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ALGEBRAIC LIE ALGEBRA BUNDLES AND DERIVATIONS OF LIE ALGEBRA BUNDLES

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Abstract. In this paper, we define algebraic Lie algebra bundles, discuss some results on algebraic Lie algebra bundles and derivations of Lie algebra bundles. Some results involving inner derivations and central derivations of Lie algebra bundles are obtained.

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1. Introduction

A Lie algebra is called algebraic if it is the Lie algebra of an algebraic group [4]. C. Chevalley in [3] gave an alternative definition of algebraic Lie algebras based on the replica of a matrix. In the second section, an algebraic Lie algebra bundle is defined and a few basic properties of algebraic Lie algebra bundles are discussed.

A derivation of a Lie algebra plays an important role in the structure of a Lie algebra. G. Hochschild [6] defined derivation algebras of a Lie algebra. B. S. Kiranagi, R. Kumar, K. Ajaykumar and B. Madhu have defined and studied derivation algebra bundle of a Lie algebra bundle in [7]. In the third section, characteristically solvable Lie algebra bundles are characterized and the relations between inner and central derivations are discussed.

Here we assume that all the base spaces are compact Hausdorff and the underlying field is real or complex.

2. Algebraic Lie algebra bundles

We recall a few relevant definitions to define our algebraic Lie algebra bundles. If k is any integer and E_k is the unit matrix of degree k then the Kronecker sum is given by $X \oplus Y = X \otimes E_n + E_m \otimes Y$ where X, Y are matrices of degree m and n respectively. Further, for any matrix X, $X_{rs} = \underbrace{X^* \oplus X^* \oplus X^* \oplus \cdots \oplus X^*}_{\text{r times}} \oplus \underbrace{X \oplus X \oplus \cdots \oplus X}_{\text{s times}}$.

It is a matrix of degree m^{r+s}. A vector of type (r, s) is a column of m^{r+s} elements from the field. A vector $\vec{e}$ of type (r, s) is called an invariant of X if $X_{rs}\vec{e} = 0$.
C. Chevalley in [2] introduced the notion of a replica of a matrix. A matrix Y is said to be a replica of a matrix X if every invariant of X is also an invariant of Y.

A **Lie algebra bundle** is a vector bundle $\xi = (E, p, B)$ in which each fibre ξ_x is a Lie algebra and for each x in B, there is an open neighbourhood U of x, a Lie algebra L and a homeomorphism $\phi : U \times L \to p^{-1}(U)$ such that for each y in U, $\phi_y : L \to p^{-1}(y)$ is a Lie algebra isomorphism.

Definition 2.1. Let $\xi = (E, p, B)$ be a Lie algebra bundle with local trivialization $\phi : U \times L \to p^{-1}(U)$. For any endomorphism A of ξ, we define $A' \in End(\xi)$ as the **replica** of A, if any invariant of $A|\xi_x$ is also an invariant of $A'|\xi_x$. The set of all replicas of A is denoted by $\{A\}$.

A Lie algebra of matrices is said to be *algebraic* [3] if replicas of every matrix is in the same Lie algebra. Morikuni Gôtô in [5] refers to this algebraicity of Lie algebra of matrices as $l - algebraicity$.

Definition 2.2. Any subbundle of $End(\xi)$ is said to be an **l-algebraic bundle** if each of its fibre is $l - algebraic$.

The smallest l-algebraic Lie algebra which contains a subalgebra L of $gl(n, K)$ is called the algebraic hull of L and is denoted by L^*[5]. It is the smallest l-algebraic Lie algebra containing L.

Definition 2.3. Let $\xi = (E, p, B)$ be a Lie algebra bundle and $End\xi$ be the Lie algebra bundle of endomorphisms on ξ with the local trivialization $\phi : U \times EndL \to (Endp)^{-1}(U)$ such that $\phi_x : \{x\} \times EndL \to (Endp)^{-1}(x)$ is an isomorphism. By this isomorphism, for a subbundle η of $End(\xi)$ there exists a smallest algebraic Lie algebra η_x^* containing η_x for all $x \in B$. Then $\eta^* = \bigcup_{x \in B} \eta_x^*$ is a Lie algebra bundle and is called the algebraic hull of η.

