{cases}$$

then $u_1\le u_2$ a.e. in Ω.
ii) Let $\beta>0$, let $a\in L^\infty(\Omega)$ be such that $ess\,inf\,(a)>0$ and, for $i=1,2$, let $u_i\in H_0^1(\Omega)$ be such that, in weak sense,

$$\begin{cases} -\Delta u_1 \le a u_1^{-\beta} \text{ in } \Omega, \\ u_1 = 0 \text{ on } \partial\Omega \\ u_1>0 \text{ in } \Omega \end{cases} \quad \text{and} \quad \begin{cases} -\Delta u_2 \ge a u_2^{-\beta} \text{ in } \Omega, \\ u_2 = 0 \text{ on } \partial\Omega \\ u_2>0 \text{ in } \Omega, \end{cases}$$

then $u_1\le u_2$ a.e. in Ω.

Proof. To see i) observe that, in weak sense,

$$\begin{cases} -\Delta(u_2-u_1) \geq a_2 u_2^{-\beta} - a_1 u_1^{-\beta} \geq a_1\left(u_2^{-\beta} - u_1^{-\beta}\right) \text{ in } \Omega, \\ u_2 - u_1 = 0 \text{ on } \partial\Omega, \end{cases}$$

Now we use the test function $\varphi := -(u_2-u_1)^-$ to get

$$\int_\Omega \left\|\nabla\left((u_2-u_1)^-\right)\right\|^2 \leq -\int_\Omega a_1\left(u_2^{-\beta}-u_1^{-\beta}\right)(u_2-u_1)^- \leq 0.$$

Thus, by the Poincaré's inequality, $u_1 \leq u_2$.
The proof of ii) is similar. We have, in weak sense,

$$\begin{cases} -\Delta(u_2-u_1) \geq a\left(u_2^{-\beta} - u_1^{-\beta}\right) \text{ in } \Omega, \\ u_2 - u_1 = 0 \text{ on } \partial\Omega, \end{cases}$$

and so, by taking the test function $\varphi := -(u_2-u_1)^-$, we get

$$\int_\Omega \left\|\nabla\left((u_2-u_1)^-\right)\right\|^2 \leq -\int_\Omega a\left(u_2^{-\beta}-u_1^{-\beta}\right)(u_2-u_1)^- \leq 0.$$

which, as before, by the Poincaré's inequality implies $u_1 \leq u_2$ *a.e.* in Ω. □

Remark 2.2. *For $\beta \in (0,3)$ and for $a \in L^\infty(\Omega)$ such that $0 \leq a \not\equiv 0$ it is well known that there exists one and only one weak solution of the problem*

$$\begin{cases} -\Delta w = a w^{-\beta} \text{ in } \Omega, \\ w = 0 \text{ on } \partial\Omega \\ w > 0 \text{ in } \Omega \end{cases} \tag{2.2}$$

(and, in fact, this follows immediately from Lemma 2.1).
Notice also that, if in addition, $a \in C^\eta_{loc}(\Omega)$ for some $\eta \in (0,1)$ and $a \approx \mathbf{1}$ in Ω then, as a particular case of ([11], Theorem 1), problem (2.2) has a unique classical solution $w \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$. Moreover, $w \approx \Psi_\beta$ in Ω, with $\Psi_\beta : \overline{\Omega} \to \mathbb{R}$ given by the following definition:

Definition 2.3. *For $\beta \in (0,3)$ let $\Psi_\beta : \overline{\Omega} \to \mathbb{R}$ be defined by*

$$\begin{aligned} \Psi_\beta &:= \delta_\Omega \text{ if } 0<\beta<1, \\ \Psi_1 &:= \delta_\Omega\left(\log\left(\frac{\omega_0}{\delta_\Omega}\right)\right)^{\frac{1}{2}} \text{ in } \Omega \text{ and } \Psi_1 := 0 \text{ on } \partial\Omega, \\ \Psi_\beta &:= \delta_\Omega^{\frac{2}{1+\beta}} \text{ if } 1<\beta<3, \end{aligned}$$

with ω_0 an arbitrary constant such that $\omega_0 > diam(\Omega)$.

Notice that, in each case, $\Psi_\beta \in C\left(\overline{\Omega}\right)$. The functions Ψ_β, as well as the estimates from [11] quoted in Remark 2.2 will play a relevant role in our work.

Remark 2.4. *Direct computations using the definitions of the functions Ψ_β show that $\delta_\Omega \Psi_\beta^{-\beta} \in L^2(\Omega)$ and $\Psi_\beta^{1-\beta} \in L^1(\Omega)$ for any $\beta \in (0,3)$.*

Remark 2.5. *If $a \in C^\eta_{loc}(\Omega)$ for some $\eta \in (0,1)$, and $a \approx \mathbf{1}$ in Ω, then the classical solution w of problem 2.2 (given by the result quoted in Remark 2.2) belongs to $H^1_0(\Omega)$ and is a weak solution of (2.2). Indeed, since $w \approx \Psi_\beta$ and since, for $\beta = 1$, $\Psi_\beta \lessapprox d^\gamma_\Omega$ for some $\gamma \in (0,1)$, the assertion follows from ([36], Lemma 3.2), taking there $f(\lambda,.,u) = \lambda h(.,u)$ and $\lambda = 0$.*

We recall also the following lemma from [37] concerning the functions Ψ_β :

Lemma 2.6. *(See [37], Lemma 2.9) If $f \in L^\infty(\Omega)$, then $\Psi_\beta^{-\beta} f \in H^{-1}(\Omega)$ and there exists a constant $c>0$, independent of f, such that $\left\|(-\Delta)^{-1}\left(\Psi_\beta^{-\beta} f\right)\right\|_{H^1_0(\Omega)} \leq c\|f\|_\infty$ and $\left\|\Psi_\beta^{-1}(-\Delta)^{-1}\left(\Psi_\beta^{-\beta} f\right)\right\|_\infty \leq c\|f\|_\infty$.*

Lemma 2.7. $(-\Delta)^{-1}\left(\Psi_\beta^{-\beta}\right) \approx \Psi_\beta$ *in* Ω.

Proof. By Lemma 2.6 $\Psi_{\beta}^{-\beta} \in H^{-1}(\Omega)$. Let $w \in C^{2}(\Omega) \cap C(\overline{\Omega})$ be such that

$$\begin{cases} -\Delta w = w^{-\beta} \text{ in } \Omega, \\ w = 0 \text{ on } \partial\Omega, \\ w > 0 \text{ in } \Omega, \end{cases} \tag{2.3}$$

(by Remark 2.2 there exists a unique such a w). Then, By Remark 2.5, $w \in H_{0}^{1}(\Omega)$ and w is a weak solution of (2.3), and by Remark 2.2, there exist positive constants c_1 and c_2 such that

$$c_1 \Psi_{\beta} \le w \le c_2 \Psi_{\beta} \quad \text{ in } \Omega.$$

Thus $c_1^{\beta} w^{-\beta} \le \Psi_{\beta}^{-\beta} \le c_2^{\beta} w^{-\beta}$ in Ω, and so $c_1^{\beta}(-\Delta)^{-1}(w^{-\beta}) \le (-\Delta)^{-1}\left(\Psi_{\beta}^{-\beta}\right) \le c_2^{\beta}(-\Delta)^{-1}(w^{-\beta})$. Since $(-\Delta)^{-1}(w^{-\beta}) = w$ and $w \approx \Psi_{\beta}$, the lemma follows. □

The next definition introduces, for $\beta \in (0,3)$, a Banach space Y_{β} and an open set U_{β} in Y_{β} which will play a significant role in our arguments

Definition 2.8. *For $\beta \in (0,3)$, following [37], we define*

$$\begin{aligned} Y_{\beta} &:= \left\{ u \in H_{0}^{1}(\Omega) : \Psi_{\beta}^{-1} u \in L^{\infty}(\Omega) \right\}, \\ \|u\|_{Y_{\beta}} &:= \|\nabla u\|_{2} + \left\| \Psi_{\beta}^{-1} u \right\|_{\infty} \\ U_{\beta} &:= \left\{ u \in Y_{\beta} : \inf_{\Omega} \Psi_{\beta}^{-1} u > 0 \right\}. \end{aligned}$$

As observed in ([37], Lemma 3.2), $\left(Y_{\beta}, \|.\|_{Y_{\beta}}\right)$ is a Banach's space, and U_{β} is a nonempty open set in Y_{β}.

The next remark recalls a celebrated a-priori estimate for subcritical problems due to Gidas and Spruck. It reads as:

Remark 2.9. *(see [41], Theorem 1.1): Let $g : \overline{\Omega} \times [0,\infty) \to \mathbb{R}$ be a nonnegative and continuous function such that $\lim_{s\to\infty} \frac{g(x,s)}{s^{p}} = k(x)$ uniformly on x, with $p \in \left(1, \frac{n+2}{n-2}\right)$ and with $k \in C(\overline{\Omega})$ such that $\min_{\overline{\Omega}} k > 0$. Then there exists $M \in (0,\infty)$ such that $u \le M$ for any solution (in the sense of distributions on Ω) $u \in C^{1}(\Omega) \cap C(\overline{\Omega})$ of the problem*

$$\begin{cases} -\Delta u = g(.,u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \\ u > 0 \text{ in } \Omega \end{cases}$$

Notice that, although the proof of ([41], Theorem 1.1) was written for the case when $u \in C^{2}(\Omega) \cap C(\overline{\Omega})$, the proof can be adapted for solutions $u \in C^{1}(\Omega) \cap C(\overline{\Omega})$ (as said at the comments in [41] after the statement of Theorem 1.1).

Lemma 2.10. *If u is a weak solution of (1.1) for some $\lambda \ge 0$, then*
i) $u \ge \zeta$, where ζ is the (unique) weak solution of the problem

$$\begin{cases} -\Delta \zeta = a\zeta^{-\beta} \text{ in } \Omega, \\ \zeta = 0 \text{ on } \partial\Omega. \end{cases}$$

ii) There exists a positive constant c, independent of λ and u, such that $u \ge c\Psi_{\beta}$ in Ω.
iii) $u \in C(\overline{\Omega}) \cap C^{1}(\Omega)$.
iv) $u \in U_{\beta}$.