Lemma 2.4. *Let $\xi = (E, p, B)$ be a Lie algebra bundle. Then any $A \in End(\xi)$ can be uniquely expressed as $A = A^0 + A^s$ where A^0 is nilpotent and A^s on ξ_x is a matrix with simple elementary divisors.*

Proof. Let $\phi : U \times EndL \to (Endp)^{-1}(U)$ be a local trivialization of $End\xi$. Then from the isomorphism $\phi_x : \{x\} \times EndL \to (Endp)^{-1}(x)$ for any $A \in End\xi$ corresponding to eigenvalues $\lambda_1, \lambda_2, \ldots$ of A in ξ_x, the Lie algebra ξ_x can be written as a direct sum of eigenspaces, $\xi_x = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots$. Let A^s be such that $A^s|\xi_x : \xi_x \to \xi_x$

is $A^s y = \lambda_i y$ for $y \in E_{\lambda_i}$. Then A^s is well defined endomorphism on ξ. Further, A^s on ξ_x is a matrix with simple elementary divisors, $(X - \lambda_i)$. Put $A^0 = A - A^s$ on each ξ_x. Then A^0 is nilpotent. Also, A^0 and A^s commute and hence this representation is unique. □

Lemma 2.5. *In the representation of Lemma 2.4, A^0 and A^s are replicas of A.*

Proof. Let $A = A^0 + A^s$. Then by the methods of [8] , $(A_{rs})_x = (A^0_{rs})_x + (A^s_{rs})_x$. For each x, $(A^0_{rs})_x$ is nilpotent and $(A^s_{rs})_x$ is a matrix with simple elementary divisors. If V is the vector space on which $A|\xi_x$ operates. Let $(I_{rs})_x$ denote the Kronecker product $\underbrace{V^* \otimes V^* \otimes \ldots \otimes V^*}_{\text{r times}} \otimes \underbrace{V \otimes V \otimes \ldots \otimes V}_{\text{s times}}$ and $(I_k)_x$ the eigenspace of $(I_{rs})_x$ corresponding to a eigenvalue λ_k of $(A_{rs})_x$. For any $y \in (I_k)_x$, $(A_{rs})_x y = \lambda_k y$ and by Lemma 2.4 $(A^s_{rs})_x y = \lambda_k y$. Let $z \in (I_{rs})_x$ and $(A_{rs})_x z = 0$. Then $z \in (I_0)_x$ and hence $(A^s_{rs})_x z = 0$. Therefore, $(A^s_{rs})_x$ is a replica of $(A_{rs})_x$. Now, $(A^0_{rs})_x z = ((A_{rs})_x - (A^s_{rs})_x) z = 0$. This shows that $(A^0_{rs})_x$ is a replica of $(A_{rs})_x$. □

Lemma 2.6. *Any $A \in End(\xi)$ can be decomposed as $A_x = (A_0)_x + \lambda_1 (A_1)_x + \lambda_2 (A_2)_x + \cdots + \lambda_l (A_l)_x$ on each fibre ξ_x of $End(\xi)$.*

Proof. $A|\xi_x$ is an endomorphism on ξ_x. Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the distinct eigenvalues of $A|\xi_x$. Of these let l be the maximal number of λ_i's which are linearly independent over the prime field P of complex numbers, say, $\lambda_1, \lambda_2, \ldots, \lambda_l$. Then for each $i = 1, 2, \ldots, k$, $\lambda_i = \sum_{j=1}^{l} r_{ij} \lambda_j$, $r_{ij} \in P$. If V is the vector space on which $A|\xi_x$ operates, then $V = \bigoplus_{i=1}^{k} V_{\lambda_i}$. Let E_i be the projection of V on V_{λ_i}. Then for $y \in V$, $y = y_1 + y_2 + \cdots + y_k$ and $E_i y = y_i$. From Lemma 2.4, $(A')_x y_i = \lambda_i y_i$. Hence, we get $(A')_x y = (A')_x (\sum_{i=1}^{k} y_i) = \sum_{i=1}^{k} (A')_x y_i = \sum_{i=1}^{k} \lambda_i y_i = \sum_{i=1}^{k} (\lambda_i E_i) y$ from which it follows that $(A')_x = \sum_{i=1}^{k} \lambda_i E_i$. Take $(A_j)_x = \sum_{i=1}^{k} r_{ij} E_i$, $j = 1, 2, \ldots, l$. Then $(A)_x = (A_0)_x + (A')_x = (A_0)_x + \sum_{i=1}^{k} \lambda_i E_i = (A_0)_x + \sum_{i=1}^{k} \sum_{j=1}^{l} r_{ij} \lambda_j E_i = (A_0)_x + \sum_{j=1}^{l} \lambda_j (A_j)_x$. This proves the lemma. □

Theorem 2.7. *If ξ is a Lie algebra bundle then the derivation algebra bundle of ξ, $D(\xi)$ is l-algebraic.*

Proof. For any $D \in D(\xi)$ we show that $\{D\} \subset D(\xi)$. From the derivation $D|\xi_x = D_x$ we have the decomposition of ξ_x as $\xi_x = E_\alpha \oplus E_\beta \oplus \ldots$ where $\alpha, \beta, \ldots$ are the eigenvalues of $D_x, x \in B$. Then for $y \in E_\alpha$ and $z \in E_\beta$, $D_x y = \alpha y$ and $D_x z = \beta z$. $D_x[y, z] = [D_x y, z] + [y, D_x z] = [\alpha y, z] + [y, \beta z] = (\alpha + \beta)[y, z]$ so that $[y, z] \in E_{(\alpha+\beta)}$ if $\alpha + \beta$ is an eigenvalue of D_x. Therefore, $[E_\alpha, E_\beta] \subseteq E_{(\alpha+\beta)}$. From [5] D^s is a derivation of ξ. Therefore, we consider the case when D is an s-matrix on ξ_x.

Then for a suitable basis $x_1, x_2, \ldots, x_r$ of ξ_x we have $D_x x_i = a_i x_i, i = 1, 2, \ldots, r$. Let the structure constants of ξ_x be $[x_i, x_j] = \sum_h c_{ijh} x_h$. Then $D \in D(\xi)$ implies $D[x_i, x_j] = D \sum_h c_{ijh} x_h = (a_i + a_j)[x_i, x_j]$. From the equality of sums on both sides, $(a_i + a_j)c_{ijh} = a_h c_{ijh}, \forall\ i,\ j,\ h = 1, 2, \ldots, r$ or $(a_i + a_j - a_h)c_{ijh} = 0\ \forall\ i,\ j,\ h = 1, 2, \ldots, r$. Let $D = D^0 + D^s$ be a canonical decomposition of D such that $D_x = D_x^0 + \lambda_1 (D_x)_1 + \lambda_2 (D_x)_2 + \ldots + \lambda_k (D_x)_k$. Then each $(D_x)_l$ may be defined as $(D_x)_l x_i = r_i^l x_i$ where $a_i = \sum_l \lambda_l r_i^l$. Now $(a_i + a_j - a_h)c_{ijh} = 0$ is a trivial relation if $c_{ijh} = 0$. If $c_{ijh} \neq 0$, $(a_i + a_j - a_h) = 0$ which gives $\sum_l (r_i^l + r_j^l - r_h^l)\lambda_l = 0$. Since λ_l's are linearly independent $r_i^l + r_j^l - r_h^l = 0$. Thus when $a_i = r_i^l$ the condition $(a_i + a_j - a_h)c_{ijh} = 0$ is satisfied. This gives $D_l \in D(\xi)$ so that $\{D\} \subseteq D(\xi)$. Therefore, $D(\xi)$ is l-algebraic. □