Proof. i) follows immediately from the equations satisfied by u and ζ and the comparison Lemma 2.1.
To see ii) consider two positive constants k_1 and k_2 such that $k_1 \le a \le k_2$ in Ω. Since u is a weak solution of (1.1) we have, in weak sense,

$$\begin{cases} -\Delta u = au^{-\beta} + \lambda h(.,u) \ge k_1 u^{-\beta} \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

Let $u_0 \in C^{2}(\Omega) \cap C(\overline{\Omega})$ be the (unique) solution of (2.3) given by Remark 2.2. By Remark 2.5 $u_0 \in H_{0}^{1}(\Omega)$ and u_0 is a weak solution of (2.3) and, by Remark 2.2, $u_0 \ge c\Psi_{\beta}$ for some constant $c > 0$. Now, in weak sense,

$$\begin{cases} -\Delta\left(k_1^{\frac{1}{1+\beta}} u_0\right) = k_1 \left(k_1^{\frac{1}{1+\beta}} u_0\right)^{-\beta} \text{ in } \Omega, \\ u_0 = 0 \text{ on } \partial\Omega. \end{cases}$$

Thus, by (2.4), (2.4), and Lemma 2.1, we have $u \geq k_1^{\frac{1}{1+\beta}} u_0 \geq c k_1^{\frac{1}{1+\beta}} \Psi_\beta$ in Ω, and so ii) holds.
Let us prove iii). We have $0 \leq a \in L^\infty(\Omega)$ and, by ii), $u \geq c\Psi_\beta$ for some constant $c > 0$. Therefore, for some constant $c' > 0$ we have $0 \leq au^{-\beta} \leq c'\Psi_\beta^{-\beta}$. Thus $au^{-\beta} = g\Psi_\beta^{-\beta}$ for some $g \in L^\infty(\Omega)$ and then Lemma 2.6 gives that $au^{-\beta} \in \left(H_0^1(\Omega)\right)'$. Let $z := (-\Delta)^{-1}\left(au^{-\beta}\right)$. Then, for some constant $c'' > 0$,

$$0 \leq z \leq c'(-\Delta)^{-1}\left(\Psi_\beta^{-\beta}\right) \leq c''\Psi_\beta,$$

the last inequality by Lemma 2.7. Thus $z \in L^\infty(\Omega)$. Since $0 \leq au^{-\beta} \leq c'\Psi_\beta^{-\beta}$ we have also $au^{-\beta} \in L^\infty_{loc}(\Omega)$. Thus, by the inner elliptic estimates (see e.g., [44], Theorem 8.24), $z \in C^1(\Omega)$, and since $0 \leq z \leq c''\Psi_\beta$, we have also that z is continuous at $\partial\Omega$. Thus $z \in C\left(\overline{\Omega}\right) \cap C^1(\Omega)$. Now,

$$\begin{cases} -\Delta(u-z) = \lambda h(.,u) \text{ in } \Omega, \\ u - z = 0 \text{ on } \partial\Omega. \end{cases}$$

Let $w := u - z$. Since $-\Delta(u-z) = \lambda h(.,u) \geq 0$ in Ω and $u - z = 0$ on $\partial\Omega$, the weak maximum principle gives that $w \geq 0$ *a.e.* in Ω. Thus $u \geq z$ in Ω. For $(x,s) \in \overline{\Omega} \times [0,\infty)$ let $h^*(x,s) := h(x, s + z(x))$. Then $h^* \in C\left(\overline{\Omega} \times [0,\infty)\right)$ and

$$\begin{cases} -\Delta w = \lambda h^*(.,w) \text{ in } \Omega, \\ w = 0 \text{ on } \partial\Omega, \\ w > 0 \text{ in } \Omega, \end{cases} \tag{2.4}$$

Now,

$$\frac{h^*(x,s)}{s^p} = \frac{h(x,s+z(x))}{s^p} = \frac{h(x,s)}{s^p} + \frac{h(x,s+z(x)) - h(x,s)}{s^p}, \tag{2.5}$$

and the mean value theorem gives that, for some $\theta = \theta_x \in (0,1)$,

$$\frac{|h(x,s+z(x)) - h(x,s)|}{s^p} = \frac{h_s(x,s+\theta z(x))\,z(x)}{s^p} \leq \frac{h_s(x,2s)}{s^p}\|z\|_\infty = 2^p\frac{h_s(x,2s)}{(2s)^p}\|z\|_\infty \text{ for } s \geq \|z\|_\infty$$

and thus

$$\lim_{s\to\infty}\frac{|h(x,s+z(x)) - h(x,s)|}{s^p} = 0$$

uniformly on $x \in \overline{\Omega}$. Let k be as given by H3). Then, by (2.5) $\lim_{s\to\infty}\frac{h^*(x,s)}{s^p} = k(x)$ uniformly on $x \in \overline{\Omega}$, and so, by Remark 2.9 and (2.4), $w \in L^\infty(\Omega)$. Then $\lambda h^*(.,w) \in L^\infty(\Omega)$, and thus, from (2.4), $w \in W^{2,q}(\Omega)$ for any $q \in [1,\infty)$. Then $w \in C\left(\overline{\Omega}\right) \cap C^1(\Omega)$, and thus, since $z \in C\left(\overline{\Omega}\right) \cap C^1(\Omega)$ we get that $u \in C\left(\overline{\Omega}\right) \cap C^1(\Omega)$. Thus iii) holds.
To prove iv) it only remains to see that $u \in Y_\beta$, i.e., to see that, for some positive constant c, $u \leq c\Psi_\beta$ in Ω. By iii), $u \in C\left(\overline{\Omega}\right)$, and then, by our assumptions on h, we have $\lambda h(.,u) \in L^\infty(\Omega)$. Thus, for some positive constant M, $au^{-\beta} + \lambda h(.,u) \leq Mu^{-\beta}$ in Ω. Therefore $-\Delta u \leq Mu^{-\beta}$ and so $-\Delta\left(M^{-\frac{1}{1+\beta}}u\right) \leq \left(M^{-\frac{1}{1+\beta}}u\right)^{-\beta}$ and thus, by Lemma 2.1, $M^{-\frac{1}{1+\beta}}u \leq u_0$ with u_0 as in the proof of i). By Remark 2.2 $u_0 \leq c'\Psi_\beta$ for some positive constant c'. Therefore $u \leq c'M^{\frac{1}{1+\beta}}\Psi_\beta$ in Ω, which concludes the proof of iv). □

Remark 2.11. *Lemma 2.10 says that any weak solution of (1.1) belongs to U_β, and so it improves ([37], Lemma 3.5) which, applied to our actual case, only says that any weak solution in $L^\infty(\Omega)$ belongs also to U_β.*

3. On the maximal solution of problem (1.1)

Let Σ be as in Remark 1.5 and, for each $\lambda \in [0,\Sigma]$, let v_λ as given there, which, we recall, has the property that if $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is a weak solution of (1.1) and $u \geq v_\lambda$, then $u = v_\lambda$.
Notice that $u_\lambda \neq v_\lambda$ for any $\lambda \in (0,\Sigma)$. Indeed, if $\lambda \in (0,\Sigma)$, ([36], Theorems 1.2 and 1.3) give two weak solutions of (1.1). Suppose that $u_\lambda = v_\lambda$, and consider any arbitrary weak solution w of (1.1). Since u_λ is minimal we have $u_\lambda \leq w$ and so we would have $v_\lambda \leq w$, which implies $v_\lambda = w$. Then $w = v_\lambda = u_\lambda$, which contradicts existence of two weak solutions of (1.1).
Our main purpose in this section is to prove that $u \leq v_\lambda$ for any weak solution u of problem (1.1). To do it, we will proceed by contradiction, using a sub-supersolutions argument.

Definition 3.1. *Let $\zeta : \Omega \to \mathbb{R}$ be a measurable function such that $\zeta\varphi \in L^1(\Omega)$ for any $\varphi \in H_0^1(\Omega)$. As usual, a function $u : \Omega \to \mathbb{R}$ is called a weak subsolution of the problem*

$$\begin{cases} -\Delta u = \zeta \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega \end{cases} \tag{3.1}$$

if $u \in H^1(\Omega)$, $u \le 0$ *on* $\partial\Omega$, *and*

$$\int_\Omega \langle \nabla u, \nabla \varphi \rangle \le \int_\Omega \zeta \varphi \tag{3.2}$$

for any nonnegative $\varphi \in H_0^1(\Omega)$. *Weak superolutions are similarly defined by reversing the above inequalties. Following [46], we say also that u is a subsolution, in the sense of distributions, of the problem*

$$-\Delta u = \zeta \text{ in } \Omega,$$

if $u \in L^1_{loc}(\Omega)$ *and (3.2) holds for any nonnegative* $\varphi \in C_c^\infty(\Omega)$. *Supersolutions, in the sense of distributions, are similarly defined by reversing the inequality (3.2).*

Proposition 3.2. *For* $\beta \in (0,3)$ *let* U_β *be as given in Definition 2.8. If* $\underline{u} \in U_\beta$ *and* $\overline{u} \in U_\beta$ *are a weak subsolution and a weak supersolution, respectively, of problem (1.1) such that* $\underline{u} \le \overline{u}$, *then problem (1.1) has a weak solution* u^* *satisfying* $\underline{u} \le u^* \le \overline{u}$ *in* Ω.

Proof. Clearly $\underline{u}$ and $\overline{u}$ are a subsolution and a supersolution, respectively, in the sense of distributions, of (1.1). Let

$$k(x) := a(x)\underline{u}(x)^{-\beta} + \lambda h(x, \overline{u}(x))$$

Let Ψ_β be as given by Definition 2.8. Since $\underline{u}^{-\beta} \approx \Psi_\beta^{-\beta} \in L^1_{loc}(\Omega)$ and $\overline{u} \approx \Psi_\beta \in L^\infty(\Omega)$ then we have $k \in L^1_{loc}(\Omega)$. Also, $s \to a(x)s^{-\beta}$ is nonincreasing and $s \to \lambda h(x,s)$ is nondecreasing, in both cases for $a.e.\ x \in \Omega$, thus for $a.e.\ x \in \Omega$ it holds that

$$0 \le a(x)s^{-\beta} + \lambda h(x,s) \le k(x) \text{ for all } s \in [\underline{u}(x), \overline{u}(x)],$$

then, by ([46], Theorem 2.4) (1.1) has a solution $z \in W^{1,2}_{loc}(\Omega)$, in the sense of distributions, such that $\underline{u} \le u \le \overline{u}$ $a.e.$ in Ω. Since $0 \le \underline{u} \le \overline{u} \approx \Psi_\beta \in L^\infty(\Omega)$ we have that $u \in W^{1,2}_{loc}(\Omega) \cap L^\infty(\Omega)$ Since $\underline{u} \approx \Psi_\beta$ and $\overline{u} \approx \Psi_\beta$ we have $u \approx \Psi_\beta$. Now:
If $0 < \beta < 1$ we have $\Psi_\beta = \delta_\Omega$ and so $\underline{u} \approx \delta_\Omega$.
If $\beta = 1$ then $\Psi_\beta = \delta_\Omega \left(\log\left(\frac{\omega}{\delta_\Omega}\right)\right)^{\frac{1}{2}}$ and so, for any $\gamma \in (0,1)$, $\delta_\Omega \lessapprox \Psi_\beta \lessapprox d_\Omega^\gamma$ which gives $\delta_\Omega \lessapprox u \lessapprox \delta_\Omega^\gamma$.
If $1 < \beta < 2$ then $\Psi_\beta = \delta_\Omega^{\frac{2}{1+\beta}}$ and then $u \approx \delta_\Omega^{\frac{2}{1+\beta}}$.
Thus, by ([36], Lemma 3.2), $u \in H_0^1(\Omega)$ and u is a weak solution of (1.1). □

Remark 3.3. *(see [42], Proposition 5.9) Let U be a domain in* $\mathbb{R}^n$. *Let* $f_1, f_2 \in L^1(U)$. *If* $u_1, u_2 \in L^1(U)$ *are such that* $\Delta u_1 \ge f_1$ *and* $\Delta u_2 \ge f_2$ *in the sense of distributions in* U, *then*

$$\Delta \max\{u_1, u_2\} \ge \chi_{\{u_1 > u_2\}} f_1 + \chi_{\{u_2 > u_1\}} f_2 + \chi_{\{u_1 = u_2\}} \frac{f_1 + f_2}{2}$$

in the the sense of distributions in U.

Lemma 3.4. *If* u, v *are weak subsolutions (repectively weak supersolutions) of (1.1) then* $w := \max\{u,v\}$ *(resp.* $w := \min\{u,v\}$*) is a weak subsolution (resp. a weak supersolution) of (1.1).*

Proof. Suppose that u, v are weak subsolutions of (1.1) and consider an arbitrary $\varphi \in C_c^\infty(\Omega)$ and an open domain U such that $supp(\varphi) \subset U \subset \overline{U} \subset \Omega$. Since $u, v \in H_0^1(\Omega)$ we have $u, v \in L^1(U)$ and by Lemma 2.10, there exists a positive constant c such that $u \ge c\Psi_\beta$ and $v \ge c\Psi_\beta$ $a.e$ in Ω. Thus $au^{-\beta}$ and $av^{-\beta}$ belong to $L^1(U)$. Also, again by Lemma 2.10, u and v belong to $C(\overline{\Omega})$ and so, since h is nonnegative and $s \to h(x,s)$ is nondecreasing for $a.e.\ x \in \Omega$, we have $0 \le h(.,u) \le h(., \|u\|_\infty) \in L^1(U)$ and thus $h(.,u) \in L^1(U)$. Similarly, $h(.,v) \in L^1(U)$ and so $au^{-\beta} + h(.,u)$ and $av^{-\beta} + h(.,v)$ belong to $L^1(U)$. Thus, by Remark 3.3 i) used with $u_1 = u$, $u_2 = v$, $f_1 = au^{-\beta} + h(.,u)$ and $f_2 = av^{-\beta} + h(.,v)$, we have

$$\int_U \langle \nabla w, \nabla \varphi \rangle \le \int_U \left(aw^{-\beta} + h(.,w)\right)\varphi.$$

Since u, v belong to $C(\overline{\Omega})$ we have $w \in C(\overline{\Omega})$, and so $0 \le h(.,w) \le h(., \|w\|_\infty)$ and thus the mapping $\psi \to \int_\Omega h(.,w)\psi$ is continuous on $H_0^1(\Omega)$, and since $w \in H_0^1(\Omega)$, the mapping $\psi \to \int_\Omega \langle \nabla w, \nabla \psi \rangle$ is also continuous on $H_0^1(\Omega)$. On the other hand, since $w \ge c\Psi_\beta$ $a.e$ in Ω, Lemma 2.6 gives the continuity of $\psi \to \int_\Omega aw^{-\beta}\psi$ on $H_0^1(\Omega)$. Thus, by density, (3.3) holds for any $\varphi \in H_0^1(\Omega)$ and so w is a subsolution of (1.1).
The assertion of the lemma in the case when u, v are supersolutions of (1.1) follows from the previous one and from the fact that $\min(u,v) = -\max(-u,-v)$. □

Lemma 3.5. *For any $k>1$ the following two statements are equivalent:*
i) The problem

$$-\Delta u = kau^{-\beta} + \lambda h(.,u) \text{ in } \Omega,\ u=0 \text{ on } \partial\Omega,\ u>0 \text{ in } \Omega, \tag{3.3}$$

has at least a weak solution.
ii) The problem

$$-\Delta u = au^{-\beta} + \lambda h(.,u) \text{ in } \Omega,\ u=0 \text{ on } \partial\Omega,\ u>0 \text{ in } \Omega, \tag{3.4}$$

has at least a weak solution.

Proof. Suppose that i) holds and let z be a solution of problem (3.3). Thus z is a supersolution of problem (3.4). Let u_0 be the (unique) solution of the problem

$$\begin{cases} -\Delta u_0 = au_0^{-\beta} \text{ in } \Omega, \\ u_0 = 0 \text{ on } \partial\Omega, \\ u_0 > 0 \text{ in } \Omega. \end{cases}$$

Then u_0 is a subsolution of (3.4). Also,

$$-\Delta\left(k^{-\frac{1}{1+\beta}}z\right) = k^{-\frac{1}{1+\beta}}kaz^{-\beta} + k^{-\frac{1}{1+\beta}}h(\lambda,.,z) \geq a\left(k^{-\frac{1}{1+\beta}}z\right)^{-\beta}$$

and so, by Lemma 2.1, $k^{-\frac{1}{1+\beta}}z \geq u_0$ in Ω. Thus $u_0 \leq k^{-\frac{1}{1+\beta}}z \leq z$ in Ω. Then, by Proposition 3.2, (3.4) has a solution u such that $u_0 \leq u \leq z$ *a.e* in Ω. Thus i) implies ii).
Suppose now that ii) holds, and let u be a solution of (3.4). Then $-\Delta ku = kau^{-\beta} + k\lambda h(.,u) \geq kau^{-\beta} + \lambda h(.,u)$ and so ku is a supersolution of (3.3). Also, $-\Delta\left(\frac{1}{2}u\right) = \frac{1}{2}au^{-\beta} + \frac{1}{2}\lambda h(.,u) \leq kau^{-\beta} + \lambda h(.,u)$ and so $\frac{1}{2}u$ is a subsolution of (3.3) which satisfies $\frac{1}{2}u \leq ku$. Then, by Proposition 3.2, (3.3) has a solution $\widetilde{u}$ such that $\frac{1}{2}u \leq \widetilde{u} \leq ku$ *a.e* in Ω. Thus ii) implies i). □

Lemma 3.6. *Let Σ be as in Remark 1.5 and let $\lambda \in (0,\Sigma)$. Then $w \leq v_\lambda$ for any weak solution w of problem (1.1).*

Proof. We proceed by the way of contradiction. Suppose that w is a weak solution of problem (1.1) such that

$$|\{x \in \Omega : w(x) > v_\lambda(x)\}| > 0.$$

For $k>1$, by Remark 3.5, the problem

$$\begin{cases} -\Delta u = kau^{-\beta} + \lambda h(.,u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \\ u > 0 \text{ in } \Omega. \end{cases}$$

has a weak solution z, and by Lemma 2.10, $z \in U_\beta$. Since $k>1$, $\overline{u} := z$ is a supersolution of problem (1.1). On the other hand, by Remark 3.3, $\underline{u} := \max(v_\lambda, w)$ is a subsolution of problem (1.1) and clearly $\underline{u} \geq v_\lambda$ and $\underline{u} \neq v_\lambda$. Observe that, for k large enough,

$$\underline{u} \leq \overline{u} \ a.e. \text{ in } \Omega. \tag{3.5}$$

Indeed,

$$-\Delta\left(k^{-\frac{1}{1+\beta}}z\right) = k^{-\frac{1}{1+\beta}}kaz^{-\beta} + k^{-\frac{1}{1+\beta}}\lambda h(.,z) \geq a\left(k^{-\frac{1}{1+\beta}}z\right)^{-\beta}.$$

Let u_0 be the (unique) solution of the problem

$$\begin{cases} -\Delta u_0 = au_0^{-\beta} \text{ in } \Omega, \\ u_0 = 0 \text{ on } \partial\Omega. \end{cases}$$

Then, by Lemma 2.1, $k^{-\frac{1}{1+\beta}}z \geq u_0$ in Ω and so $z \geq k^{\frac{1}{1+\beta}}u_0$ in Ω. On the other hand, since (by Lemma 2.10) u_0, w and v_λ belong to U_β, there exist positive constants c_0, c_1 and c_2 such that $u_0 \geq c_0\Psi_\beta$, $w \leq c_1\Psi_\beta$ and $v_\lambda \leq c_2\Psi_\beta$. Thus $z \geq k^{\frac{1}{1+\beta}}u_0 \geq k^{\frac{1}{1+\beta}}c_0\Psi_\beta \geq k^{\frac{1}{1+\beta}}c_0c_1^{-1}w$ and, similarly, $z \geq k^{\frac{1}{1+\beta}}c_0c_2^{-1}v_\lambda$. Then (3.5) holds for $k > \max\left(1, \left(c_0^{-1}c_1\right)^{1+\beta}, \left(c_0^{-1}c_2\right)^{1+\beta}\right)$.
Notice that, by the assumptions on h, $\lambda h(.,s) \in L^2(\Omega)$ for any $s>0$. Thus, by Proposition 3.2, problem (1.1) has a solution u^* such that $\underline{u} \leq u^* \leq \overline{u}$, which, since $\underline{u} \geq v_\lambda$ and $\underline{u} \neq v_\lambda$, contradicts the property of v_λ stated at the beggining of the section. □

4. Some facts about a class of principal eigenvalue problems

In this brief section we recall some facts concerning a class of principal eigenvalue problems with singular potential and weight function, which we will need to prove Theorem 1.6.

Definition 4.1. *Let* $\mathcal{B} := \left\{b : \Omega \to \mathbb{R} : \delta_\Omega^2 b \in L^\infty(\Omega)\right\}$, *and for* $b \in \mathcal{B}$ *let* $\|b\|_{\mathcal{B}} := \left\|\delta_\Omega^2 b\right\|_\infty$, *and let* $\mathcal{B}^+ := \{b \in \mathcal{B} : b \geq 0 \text{ in } \Omega\}$ *and* $P := \{m \in L^\infty(\Omega) : m > 0 \text{ a.e. in } \Omega\}$.

Notice that $\mathcal{B}$ provided with the norm $\|.\|_{\mathcal{B}}$ is a Banach space.

Remark 4.2. *For* $b \in \mathcal{B}^+$ *and* $m \in P$ *consider the principal eigenvalue problem*

$$\begin{cases} -\Delta z + bz = \rho m z \text{ in } \Omega, \\ z = 0 \text{ on } \partial\Omega, \\ z > 0 \text{ in } \Omega \end{cases} \tag{4.1}$$

where $\rho \in \mathbb{R}$ *and the equation is understood in weak sense, i.e.,* $z \in H_0^1(\Omega)$ *and*

$$\int_\Omega \left(\langle \nabla z, \nabla \varphi\rangle + bz\varphi\right) = \rho \int_\Omega m z \varphi$$

for any $\varphi \in H_0^1(\Omega)$. *Notice that, although b may be singular at* $\partial\Omega$ *(for instance* $\delta_\Omega^{-2} \in \mathcal{B}^+$*), by ([43], Theorem 4.1), the principal eigenvalue of (4.1) exists, is unique, positive, and simple. In order to emphasize its dependence on m and b, we will denote such a* ρ *by* $\rho_{m,b}$. *Similarly, we will denote by* $\phi_{m,b}$ *its positive eigenfunction normalized by* $\left\|\phi_{m,b}\right\|_2 = 1$. *In addition, by ([43], Theorem 4.3),* $\rho_{m,b}$ *is given by the usual Rayleigh's variational formula*

$$\rho_{m,b} = \inf_{w \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega \left(|\nabla w|^2 + bw^2\right)}{\int_\Omega m w^2} \tag{4.2}$$

Remark 4.3. *Let P and* $\mathcal{B}^+$ *be as in Definition 4.1, with P provided with the topology inherited from* $L^\infty(\Omega)$ *and* $\mathcal{B}^+$ *endowed with the topology inherited from the Banach space* $\mathcal{B}$. *Then, by ([43], Theorem 4.5) we have:*
i) The map $(m,b) \to \rho_{m,b}$ *is continuous from* $P \times \mathcal{B}^+$ *into* $\mathbb{R}$.
ii) The map $(m,b) \to \phi_{m,b}$ *is continuous from* $P \times \mathcal{B}^+$ *into* $H_0^1(\Omega)$.

Definition 4.4. *Let* Σ *be as in Remark 1.5 and let* U_β *be as given by Definition 2.8. For* $u \in U_\beta$, $x \in \Omega$, *and* $\sigma \in (0,\Sigma)$, *let*

$$b_u(x) := \beta a(x) u^{-\beta-1}(x)$$

and let

$$N^{\sigma,u}(x) := \sigma h_s(x, u(x))$$

where $h_s(x,t) := \frac{\partial h(x,s)}{\partial s}_{|s=t}$.