Proposition 2.8. *Let η be a Lie subbundle of $End(\xi)$ and η_1 be a subbundle of η. Then $I(\eta_1)^* = I(\eta_1^*)$.*

Proof. Let $\phi_x : \{x\} \times ad(L_1)^* \to I_x(\eta_1)^*$ be the local trivialization of $I(\eta_1)^*$ at $x \in B$. By [10] Proposition 4, $ad(L_1)^*$ is spanned by the replicas, $\{ady\}$ with y in L_1. For any element A of $End(L)$ and a subalgebra H of $End(L)$ such that $[A, H] \subset H$, $\{adA\} = ad\{A\}$. Therefore, $ad(L_1)^* = span\{\{ady\} : y \in L_1\} = ad\{span\{y\} : y \in L_1\} = ad(L_1^*)$. Then $\psi_x : \{x\} \times ad(L_1^*) \to I_x(\eta_1^*)$ is the local trivialization of $I(\eta_1^*)$ and hence $I(\eta_1)^* = I(\eta_1^*)$. □

Remark 2.9. For a Lie subalgebra L of $gl(V)$, $R(L^*) = R(L)^*$[5].

For a subbundle η of $End(\xi)$, $R(\eta) = \bigcup_{x \in B} R(\eta_x)$ and hence $R(\eta^*) = R(\eta)^*$.

Proposition 2.10. *Let η^* be the algebraic hull of a subbundle η of $End(\xi)$. Then*

(1) *Every ideal bundle of η is also an ideal bundle in η^*.*
(2) *Center of η is contained in the center of η^*.*
(3) *η and η^* have the same derived algebra.*
(4) *If η is an ideal bundle in a subbundle η_1 of $End(\xi)$ then $[\eta_1^*, \eta^*] \subset \eta$.*

Proof. Let $\phi_x : \{x\} \times \eta_x^* \to p^{-1}(x)$ be the local trivialization of η^* at $x \in B$. Then by [3] every ideal in η_x is an ideal in η_x^*. For any ideal bundle $\eta^{'}$ of η, $\eta^{'}$ is an ideal bundle of η^* by the local trivialization $\phi_x : p^{-1}(x) \to x \times \eta_x^{'}$ which proves (1). (2) follows from $Z(\eta) = \bigcup_{x \in B} Z(\eta_x) \subset \bigcup_{x \in B} Z(\eta_x^*) = Z(\eta^*)$. For any k, $\eta^k = \bigcup_{x \in B} \eta_x^k = \bigcup_{x \in B} (\eta_x^*)^k = (\eta^*)^k$ which proves (3). $[\eta_1^*, \eta^*] = \bigcup_{x \in B} [(\eta_1^*)_x, (\eta^*)_x] \subset \bigcup_{x \in B} \eta_x = \eta$ from which (4) holds. □

Corollary 2.11. *If η^* is the algebraic hull of a subbundle η of $End(\xi)$, then η is an ideal in η^* and η^*/η is an abelian subbundle. If η is solvable, then η^* is solvable.*

Proof. By Proposition 2.10, η is an ideal bundle in η^*. From Proposition 2.10(4), $[\eta^*, \eta^*] \subset \eta$ implies $\forall x \in B, [\eta^*_x, \eta^*_x] \subset \eta_x$. For all $k, k' \in \eta^*_x, [k + \eta_x, k' + \eta_x] = [k.k'] + \eta_x = 0 + \eta_x$ so that η^*_x/η_x is abelian for all $x \in B$. Hence η^*/η is an abelian Lie subbundle. η is solvable and hence η_x is solvable for every $x \in B$. η^*_x/η_x is abelian as a Lie algebra and hence η^*_x is solvable. Therefore, η^* is solvable. □

Proposition 2.12. *Let η be an algebraic Lie subbundle of $End(\xi)$. If $R(\eta)$ is the radical of η and $\eta = \mathfrak{S} + R(\eta)$, then $R(\eta)$ is algebraic.*