Remark 4.5. *Let* Σ *be as in Definition 4.4 and for* $\sigma \in (0,\Sigma)$ *and* $u \in U_\beta$, *consider the principal eigenvalue problem*

$$\begin{cases} -\Delta z + \beta a u^{-\beta-1} z = r\sigma h_s(.,u) z \text{ in } \Omega, \\ z = 0 \text{ on } \partial\Omega, \\ z > 0 \text{ in } \Omega, \end{cases} \tag{4.3}$$

which is, with the above notations, the problem

$$\begin{cases} -\Delta z + b_u z = rN^{\sigma,u} z \text{ in } \Omega, \\ z = 0 \text{ on } \partial\Omega, \\ z > 0 \text{ in } \Omega, \end{cases} \tag{4.4}$$

Notice that since $u \in U_\beta$ *and* $\sigma \in (0,\Sigma)$ *then* $b_u \in \mathcal{B}^+$ *and* $N^{\sigma,u} \in P$. *Indeed, since* $0 \leq a \in L^\infty(\Omega)$ *and* $u \in U_\beta$ *there exists a constant* $c > 0$ *such that* $0 \leq b_u \leq c\Psi_\beta^{-\beta-1}$. *Thus* $b_u \in \mathcal{B}^+$. *In fact, let* Ψ_β *be as defined by Definition 2.3 and let* δ_Ω *be defined by (2.1). Then (since* $0 < a \in L^\infty(\Omega)$*):*

i) If $0 < \beta < 1$ *then* $\Psi_\beta = \delta_\Omega$ *and so* $\delta_\Omega^2 b_u \leq c\delta_\Omega^2 \Psi_\beta^{-\beta-1} = c\delta_\Omega^{1-\beta} \in L^\infty(\Omega)$,

ii) If $\beta = 1$ *then* $\Psi_\beta = \delta_\Omega \left(\log\left(\frac{\omega_0}{\delta_\Omega}\right)\right)^{\frac{1}{2}}$ *and so* $\delta_\Omega^2 b_u \leq c\delta_\Omega^2 \Psi_\beta^{-2} = c\left(\log\left(\frac{\omega_0}{\delta_\Omega}\right)\right)^{-1} \in L^\infty(\Omega)$, *iii) If* $1<\beta<3$ *then* $\Psi_\beta=\delta_\Omega^{\frac{2}{1+\beta}}$ *and so* $\delta_\Omega^2 b_u\le c\delta_\Omega^2\Psi_\beta^{-\beta-1}=c\delta_\Omega^2\delta_\Omega^{--(\beta+1)\frac{1}{1+\beta}}\in L^\infty(\Omega)$.

Therefore, for any $\beta\in(0,3)$ *and* $u\in U_\beta$, *we have* $b_u\in\mathcal{B}^+$. *On the other hand,* $N^{\sigma,u}=\sigma h_s(.,u(.))$ *and so, from the assumptions on h stated at the introduction, it is clear that* $N^{\sigma,u}>0$ *in* Ω *and that* $N^{\sigma,u}\in L^\infty(\Omega)$, *and so* $N^{\sigma,u}\in P$.
Then, by Remark 4.2, problem (4.3) has a unique principal eigenvalue $r=\rho_{N^{\sigma,u},b_u}$ *which is unique, positive, simple, and it is given by the corresponding Rayleigh's variational formula.*

Remark 4.6. *In order to simplify the notation the principal eigenvalue of problem (4.3) will be denoted, from now on, by* $r_{\sigma,u}$ *(instead of* $\rho_{N^{\sigma,u},b_u}$*), and its normalized positive principal eigenfunction (normalized by requiring* $\|.\|_2=1$*) will be denoted by* $\phi_{\sigma,u}$ *(instead of* $\phi_{N^{\sigma,u},b_u}$*).*

We will need also the following lemma

Lemma 4.7. *Let* Y_β *and* U_β *be as given in Definition 2.8 and let* $u\in U_\beta$. *Let* Σ *be as in Remark 1.5 and let* $\sigma\in(0,\Sigma)$. *Let* $\{\sigma_j\}_{j\in\mathbb{N}}$ *and* $\{u_j\}_{j\in\mathbb{N}}$ *be sequences in* $(0,\Sigma)$ *and* U_β *respectively, and assume that* $\{\sigma_j\}_{j\in\mathbb{N}}$ *converges to* σ *and that* $\{u_j\}_{j\in\mathbb{N}}$ *converges to u in* Y_β. *Then*
i) $\{b_{u_j}\}_{j\in\mathbb{N}}$ *converges to* b_u *in* $\mathcal{B}$.
ii) $\{N^{\sigma_j,u_j}\}_{j\in\mathbb{N}}$ *converges to* N^{σ_j,u_j} *in* $L^\infty(\Omega)$.
iii) $\{r_{\sigma_j,u_j}\}_{j\in\mathbb{N}}$ *converges to* $r_{\sigma,u}$ *in* $\mathbb{R}$ *and* $\{\phi_{\sigma_j,u_j}\}_{j\in\mathbb{N}}$ *converges to* $\phi_{\sigma,u}$ *in* $H_0^1(\Omega)$.

Proof. Let Ψ_β be as given by Definition 2.3. Since $u\in U_\beta$ there exists $c>0$ such that

$$u\ge c\Psi_\beta \text{ in } \Omega. \tag{4.5}$$

Let Y_β and $\|.\|_{Y_\beta}$ be as given by Definition 2.8, and let $B^{Y_\beta}\left(u,\frac{c}{2}\right)$ be the open ball in Y_β centered at u and with radius $\frac{c}{2}$. Thus for any $z\in B^{Y_\beta}\left(u,\frac{c}{2}\right)$ we have $\left\|\Psi_\beta^{-1}(z-u)\right\|_\infty<\frac{c}{2}$ and so $z>u-\frac{c}{2}\Psi_\beta\ge\left(c-\frac{c}{2}\right)\Psi_\beta=\frac{c}{2}\Psi_\beta$ in Ω. Now, $\{u_j\}_{j\in\mathbb{N}}$ converges to u in Y_β and so there exists $j_0\in\mathbb{N}$ such that $u_j\in B^{Y_\beta}\left(u,\frac{c}{2}\right)$ for any $j\ge j_0$. Then

$$u_j\ge\frac{c}{2}\Psi_\beta \text{ in } \Omega \text{ for any } j\ge j_0. \tag{4.6}$$

Let b_{u_j} and b_u be defined by Definition 4.4. Observe that, for $j\in\mathbb{N}$,

$$\left|\delta_\Omega^2 b_{u_j}-\delta_\Omega^2 b_u\right|=\left|\beta a\delta_\Omega^2\left(\left(u_j^{-\beta-1}-u^{-\beta-1}\right)\right)\right|\le c\left|\delta_\Omega^2\left(u_j^{-\beta-1}-u^{-\beta-1}\right)\right| \text{ in } \Omega, \tag{4.7}$$

where $c=\beta\|a\|_\infty$ is a positive constant independent of j. Now, for $x\in\Omega$, the mean value theorem gives that

$$u_j^{-\beta-1}(x)-u^{-\beta-1}(x)=-(\beta+1)\,\theta_{j,x}^{-\beta-2}(u_j(x)-u(x)) \tag{4.8}$$

for some number $\theta_{j,x}$ belonging to the open segment with endpoints $u_j(x)$ and $u(x)$, and so, by (4.5) and (4.6),

$$\theta_{j,x}\ge\frac{c}{2}\Psi_\beta(x) \text{ in } \Omega \text{ for any } x\in\Omega \text{ whenever } j\ge j_0. \tag{4.9}$$

Therefore, from (4.7), (4.8), and (4.9), we have, for any $j\ge j_0$,

$$\left|\delta_\Omega^2 b_{u_j}-\delta_\Omega^2 b_u\right|\le\frac{c(\beta+1)\,\delta_\Omega^2\left|u_j-u\right|}{\left(\frac{c}{2}\Psi_\beta\right)^{\beta+2}}=c'\frac{\delta_\Omega^2\Psi_\beta\left|\Psi_\beta^{-1}(u_j-u)\right|}{\Psi_\beta^{\beta+2}}=c'\delta_\Omega^2\Psi_\beta^{-\beta-1}\left|\Psi_\beta^{-1}(u_j-u)\right| \text{ in } \Omega, \tag{4.10}$$

with c' a positive constant independent of j. Direct computations using the definition of the functinos Ψ_β give that

$$\delta_\Omega^2\Psi_\beta^{-\beta-1}\in L^\infty(\Omega). \tag{4.11}$$

Then, by (4.10) and (4.11) we get

$$\left|\delta_\Omega^2 b_{u_j}-\delta_\Omega^2 b_u\right|\le c''\left|\Psi_\beta^{-1}(u_j-u)\right| \text{ in } \Omega,$$

with c'' a positive constant independent of j, and since $\{u_j\}_{j\in\mathbb{N}}$ converges to u in Y_β we have also that $\lim_{j\to\infty}\left\|\Psi_\beta^{-1}(u_j-u)\right\|_\infty=0$ and then

$$\lim_{j\to\infty}\left\|\delta_\Omega^2 b_{u_j}-\delta_\Omega^2 b_u\right\|_\infty=0$$

which gives that $\{b_{u_j}\}_{j\in\mathbb{N}}$ converges to b_u in $\mathcal{B}$. Thus i) holds

Let us see that $\{N^{\sigma_j,u_j}\}_{j\in\mathbb{N}}$ converges to $N^{\sigma,u}$ in $L^\infty(\Omega)$. Since each Ψ_β is bounded, $\lim_{j\to\infty}\left\|\Psi_\beta^{-1}(u_j-u)\right\|_\infty=0$ implies that

$$\lim_{j\to\infty}\left\|u_j-u\right\|_\infty=0.$$

Also $u\in L^\infty(\Omega)$ (because $\Psi_\beta^{-1}u\in L^\infty(\Omega)$). Then $\{u_j\}_{j\in\mathbb{N}}$ is bounded in $L^\infty(\Omega)$. Thus there exists $M>0$ such that $\|u\|_\infty\le M$ and $\|u_j\|_\infty\le M$ for all $j\in\mathbb{N}$. Then for each j there exists $E_j\subset\Omega$ such that $|E_j|=0$ and $0\le u_j\le M$ in $\Omega\setminus E_j$, and there exists $E\subset\Omega$ such that $|E|=0$ and $0\le u\le M$ in $\Omega\setminus E$. Let $F:=E\cup\cup_{j\in\mathbb{N}}E_j$. Then $|F|=0$, $0\le u\le M$ in $\Omega\setminus F$ and $0\le u_j\le M$ in $\Omega\setminus F$ for all $j\in\mathbb{N}$. Now, by our assumptions on h stated at the introduction, there exists a constant $M^*>0$ such that $|h_s(x,t)|\le M^*$ and $|h_{ss}(x,t)|\le M^*$ for any $(x,t)\in\Omega\times[0,M]$. Then by the triangle inequality and the mean value theorem we have, for any $x\in\Omega\setminus F$ and for all $j\in\mathbb{N}$,

$$\begin{aligned}&|N^{\sigma_j,u_j}(x)-N^{\sigma,u}(x)|=\left|\sigma_j h_s(x,u_j(x))-\sigma h_s(x,u(x))\right|\\&\le\left|(\sigma_j-\sigma)h_s(x,u_j(x))\right|+\left|\sigma h_s(x,u_j(x))-\sigma h_s(x,u(x))\right|\\&\le\left|(\sigma_j-\sigma)\right|\left|h_s(x,u_j(x))\right|+\sigma\left|h_{ss}(x,\zeta_{j,x})\right|\left|u_j(x)-u(x)\right|\end{aligned}\tag{4.12}$$

where $\zeta_{j,x}$ is a number belonging to the open segment with endpoints $u_j(x)$ and $u(x)$. Then, for $x\in\Omega\setminus F$ and for all $j\in\mathbb{N}$, $\left|h_{ss}(x,\zeta_{j,x})\right|\le M^*$ and so, for such x and j, (4.12) gives

$$|N^{\sigma_j,u_j}(x)-N^{\sigma,u}(x)|\le M^*\left|\sigma_j-\sigma\right|+\sigma M^*\left|u_j(x)-u(x)\right|$$

which, since $\lim_{j\to\infty}\left\|u_j-u\right\|_\infty=0$ and $\lim_{j\to\infty}\sigma_j=\sigma$, implies that $\{N^{\sigma_j,u_j}\}_{j\in\mathbb{N}}$ converges to $N^{\sigma,u}m$ in $L^\infty(\Omega)$. Thus ii) holds. Now, iii) follows from i), ii), and Remark 4.3. □

5. Proof of the main results

We fix, for the whole section, Σ as given by Remark 1.5.

Definition 5.1. *Let Y_β and U_β be as in Definition 2.8, and let $S:(0,\Sigma)\times U_\beta\to Y_\beta$ be defined by*

$$S(\lambda,u):=u-(-\Delta)^{-1}\left(au^{-\beta}+\lambda h(.,u)\right).\tag{5.1}$$

By ([37], Lemma 3.3) we have $au^{-\beta}+\lambda h(.,u)\in H^{-1}(\Omega)$, and $(-\Delta)^{-1}\left(au^{-\beta}+\lambda h(\lambda,u)\right)\in Y_\beta$ for any $(\lambda,u)\in(0,\Sigma)\times U_\beta$, therefore S is well defined. Moreover, by ([37], Lemma 3.7), ([37], Corollary 3.8), and ([37], Lemma 3.9) (all of them applied with $f(\lambda,.,s):=\lambda h(.,s)$) the operator S is continuously Fréchet differentiable in $(0,\Sigma)\times U_\beta$, and its differential at $(\lambda,u)\in(0,\Sigma)\times U_\beta$, denoted by $DS_{(\lambda,u)}$, is given by

$$DS_{(\lambda,u)}(\tau,\psi)=\psi-(-\Delta)^{-1}\left(-\beta a\psi u^{-\beta-1}+\tau h(.,u)+\psi\lambda\tfrac{\partial h}{\partial s}(.,u)\right),\tag{5.2}$$

and its partial derivative $D_2S_{(\lambda,u)}$ at (λ,u) (i.e. the Fréchet differential at u, of the mapping $v\to S(\lambda,v)$) is given by

$$D_2S_{(\lambda,u)}(\psi)=\psi-(-\Delta)^{-1}\left(\left(-\beta au^{-\beta-1}+\lambda\frac{\partial h}{\partial s}(.,u)\right)\psi\right)\tag{5.3}$$

We recall that, as said in Remark 4.6, the principal eigenvalue of a problem of the form (4.3) will be denoted by $r_{\sigma,u}$.

Remark 5.2. *In ([37], Lemma 5.17) it is proved that if $\lambda\in(0,\Sigma)$ and if u_λ is the minimal solution (as provided by Remark 1.2) of (1.1) then,*

$$r_{\lambda,u_\lambda}>1.$$

where r_{λ,u_λ} denotes the principal eigenvalue of problem 4.3, taking there $u=u_\lambda$. By using this fact and a maximum principle with weight function given by ([37], Lemma 4.4), in ([37], Lemma 5.18) it was proved that

$$D_2S_{(\lambda,u_\lambda)}:Y_\beta\to Y_\beta\text{ is bijective.}$$

An inspection of the proofs of lemmas ([37], Lemma 5.17) and ([37], Lemma 5.18) shows that they work also if u_λ is replaced by any weak solution u of (1.1) such that $u\le v_\lambda$ and $u\ne v_\lambda$.

Lemma 5.3. *Let $\lambda\in(0,\Sigma)$, and let u be a weak solution of (1.1) such that $u\ne v_\lambda$, then:*
ii) $r_{\lambda,u}>1$.
ii) $D_2S_{(\lambda,u)}:Y_\beta\to Y_\beta$ is bijective.

Proof. By Lemma 3.6) we actually know that $u \leq v_\lambda$ for any weak solution of (1.1), then the lemma follows from Remark 5.2 □

Now we can prove the following

Lemma 5.4. *Let $\lambda \in (0,\Sigma)$, and let u be a weak solution of (1.1) such that $r_{\lambda,u} > 1$, then there exist $\varepsilon > 0$ and an open set $V \subset Y_\beta$ such that $u \in V \subset U_\beta$ and if $J := (\lambda - \varepsilon, \lambda + \varepsilon)$ then:*
i) $J \subset (0,\Sigma)$ and for any $\sigma \in J$ there exists a unique $\xi(\sigma) \in V$ such that

$$\begin{cases} S(\sigma,\xi(\sigma)) = 0, \\ \xi(\lambda) = u. \end{cases}$$

Moreover, $\xi : J \to Y_\beta$ is continuously differentiable, and its derivative ξ' satisfies, in weak sense, for any $\sigma \in J$,

$$\begin{cases} -\Delta(\xi'(\sigma)) = -\beta a(\xi(\sigma))^{-(1+\beta)}\xi'(\sigma) + h(.,\xi(\sigma)) + \sigma \frac{\partial h}{\partial s}(.,\xi(\sigma))\xi'(\sigma) \text{ in } \Omega, \\ \xi'(\sigma) = 0 \text{ on } \partial\Omega.. \end{cases} \tag{5.4}$$

ii) $r_{\sigma,\xi(\sigma)} > 1$ for any $\sigma \in J$.
iii) $\sigma \to \xi(\sigma)$ is nondecreasing on J.
iv) $\sigma \to r_{\sigma,\xi(\sigma)}$ is nonincreasing on J.

Proof. The first assertion of i) follows from Lemma 5.3 and the implicit function theorem and, since $S(\sigma,\xi(\sigma)) = 0$ for any $\sigma \in J$, (5.4) follows from (5.2) and the chain rule.
To see ii), observe that, by i), $\sigma \to \xi(\sigma)$ is continuous from J into Y_β. Then, by lemma 4.7, $\sigma \to r_{\sigma,\xi(\sigma)}$ is continuous on J. Thus, since $r_{\lambda,u} > 1$, by diminishing ε if necessary, we get that $r_{\sigma,\xi(\sigma)} > 1$ for all $\sigma \in J$. Thus ii) holds.
Let us see iii). We rewrite (5.4) as

$$\begin{cases} -\Delta(\xi'(\sigma)) + \beta a(\xi(\sigma))^{-\beta-1}\xi'(\sigma) = N^{\sigma,\xi(\sigma)}\xi'(\sigma) + h(.,\xi(\sigma)) \text{ in } \Omega, \\ \xi'(\sigma) = 0 \text{ on } \partial\Omega. \end{cases}$$

Then, since $h(.,\xi(\sigma)) \geq 0$ and $r_{\sigma,\xi(\sigma)} > 1$, the maximum principle with weight stated in ([37] Lemma 4.4 ii)) gives that $\xi'(\sigma) \geq 0$ for any $\sigma \in J$. Thus $\sigma \to \xi(\sigma)$ is nondecreasing on J, and so iii) holds.
To see iv), observe that for σ, $\tau \in J$ such that $\sigma \leq \tau$ we have, by iii), $\xi(\sigma) \leq \xi(\tau)$ in Ω, and so, by the assumptions on h stated at the introduction,

$$N^{\sigma,\xi(\sigma)} = \sigma\frac{\partial h}{\partial s}(.,\xi(\sigma)) \leq \sigma\frac{\partial h}{\partial s}(.,\xi(\tau)) \leq \tau\frac{\partial h}{\partial s}(.,\xi(\tau)) = N^{\tau,\xi(\tau)} \text{ in } \Omega.$$

Then

$$r_{\sigma,\xi(\sigma)} = \inf_{z\in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega \left[|\nabla z|^2 + \beta a\xi(\sigma)^{-\beta-1}z^2\right]}{\int_\Omega N^{\sigma,\xi(\sigma)}z^2} \geq \inf_{z\in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega \left[|\nabla z|^2 + \beta a\xi(\tau)^{-\beta-1}z^2\right]}{\int_\Omega N^{\tau,\xi(\tau)}z^2.} = r_{\tau,\xi(\tau)},$$

and thus iv) holds.

□

Let us recall the Hardy's inequality (see e.g., [45], p. 313):

$$\sup_{0\neq\varphi\in H_0^1(\Omega)} \frac{\left\|\varphi\delta_\Omega^{-1}\right\|_{L^2(\Omega)}}{\|\nabla\varphi\|_{L^2(\Omega)}} < \infty$$

Lemma 5.5. *Let Σ be as given by Remark 1.5 and let Y_β and U_β be as given in Definition 2.8. Let $\lambda_0 \in (0,\Sigma)$ and let w_0 be a weak solution of problem (1.1) such that $w_0 \neq v_{\lambda_0}$, where v_{λ_0} is the maximal solution of (1.1) corresponding to $\lambda = \lambda_0$. Then there exists $\varepsilon > 0$ and a function $\Theta : [0,\lambda_0+\varepsilon) \to U_\beta$ such that*
i) $\Theta(\lambda_0) = w_0$.
ii) $\Theta \in C^1\left((0,\lambda_0+\varepsilon), Y_\beta\right) \cap C\left([0,\lambda_0+\varepsilon), Y_\beta\right)$.
iii) $S(\sigma,\Theta(\sigma)) = 0$ for any $\sigma \in (0,\lambda_0+\varepsilon)$
iv) $\Theta(0) = u_0$ with convergence in Y_β, where u_0 is the (unique) weak solution of (1.1) corresponding to $\lambda = 0$.

Proof. Let $\mathcal{G}$ be the family of the pairs (J,ξ_J) such that:
1) J is an open interval in $\mathbb{R}$, $J \subset (0,\Sigma)$, and $\lambda_0 \in J$.
2) $\xi_J \in C^1\left(J,U_\beta\right)$, $\xi_J(\lambda_0) = w_0$, and $S(\sigma,\xi_J(\sigma)) = 0$ for all $\sigma \in J$.
3) $r_{\sigma,\xi_J(\sigma)} > 1$ for any $\sigma \in J$.
4) $\sigma \to \xi_J(\sigma)$ is nondecreasing on J

5) $\sigma \to r_{\sigma,\xi_J(\sigma)}$ is nonincreasing on J.
By Lemma 5.4, $\mathcal{G} \neq \varnothing$. Notice that, since $\xi_J \in C^1\left(J, U_\beta\right)$ then, by Lemma 4.7,

$$\sigma \to r_{\sigma,\xi_J(\sigma)} \text{ is continuous on } J. \tag{5.5}$$

We claim that:

$$\text{If } (J,\xi_J) \in \mathcal{G}, (J_*,\xi_{J_*}) \in \mathcal{G}, \text{ and } J \cap J_* \neq \varnothing, \text{ then } \xi_J = \xi_{J_*} \text{ in } J \cap J_*. \tag{5.6}$$

Indeed, let $F := \{\sigma \in J \cap J_* : \xi_J(\sigma) = \xi_{J_*}(\sigma)\}$ Then $\lambda_0 \in F$ and so $F \neq \varnothing$. Also, since ξ_J and ξ_{J_*} are continuous in their respective domains, F is closed in $J \cap J_*$. Moreover, if $\sigma \in F$ then, by the uniqueness assertion of Lemma 5.4 i) (used taking there $\lambda = \sigma$), there exists $\varepsilon > 0$ such that $(\sigma - \varepsilon, \sigma + \varepsilon) \subset F$, and so F is open in $J \cap J_*$. Then, since $J \cap J_*$ is a connected set, we conclude that $F = J \cap J_*$, and thus $\xi_J = \xi_{J_*}$ in $J \cap J_*$.
Let $I := \cup_{J:(J,\xi_J)\in\mathcal{G}} J$. Since I is a union of open intervals contained in $(0,\Sigma)$, and all of them contain λ_0, it follows that

$$I \text{ is an open interval, } I \subset (0,\Sigma), \text{ and } \lambda_0 \in I. \tag{5.7}$$