Proof. Let $R(\eta)^*$ be the algebraic hull of $R(\eta)$. Since η is algebraic, $R(\eta)^* \subset \eta$ and $[\eta, R(\eta)^*] = \bigcup_{x\in B}[\eta_x, R(\eta)^*_x] \subset R(\eta) \subset R(\eta)^*$. Therefore, it follows that $R(\eta)^*$ is an ideal subbundle of η. By Proposition 2.10, $[R(\eta)^*, R(\eta)^*] \subset R(\eta)$ so that by Corollary 2.11 $R(\eta)^*/R(\eta)$ is an abelian Lie subbundle of η and hence $R(\eta)^*$ is solvable. Therefore, $R(\eta)^* = R(\eta)$ which shows that $R(\eta)$ is algebraic. □

3. Derivations of Lie algebra bundles

Let ξ be a Lie algebra bundle. A vector bundle morphism $D : \xi \to \xi$ is called a **derivation** if $D([u, v]) = [u, D(v)] + [D(u), v]$ for all $u, v \in \xi_x$.

A derivation D is called **inner** if there exists a section s of ξ such that for all u in ξ_x and x in B, $D(u) = [u, s(x)]$[7].

A derivation D is **central** if for each x in B, $D\xi_x \subseteq Z(\xi_x)$. The set of all derivations of ξ form a Lie algebra bundle denoted by $D(\xi)$. Then the set of all inner derivations of ξ denoted by $I(\xi)$ and the set of all central derivations denoted by $C(\xi)$ form subbundles of $D(\xi)$.

Let $\xi = (E, p, B)$ be a Lie algebra bundle, $\phi : U \times L \to \bigcup\limits_{x\in B} \xi_x$ be a local triviality of ξ where L is a Lie algebra, let R be the radical of L, ${\xi_x}^r$ be the radical of ξ_x. Then $\phi : U \times R \to {\xi_x}^r$ is an isomorphism. We call the bundle as **radical bundle** of ξ.

Let $\xi = (E, p, B)$ be a Lie algebra bundle and ξ be the direct sum of the ideal bundles $\xi_1, \xi_2, \dots, \xi_n$. Let p_i be the projection morphism of ξ onto ξ_i. Identify an element ϕ_{ij} of $End(\xi_i, \xi_j)$ with an element $\phi_{ij}p_i$ of $End(\xi)$. Then $End(\xi_i, \xi_j) \subset End(\xi)$. Let $D(\xi_i, \xi_j) = D(\xi) \cap End(\xi_i, \xi_j)$. So that $D(\xi_i, \xi_i) = D(\xi_i)$.

Theorem 3.1. *Let ξ be the direct sum of the ideal bundles ξ_1, ξ_2,…, ξ_n. Then*

(1) $D(\xi) = \sum_{i,j=1}^{n} D(\xi_i, \xi_j)$.
(2) *For $i \neq j, D(\xi_i, \xi_j)$ consists of $\phi_{ij} \in End(\xi_i, \xi_j)$ such that $\phi_{ij}\xi_i \subset Z(\xi_j)$ and $\phi_{ij}[\xi_i, \xi_j] = 0$.*
(3) *For $i \neq j, D(\xi_i, \xi_j)$ is abelian.*

Proof. We shall first prove (2). Let $D \in D(\xi_i, \xi_j)$. Then $D \in End(\xi_i, \xi_j)$. For $b \in B$ and x_i, x_j in ξ_i and ξ_j respectively, $D[x_i, x_j] = 0$ and hence $D(\xi_i)_b \subset Z(\xi_j)_b$. It follows that $D(\xi_i) \subset Z(\xi_j)$. Also, $D[\xi_i, \xi_i] = 0$. Conversely, let ϕ_{ij} be an element of $End(\xi_i, \xi_j)$ which satisfies the conditions in (2). Then ϕ_{ij} is identified with $\phi_{ij} p_i$ of $End(\xi)$ so that $\phi_{ij}[x_k, x_l] = 0 = [\phi_{ij}(x_k), x_l] + [x_k, \phi_{ij}(x_l)]$ for all x_k, x_l in $\xi_x, x \in B$. This shows that ϕ_{ij} is a derivation of ξ. This proves (1).