Let $\Theta : I \to U_\beta$ be defined by

$$\Theta(\sigma) := \xi_J(\sigma) \text{ if } \sigma \in J \text{ for some } (J,\xi_J) \in \mathcal{G}. \tag{5.8}$$

By (5.6) Θ is well defined on I and, from 2) and (5.8),

$$\Theta \in C^1\left(I, U_\beta\right), \text{ and } S(\sigma, \Theta(\sigma)) = 0 \text{ for all } \sigma \in I, \tag{5.9}$$

(Later, within the proof of the lemma, we will define also $\Theta(\lambda_*)$, where λ_* is the left endpoint of I, and we will show that Θ is continuous at λ_*. and that $\lambda_* = 0$). For $\sigma \in I$, let $(J,\xi_J) \in \mathcal{G}$ such that $\sigma \in J$. From (5.8) we have $r_{\sigma,\Theta(\sigma)} = r_{\sigma,\xi_J(\sigma)}$ and, by 3), $r_{\sigma,\xi_J(\sigma)} > 1$. Then,

$$r_{\sigma,\Theta(\sigma)} > 1 \text{ for any } \sigma \in J. \tag{5.10}$$

Suppose that $t \in I$, $s \in I$, and $t \leq s \leq \lambda_0$, and let $(J,\xi_J) \in \mathcal{G}$ such that $t \in J$. Then $s \in J$, and so, since by 4) ξ_J is nondecreasing on J, and taking into account the definition (5.8) of Θ we have $\Theta(t) \leq \Theta(s)$. Then

$$\Theta \text{ is nondecreasing on } I. \tag{5.11}$$

For $\sigma \in I$, let $N^{\sigma,\Theta(\sigma)}$ be defined as in Definition 4.4, that is,

$$N^{\sigma,\Theta(\sigma)}(x) = \sigma h_s(x, \Theta(\sigma)(x)) \text{ for any } x \in \Omega.$$

Now, Θ is nondecreasing on I and, by the assumptions on h stated at the introduction, for any $x \in \Omega$, the mapping $s \to h_s(x,s)$ is nondecreasing , then, for any $x \in \Omega$,

$$\sigma \to N^{\sigma,\Theta(\sigma)}(x) \text{ is nondecreasing on } I. \tag{5.12}$$

On the other hand, by the Rayleigh's variational formula for principal eigenvalues we have, for $\sigma \in I$,

$$r_{\sigma,\Theta(\sigma)} = \inf_{z \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega \left[|\nabla z|^2 + \beta a \Theta(\sigma)^{-\beta-1} z^2\right]}{\int_\Omega N^{\sigma,\Theta(\sigma)} z^2}. \tag{5.13}$$

From this expression, and taking into account (5.11), (5.12), and that $N^{\sigma,\Theta(\sigma)}$ is nonnegative for any $\sigma \in I$, it follows that

$$\text{the mapping } \sigma \to r_{\sigma,\Theta(\sigma)} \text{ is nonincreasing on } I. \tag{5.14}$$

Notice that, from (5.7), (5.9), (5.10), (5.11), and (5.14) it follows that

$$(I,\Theta) \in \mathcal{G} \tag{5.15}$$

Notice also that, since $\Theta : I \to U_\beta$ is continuous then, by Lemma 4.7,

$$\sigma \to r_{\sigma,\Theta(\sigma)} \text{ is continuous from } I \text{ into } \mathbb{R} \tag{5.16}$$

In addition, since $\Theta \in C^1\left(I, U_\beta\right)$ and $S(\sigma, \Theta(\sigma)) = 0$ for all $\sigma \in I$, then from (5.2) and the chain rule we have, for any $\sigma \in I$,

$$\begin{cases} -\Delta(\Theta'(\sigma)) = -\beta a (\Theta(\sigma))^{-(1+\beta)} \Theta'(\sigma) + h(.,\Theta(\sigma)) + \sigma \frac{\partial h}{\partial s}(.,\Theta(\sigma)) \Theta'(\sigma) \text{ in } \Omega, \\ \Theta'(\sigma) = 0 \text{ on } \partial\Omega.. \end{cases} \tag{5.17}$$

Since for any $\sigma \in I$, $\Theta(\sigma)$ is a weak solution of (1.1) (taking there $\lambda = \sigma$) we have, in weak sense,

$$\begin{cases} -\Delta(\Theta(\sigma)) = a(\Theta(\sigma))^{-\beta} + \sigma h(.,\Theta(\sigma)) \geq a(\Theta(\sigma))^{-\beta} \text{ in } \Omega, \\ \Theta(\sigma) = 0 \text{ on } \partial\Omega \\ \Theta(\sigma) > 0 \text{ in } \Omega. \end{cases} \tag{5.18}$$

On the other hand, the (unique) weak solution u_0 of (1.1) corresponding to $\lambda = 0$ satisfies

$$\begin{cases} -\Delta u_0 = a(u_0)^{-\beta} \text{ in } \Omega, \\ u_0 = 0 \text{ on } \partial\Omega \\ u_0 > 0 \text{ in } \Omega. \end{cases} \tag{5.19}$$

Then, by Lemma 2.1 ii), $\Theta(\sigma) \geq u_0$ for any $\sigma \in I$ and, by Lemma 2.10 iv), there exists a positive constant c_* such that $u_0 \geq c_* \Psi_\beta$ (with Ψ_β given by Definition 2.3). Then, for any $\sigma \in I$,

$$\Theta(\sigma) \geq c_* \Psi_\beta \text{ in } \Omega. \tag{5.20}$$

Let λ_* and λ^* be such that $I = (\lambda_*, \lambda^*)$. By (5.11), $\sigma \to \Theta(\sigma)$ is nondecreasing on I, and clearly $\Theta(\sigma) \geq 0$ in Ω (because $\Theta(\sigma) \in U_\beta$). Then there exists the pointwise limit $\lim_{\sigma \to \lambda_*} \Theta(\sigma)$. Define, for $x \in \Omega$

$$\Theta(\lambda_*)(x) := \lim_{\sigma \to \lambda_*} \Theta(\sigma)(x). \tag{5.21}$$

We are going to show the following three facts:
A) $\Theta(\lambda_*) \in U_\beta$.
B) $\lim_{\sigma \to \lambda_*} \Theta(\sigma) = \Theta(\lambda_*)$ with convergence in Y_β.
C) $\Theta(\lambda_*)$ is a weak solution of (1.1) corresponding to $\lambda = \lambda_*$.
From (5.20),

$$\Theta(\lambda_*) \geq c_* \Psi_\beta \text{ in } \Omega. \tag{5.22}$$

Also, $\Theta(\sigma) \leq \Theta(\lambda_0)$ for any $\sigma \in (\lambda_*, \lambda_0)$ and, since $\Theta(\lambda_0) \in U_\beta$ we have $\Theta(\lambda_0) \leq c_{**} \Psi_\beta$ for some positive constant c_{**}, then

$$\Theta(\lambda_*) \leq c_{**} \Psi_\beta \text{ in } \Omega. \tag{5.23}$$

Now, for any $\sigma \in I$,

$$\begin{cases} -\Delta(\Theta(\sigma)) = a(\Theta(\sigma))^{-\beta} + \sigma h(.,\Theta(\sigma)) \text{ in } \Omega, \\ \Theta(\sigma) = 0 \text{ on } \partial\Omega, \end{cases} \tag{5.24}$$

and so, for $\lambda_* < \sigma < \tau < \lambda_0$ we have, in weak sense,

$$\begin{cases} -\Delta(\Theta(\tau) - \Theta(\sigma)) = a\left(\Theta^{-\beta}(\tau) - \Theta^{-\beta}(\sigma)\right) + \tau h(.,\Theta(\tau)) - \sigma h(.,\Theta(\sigma)) \text{ in } \Omega, \\ \Theta(\tau) - \Theta(\sigma) = 0 \text{ on } \partial\Omega. \end{cases}$$

Then, by taking $\Theta(\tau) - \Theta(\sigma)$ as a test function in the above equation we get

$$\begin{aligned} \int_\Omega |\nabla(\Theta(\tau) - \Theta(\sigma))|^2 &= \int_\Omega a\left(\Theta^{-\beta}(\tau) - \Theta^{-\beta}(\sigma)\right)(\Theta(\tau) - \Theta(\sigma)) + \int_\Omega (\Theta(\tau) - \Theta(\sigma))(\tau h(.,\Theta(\tau)) - \sigma h(.,\Theta(\sigma))) \\ &\leq \int_\Omega (\Theta(\tau) - \Theta(\sigma))(\tau h(.,\Theta(\tau)) - \sigma h(.,\Theta(\sigma))) \end{aligned} \tag{5.25}$$

where, in the last inequality, we have used that $\Theta(\sigma) \leq \Theta(\tau)$. Now, $0 \leq \Theta(\sigma) \leq \Theta(\tau) \leq \Theta(\lambda_0)$ then, by our assumptions on h,

$$0 \leq \tau h(.,\Theta(\tau)) - \sigma h(.,\Theta(\sigma)) \leq \lambda_0 h(.,\Theta(\lambda_0)) \in L^\infty(\Omega),$$

and thus, since there exists the (finite) pointwise limit $\lim_{\sigma \to \lambda_*} \Theta(\sigma)$, we have

$$\lim_{\sigma,\tau \to \lambda_*} (\Theta(\tau) - \Theta(\sigma))(\tau h(.,\Theta(\tau)) - \sigma h(.,\Theta(\sigma))) = 0 \text{ a.e. in } \Omega.$$

Also,

$$0 \leq (\Theta(\tau) - \Theta(\sigma))(\tau h(.,\Theta(\tau)) - \sigma h(.,\Theta(\sigma))) \leq \Theta(\lambda)\lambda h(.,\Theta(\lambda)) \text{ a.e. in } \Omega,$$

and, by our assumptions on h stated at the introduction and by Lemma 2.10, $\Theta(\lambda)\lambda h(.,\Theta(\lambda)) \in L^1(\Omega)$. Then, by the L:ebesgue's dominated convergence theorem,

$$\lim_{\sigma,\tau\to\lambda_*}\int_\Omega (\Theta(\tau)-\Theta(\sigma))(\tau h(.,\Theta(\tau))-\sigma h(.,\Theta(\sigma)))=0$$

Thus, by (5.25), $\lim_{\sigma,\tau\to\lambda_*}\|\Theta(\tau)-\Theta(\sigma)\|_{H_0^1(\Omega)}=0$ and so, by the Cauchy criterion, there exists $\zeta\in H_0^1(\Omega)$ such that $\lim_{\sigma\to\lambda_*^+}\Theta(\sigma)=\zeta$ with convergence in $H_0^1(\Omega)$. Since $\lim_{\sigma\to\lambda_*^+}\Theta(\sigma)=\Theta(\lambda_*)$ with pointwise convergence in Ω, we have $\zeta=\Theta(\lambda_*)$. Then

$$\Theta(\lambda_*)\in H_0^1(\Omega) \text{ and } \lim_{\sigma\to\lambda_*^+}\Theta(\sigma)=\Theta(\lambda_*) \text{ with convergence in } H_0^1(\Omega). \tag{5.26}$$

Moreover, from (5.22), (5.23) and (5.26), we have

$$\Theta(\lambda_*)\in U_\beta.$$

Let us show that, in weak sense,

$$\begin{cases} -\Delta(\Theta(\lambda_*))=a(\Theta(\lambda_*))^{-\beta}+\lambda_* h(.,\Theta(\lambda_*)) \text{ in } \Omega,\\ \Theta(\lambda_*)=0 \text{ on } \partial\Omega.\end{cases} \tag{5.27}$$

Indeed, let $\varphi\in H_0^1(\Omega)$. Since $\lim_{\sigma\to\lambda_*^+}\Theta(\sigma)=\Theta(\lambda_*)$ with convergence in $H_0^1(\Omega)$, we have

$$\lim_{\sigma\to\lambda_*^+}\int_\Omega\langle\nabla\Theta(\sigma),\nabla\varphi\rangle=\int_\Omega\langle\nabla\Theta(\lambda_*),\nabla\varphi\rangle.$$

Also,

$$\lim_{\sigma\to\lambda_*^+} a(\Theta(\lambda_*))^{-\beta}\varphi=a(\Theta(\lambda_*))^{-\beta}\varphi \ a.e. \text{ in } \Omega$$

and, since $a\in L^\infty(\Omega)$ and $\Theta(\lambda_*)\geq c'\Psi_\beta$ with c' a positive constant, we have, for $\sigma\in(\lambda_*,\lambda)$,

$$\left|a(\Theta(\lambda_*))^{-\beta}\varphi\right|\leq\left|a(\Theta(\lambda_*))^{-\beta}\varphi\right|\leq c\Psi_\beta^{-\beta}|\varphi|=c\delta_\Omega\Psi_\beta^{-\beta}\left|\frac{\varphi}{\delta_\Omega}\right| \ a.e. \text{ in } \Omega,$$

with c a positive constant independent of σ. By Remark 2.4 we have $\delta_\Omega\Psi_\beta^{-\beta}\in L^2(\Omega)$, and then, by the Hölder's and the Hardy's inequalities,

$$\int_\Omega\delta_\Omega\Psi_\beta^{-\beta}\left|\frac{\varphi}{\delta_\Omega}\right|\leq\left\|\delta_\Omega\Psi_\beta^{-\beta}\right\|_2\left\|\frac{\varphi}{\delta_\Omega}\right\|_2\leq c'\left\|\delta_\Omega\Psi_\beta^{-\beta}\right\|_2\|\varphi\|_{H_0^1(\Omega)}<\infty$$

and so $\delta_\Omega\Psi_\beta^{-\beta}\left|\frac{\varphi}{\delta_\Omega}\right|\in L^1(\Omega)$. Thus, by the Lebesgue's dominated convergence theorem, $\int_\Omega a(\Theta(\lambda_*))^{-\beta}\varphi\in L^1(\Omega)$ and

$$\lim_{\sigma\to\lambda_*^+}\int_\Omega a\Theta^{-\beta}(\sigma)\varphi=\int_\Omega a\Theta^{-\beta}(\lambda_*)\varphi. \tag{5.28}$$

Also, by the assumptions on h stated at the introduction, and since $\lim_{\sigma\to\lambda_*^+}\Theta(\sigma)=\Theta(\lambda_*)$ pointwise in Ω, we have

$$\lim_{\sigma\to\lambda_*^+}\sigma h(.,\Theta(\sigma))\varphi=\lambda_* h(.,\Theta(\lambda_*))\varphi \ a.e. \text{ in } \Omega. \tag{5.29}$$

In addition, for $\lambda_*<\sigma<\lambda_0$, we have $|\sigma h(.,\Theta(\sigma))\varphi|\leq\lambda_0 h(.,\Theta(\lambda_0))|\varphi|$. By Lemma 2.10, $\Theta(\lambda_0)\in C(\overline{\Omega})$ and then, by our assumptions on h, $\lambda_0 h(.,\Theta(\lambda_0))\in C(\overline{\Omega})$. Therefore $\lambda_0 h(.,\Theta(\lambda_0))|\varphi|\in L^1(\Omega)$ and thus, by the Lebesgue's dominated convergence theorem, $\lambda_* h(.,\Theta(\lambda_*))\varphi\in L^1(\Omega)$ and

$$\lim_{\sigma\to\lambda_*^+}\int_\Omega\sigma h(.,\Theta(\sigma))\varphi=\int_\Omega\lambda_* h(.,\Theta(\lambda_*))\varphi. \tag{5.30}$$

By (5.24) we have, for any $\sigma\in I$

$$\int_\Omega\langle\nabla\Theta(\sigma),\nabla\varphi\rangle=\int_\Omega a\Theta^{-\beta}(\sigma)\varphi+\int_\Omega\sigma h(.,\Theta(\sigma))\varphi$$

and then, from (5.28), (5.29), and (5.30), by taking the limit as $\sigma \to \lambda_*^+$ we get

$$\int_\Omega \langle \nabla\Theta(\lambda_*), \nabla\varphi\rangle = \int_\Omega a\Theta^{-\beta}(\lambda_*)\varphi + \int_\Omega \lambda_* h(.,\Theta(\lambda_*))\varphi.$$

Thus $\Theta(\lambda_*)$ is a weak solution of (5.27).
Now we show that $\lim_{\sigma\to\lambda_*^+}\Theta(\sigma) = \Theta(\lambda_*)$ with convergence in Y_β. To do it, it is enough to see that

$$\sup_{\sigma\in(\lambda_*,\lambda)} \left\|\Theta'(\sigma)\right\|_{Y_\beta} < \infty. \tag{5.31}$$

Indeed, if (5.31) holds, then, for $\lambda_* < \sigma < \tau < \lambda$,

$$\|\Theta(\tau)-\Theta(\sigma)\|_{Y_\beta} = \left\|\int_\sigma^\tau \Theta'(s)\,ds\right\|_{Y_\beta} \le \int_\sigma^\tau \left\|\Theta'(s)\right\|_{Y_\beta} ds \le |\tau-\sigma| \sup_{\sigma\in(\lambda_*,\lambda)}\left\|\Theta'(\sigma)\right\|_{Y_\beta},$$

and so by (5.31) and the Cauchy's criterion, there exists $\xi \in Y_\beta$ such that $\lim_{\sigma\to\lambda_*^+}\Theta(\sigma) = \xi$ with convergence in Y_β, and since $\Theta(\sigma)$ converges pointwise to $\Theta(\lambda_*)$ we have $\xi = \Theta(\lambda_*)$ and so $\lim_{\sigma\to\lambda_*^+}\Theta(\sigma) = \Theta(\lambda_*)$ with convergence in Y_β.
To prove (5.31) observe that, for $\sigma \in (\lambda_*,\lambda)$, $\Theta'(\sigma)$ satisfies, in weak sense,

$$\begin{cases} -\Delta\Theta'(\sigma) + \beta a\Theta^{-\beta-1}(\sigma)\Theta'(\sigma) - \sigma\frac{\partial h}{\partial s}(.,\Theta(\sigma))\Theta'(\sigma) = h(.,\Theta(\sigma)) \text{ in } \Omega, \\ \Theta'(\sigma) = 0 \text{ on } \partial\Omega. \end{cases} \tag{5.32}$$

Since $\Theta(\sigma) \le \Theta(\lambda_0)$, $\sigma\frac{\partial h}{\partial s}(.,\Theta(\sigma)) \le \lambda_0\frac{\partial h}{\partial s}(.,\Theta(\lambda_0))$, and $h(.,\Theta(\sigma)) \le h(.,\Theta(\lambda_0))$, (5.32) gives that, in weak sense,

$$\begin{cases} -\Delta\Theta'(\sigma) + \beta a\Theta^{-\beta-1}(\lambda_0)\Theta'(\sigma) - \lambda_0\frac{\partial h}{\partial s}(.,\Theta(\lambda_0))\Theta'(\sigma) \le h(.,\Theta(\lambda_0)) \text{ in } \Omega, \\ \Theta'(\sigma) = 0 \text{ on } \partial\Omega. \end{cases} \tag{5.33}$$

Also,

$$\begin{cases} -\Delta\Theta'(\lambda_0) + \beta a\Theta^{-\beta-1}(\lambda_0)\Theta'(\lambda_0) - \lambda_0\frac{\partial h}{\partial s}(.,\Theta(\lambda_0))\Theta'(\lambda_0) = h(.,\Theta(\lambda_0)) \text{ in } \Omega, \\ \Theta'(\lambda_0) = 0 \text{ on } \partial\Omega, \end{cases} \tag{5.34}$$

and so, since $N^{\lambda_0,\Theta(\lambda_0)} = \lambda_0 h(.,\Theta(\lambda_0))$ and $\rho_{N^{\lambda_0,\Theta(\lambda_0)},\Theta(\lambda_0)} > 1$, from (5.33), (5.34) and the maximum principle of ([37] Lemma 4.4 ii)) it follows that $\Theta'(\sigma) \le \Theta'(\lambda_0)$ $a.e$ in Ω. We have also $\Theta'(\sigma) \ge 0$ $a.e$ in Ω (because $\sigma \to \Theta(\sigma)$ is nondecreasing). Therefore

$$\left\|\Psi_\beta^{-1}\Theta'(\sigma)\right\|_\infty \le c \text{ for any } \sigma \in (\lambda_*,\lambda_0) \tag{5.35}$$

where $c := \left\|\Psi_\beta^{-1}\Theta'(\lambda_0)\right\|_\infty$ (note that c is finite because $\Theta'(\lambda_0) \in Y_\beta$). In particular, (5.35) gives that for some constant $c' > 0$,

$$\left\|\Theta'(\sigma)\right\|_\infty \le c' \text{ for any } \sigma \in (\lambda_*,\lambda_0). \tag{5.36}$$

From (5.32) we have also

$$\begin{aligned} \int_\Omega \left|\nabla\Theta'(\sigma)\right|^2 &= -\int_\Omega \beta a\Theta^{-\beta-1}(\sigma)\left(\Theta'(\sigma)\right)^2 + \int_\Omega \sigma\frac{\partial h}{\partial s}(.,\Theta(\sigma))\left(\Theta'(\sigma)\right)^2 + \int_\Omega h(.,\Theta(\sigma))\Theta'(\sigma) \\ &\le \int_\Omega \lambda_0\frac{\partial h}{\partial s}(.,\Theta(\lambda_0))\left(\Theta'(\sigma)\right)^2 + \int_\Omega h(.,\Theta(\lambda_0))\Theta'(\sigma) \\ &\le M\int_\Omega \left(\Theta'(\sigma)\right)^2 + M\int_\Omega \Theta'(\sigma) \end{aligned}$$

where $M; = \left\|\lambda_0\frac{\partial h}{\partial s}(.,\Theta(\lambda_0))\right\|_\infty + \|h(.,\Theta(\lambda_0))\|_\infty$. Then, taking into account (5.36), we conclude that for some constant $c'' > 0$,

$$\left\|\Theta'(\sigma)\right\|_{H_0^1(\Omega)} \le c'' \text{ for any } \sigma \in (\lambda_*,\lambda_0).$$

which jointly with (5.35) gives (5.31).
Now we show that $\lambda_* = 0$. We proceed by the way of contradiction. Suppose $\lambda^* > 0$. Then, since $r_{\lambda_0,\Theta(\lambda_0)} > 1$ and since, by 5'), $\sigma \to r_{\sigma,\Theta(\sigma)}$ is nonincreasing on I, we have $r_{\lambda_*,\Theta(\lambda_*)} > 1$. Thus, by Lemma 5.4 there exists $\varepsilon > 0$ such that Θ has an extension (still denoted by Θ) to $I_\varepsilon := (\lambda_* - \varepsilon,\ \lambda^*)$ such that $(I_\varepsilon,\Theta) \in \mathcal{G}$, which contradicts the definition of λ_*.
Then $\lambda_* = 0$, and so, by (5.27), $\Theta(\lambda_*) = u_0$ (where u_0 is the unique solution of (1.1) for $\lambda = 0$).

□

Proof of Theorem 1.6. We proceed by the way of contradiction. Let Σ be as given by Remark 1.5. Let $\lambda_0 \in (0,\Sigma)$ and let u_{λ_0} and v_{λ_0} be the minimal weak solution and the maximal weak solution, respectively, of problem (1.1) corresponding to $\lambda = \lambda_0$. Then $u_{\lambda_0} \neq v_{\lambda_0}$. Suppose, by contradiction, that there exists a weak solution w of (1.1, corresponding to $\lambda = \lambda_0$ such that $u_\lambda \neq w \neq v_\lambda$. By Lemma 5.5, applied with $w_0 = w$, there exists a function $\Theta \in C^1\left((0,\lambda_0+\varepsilon), U_\beta\right) \cap C\left([0,\lambda_0+\varepsilon), U_\beta\right)$ such that $\Theta(\lambda_0) = w$, $\Theta(0) = u_0$ (where u_0 is the unique weak solution of (1.1) corresponding to $\lambda = 0$), and such that $r_{\sigma,\Theta(\sigma)} > 1$ for any $\sigma \in (0,\lambda_0)$, and with Θ satisfying, in weak sense and for any $\sigma \in (0,\lambda_0)$,

$$\begin{cases} -\Delta(\Theta(\sigma)) = a\Theta^{-\beta}(\sigma) + \sigma h(.,\Theta(\sigma)) \text{ in } \Omega, \\ \Theta(\sigma) = 0 \text{ on } \partial\Omega, \\ \Theta(\sigma) > 0 \text{ in } \partial\Omega. \end{cases}$$

Again by Lemma 5.5, but applied now with $w_0 = u_{\lambda_0}$, we have that, for some $\varepsilon' > 0$, there exists a function $\Phi \in C^1\left((0,\lambda_0+\varepsilon'), U_\beta\right) \cap C\left([0,\lambda_0+\varepsilon'), U_\beta\right)$ such that $\Phi(\lambda_0) = u_{\lambda_0}$,
$\Phi(0) = u_0$ where, as above, u_0 is the weak solution of (1.1) for $\lambda = 0$, and such that $r_{\sigma,\Phi(\sigma)} > 1$ for any $\sigma \in (0,\lambda_0)$, and satisfying, in weak sense and for any $\sigma \in (0,\lambda)$,

$$\begin{cases} -\Delta(\Phi(\sigma)) = a\Phi^{-\beta}(\sigma) + \sigma h(.,\Phi(\sigma)) \text{ in } \Omega, \\ \Phi(\sigma) = 0 \text{ on } \partial\Omega, \\ \Phi(\sigma) > 0 \text{ in } \partial\Omega. \end{cases}$$

Observe that, since $w \neq u_{\lambda_0}$, then

$$\Theta(\sigma) \neq \Phi(\sigma) \text{ for any } \sigma \in (0,\lambda_0). \tag{5.37}$$

Indeed, let

$$\lambda_{**} := \sup\{\eta \in [0,\lambda_0] : \Theta(\eta) = \Phi(\eta)\}. \tag{5.38}$$

We claim that $\lambda_{**} = 0$. In fact, since $\Theta(\lambda_0) = w \neq u_{\lambda_0} = \Phi(\lambda_0)$, and since Θ and Φ are continuous at λ_0 we have, necessarily, $\lambda_{**} < \lambda_0$. If $\lambda_{**} > 0$ then $\lambda_{**} \in (0,\lambda_0)$ and so $r_{\lambda_{**},\Theta(\lambda_{**})} > 1$. Thus Lemma 5.4 can be applied taking there $\lambda = \lambda_{**}$ and $u = \Theta(\lambda_{**})$ to obtain a number $\varepsilon > 0$ and an open neighborhood V of $\Theta(\lambda_{**})$ in Y_β such that for any $\sigma \in (\lambda_{**}-\varepsilon, \lambda_{**}+\varepsilon)$ there exists a unique $\xi(\sigma) \in V$ such that $S(\sigma,\xi(\sigma)) = 0$. By diminishing ε if necessary, we can assume that $(\lambda_{**}-\varepsilon, \lambda_{**}+\varepsilon) \subset (0,\lambda_0)$. From the continuity of Θ and Φ at λ_{**} and from (5.38), we have that $\Theta(\lambda_{**}) = \Phi(\lambda_{**}) \in V$ and so, δ positive and small enough, we have that if $\lambda_{**} < \sigma < \lambda_{**} + \delta$ then $\Theta(\sigma) \neq \Phi(\sigma)$ and also $S(\sigma,\Theta(\sigma)) = S(\sigma,\Phi(\sigma)) = 0$, which contradicts the uniqueness assertion of Lemma 5.4. Thus $\lambda_{**} = 0$ and so (5.37) holds.
Now, for $\sigma \in (0,\lambda_0)$,

$$\begin{cases} -\Delta(\Theta(\sigma)-\Phi(\sigma)) = a\left((\Theta(\sigma))^{-\beta} - (\Phi(\sigma))^{-\beta}\right) + \sigma(h(.,\Theta(\sigma)) - h(.,\Phi(\sigma))) \text{ in } \Omega, \\ \Theta(\sigma)-\Phi(\sigma) = 0 \text{ on } \partial\Omega, \end{cases} \tag{5.39}$$

and, by the mean value theorem, $\sigma(h(.,\Theta(\sigma)) - h(.,\Phi(\sigma))) = \sigma\frac{\partial h}{\partial s}(.,\eta_\sigma)(\Theta(\sigma)-\Phi(\sigma))$ for some function η_σ such that, for $x \in \Omega$, $\eta_\sigma(x)$ belongs to the open segment with endpoints $\Phi(\sigma)(x)$ and $\Theta(\sigma)(x)$.
Since $0 \leq \Theta(\sigma) \leq \Theta(\lambda_0)$ and $0 \leq \Phi(\sigma) \leq \Phi(\lambda_0)$, and since $\Theta(\lambda_0)$ and $\Phi(\lambda_0)$ belong to $L^\infty(\Omega)$ (because they belong to Y_β) then there exists a positive constant M_1 such that $0 \leq \eta_\sigma \leq M_1$ for any $\sigma \in (0,\lambda_0)$. Then, from our assumptions on h, it follows that there exists a constant M such that $\left|\frac{\partial h}{\partial s}(.,\eta_\sigma)\right| \leq M$ for any $\sigma \in (0,\lambda_0)$. Then, for such σ,
Now we take the test function $\varphi = \Theta(\sigma) - \Phi(\sigma)$ in (5.39) to obtain

$$\begin{aligned} \|\Theta(\sigma)-\Phi(\sigma)\|^2_{H_0^1(\Omega)} &= \int_\Omega \left((\Theta(\sigma))^{-\beta} - (\Phi(\sigma))^{-\beta}\right)(\Theta(\sigma)-\Phi(\sigma)) + \int_\Omega \sigma(h(.,\Theta(\sigma)) - h(.,\Phi(\sigma)))(\Theta(\sigma)-\Phi(\sigma)) \\ &\leq \int_\Omega \sigma|h(.,\Theta(\sigma)) - h(.,\Phi(\sigma))||\Theta(\sigma)-\Phi(\sigma)| \\ &\leq \sigma M \int_\Omega (\Theta(\sigma)-\Phi(\sigma))^2 \\ &\leq \sigma M c_P^2 \|\Theta(\sigma)-\Phi(\sigma)\|^2_{H_0^1(\Omega)} \end{aligned}$$

where c_P is the constant of the Poincaré's inequality in Ω, and where in the first inequality we used that $s \to as^{-\beta}$ is nonincreasing and, in the second one, the Poincaré's inequality was used. Then, since $\Theta(\sigma) \neq \Phi(\sigma)$ for any $\sigma \in (0,\lambda_0)$ we conclude that $1 \leq \sigma M c_P^2$ which, by taking $\lim_{\sigma\to 0^+}$, gives a contradiction that completes the proof of the theorem. □

Remark 5.6. *An inspection of the proof given for Theorem 1.6 shows that, if w is a weak solution of (1.1) then, in order to construct the function Θ (and to prove its properties), the assumption $w \neq v_\lambda$ was used only to guarantee that $\rho_{N^{\lambda,w},w} > 1$. From this fact one gets that if for some $\lambda \in (0,\Sigma)$, $r_{\lambda,v_\lambda} > 1$ then, proceeding as in the proof of Theorem 1.6, a contradiction is reached. Therefore necessarily $r_{\lambda,v_\lambda} \leq 1$ for any $\lambda \in (0,\Sigma)$.*

6. Conclusion

For a C^2 and bounded domain Ω in $\mathbb{R}^n$, $n \geq 2$, we considered the problem

$$-\Delta u = au^{-\beta} + \lambda h(.,u) \text{ in } \Omega, u = 0 \text{ on } \partial\Omega, u > 0 \text{ in } \Omega, \tag{6.1}$$

where λ is a nonnegative parameter and the solutions are understood in weak sense. Under the assumptions H1)-H6) stated at the introduction our main result can be readed as follows. If for some $\lambda \geq 0$ the above problem has at least two weak solutions, then it has exactly two weak solutions (which belong to $H_0^1(\Omega) \cap C^1(\Omega) \cap C(\overline{\Omega})$), namely, a minimal solution u_λ and a maximal solution v_λ, such that $u_\lambda \neq v_\lambda$ and $u_\lambda \leq v_\lambda$ in Ω. This fact, combined with known previous results leads to the following statement: There exists $\Sigma > 0$ such that:
For $\lambda = 0$ and $\lambda = \Sigma$ there exists exactly one weak solution,
For $0 < \lambda < \Sigma$ there exists exactly two weak solutions in $H_0^1(\Omega)$,
For $\lambda > \Sigma$ no weak solutions exist.
Let us stress that although there are many results concerning existence and multiplicity for solutions of singular elliptic problems, exact multiplicity results are far less abundant in the literature .
Our result complements known multiplicity results for these kind of singular problems. As an example, it applies, for instance, when $n \geq 2$, $a \in C(\overline{\Omega})$ is strictly positive in $\overline{\Omega}$ and $h(x,s) = \sum_{j=1}^{m} b_j(x) s^{p_j}$ with $b_j \in C(\overline{\Omega})$, such that $b_j > 0$ in $\overline{\Omega}$, and $1 < p_1 < p_2 < \dots < p_m < \frac{n+2}{n-2}$ (with the convention that $\frac{n+2}{n-2} = \infty$ if $n = 2$).
Some possible future directions of research include:
i) Study problem (6.1) in cases where the coefficient a of the singular term of the equation is singular at $\partial(\Omega)$ in order to obtain, again in some of these situations, exact multiplicity results.
ii) For $\beta > 0$ arbitrary search for exact multiplicity results for solutions $u \in C^2(\Omega) \cap C(\overline{\Omega})$ of problem (6.1).
iii) Investigate the situation when, under suitable assumptions, the Laplacian is replaced by the $q-$Laplacian in (6.1) for some $1 < q < \infty$.
Other interesting questions remain. For instance:
By ([36], Theorem 1.2) $\lambda \to u_\lambda$ is nondecreasing on $[0,\Sigma]$, and by ([35], Theorem 1.2), $\lambda = 0$ is a bifurcation point from ∞ for problem (6.1). Then, since for $\lambda \in (0,\lambda)$ u_λ and v_λ are the unique solutions of (6.1), one could suspect that $\lim_{\lambda \to 0^+} \|v_\lambda\|_{C(\overline{\Omega})} = \infty$, and that the map $\lambda \to v_\lambda$ is non increasing on $(0,\Sigma]$. It would interestig to prove these fact (if true). It would be also interesting to investigate the regularity properties (if any) of the mapping $\lambda \to v_\lambda$.

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