Let $D \in D(\xi)$. Set $\phi_{ij} = p_j D p_i$ where p_i, p_j are projections on ξ_i, ξ_j respectively. Then $\phi_{ij} : \xi_i \to \xi_j$ is a Lie algebra homomorphism and $D = \sum_{i,j=1}^{n} \phi_{ij}$.

For $i \neq j$, ϕ_{ij} satisfies conditions of (2). Therefore, $\phi_{ij} \in D(\xi_i, \xi_j)$. From this it follows that $D \in \sum_{i,j=1}^{n} D(\xi_i, \xi_j)$ and hence $D(\xi) \subset \sum_{i,j=1}^{n} D(\xi_i, \xi_j)$. Converse is also true. Therefore, $D(\xi) = \sum_{i,j=1}^{n} D(\xi_i, \xi_j)$ which proves (2).

Let $i \neq j$ and $D_{ij}, D'_{ij} \in D(\xi_i, \xi_j)$. Then for x in B, $[D_{ij}, D'_{ij}](x_k) = 0, x_k \in \xi_x$ which shows that $D(\xi_i, \xi_j)$ is abelian. This proves (3). □

Definition 3.2. A Lie algebra bundle $\xi = (E, p, B)$ is said to be characteristically solvable if $D(\xi)$ is solvable and $Z(\xi) \subset [\xi, \xi]$.

Theorem 3.3. *Let ξ be a solvable Lie algebra bundle such that $Z(\xi) \subset [\xi, \xi]$. If $D(\xi) = \mathfrak{S} + \mathfrak{R}(\xi)$, then ξ is characteristically solvable.*

Proof. By hypothesis, $Z(\xi) \subset [\xi, \xi]$. We need only prove that $D(\xi)$ is solvable. Since ξ is solvable, $ad\,\xi$ is solvable and hence $ad\,\xi$ is an ideal bundle of $R(\xi)$.

Let D be a derivation in $\mathfrak{S}$. Then for any $x \in B$, $[ad\xi_x, D|\xi_x] \subseteq ad\xi_x$ and $[ad\xi_x, D|\xi_x] \subseteq \mathfrak{S}_{\mathfrak{x}}$. Therefore, $[ad\xi, D] \subseteq ad\xi$ and $[ad\xi, D] \subseteq \mathfrak{S}$ so that $[ad\xi, D]$ is solvable. By hypothesis, $[ad\xi, D] = 0$ and hence $D\xi \subset Z(\xi) \subset [\xi, \xi]$. Now, $D^2\xi \subset D[\xi, \xi]$ and $D^2 = 0$ gives D is nilpotent.

Let V be a finite dimensional ample Lie subalgebra of $\Gamma(\mathfrak{S})$ and $s \in V$. Then $s(x)$ is a derivation in $\mathfrak{S}$ and is nilpotent. This implies ads is nilpotent. By Engel's Theorem for Lie algebra bundles $\mathfrak{S}$ is nilpotent and hence solvable. Since $\mathfrak{S}$ is semisimple, $\mathfrak{S} = 0$. Therefore, $D(\xi) = R(\xi)$ and we conclude that $D(\xi)$ is solvable. □

Let $\overline{D}(\xi)$ denote the subbundle of $D(\xi)$ consisting of all derivations of ξ such that $D\xi \subset Z(\xi)$.

Theorem 3.4. *Suppose $\xi = (E, p, B)$ is the direct sum of the ideal bundles $\xi_1, \xi_2, \ldots, \xi_n$. Suppose $Z(\xi_j) \subset [\xi_j, \xi_j]$ for some j, then*

(1) *$\overline{D}(\xi_j)$ is an abelian ideal bundle of $D(\xi)$.*
(2) *$[D(\xi_i, \xi_j), D(\xi_j, \xi_i)] \in \overline{D}(\xi_j)$ for all $i \neq j$.*

Proof. For all $x \in B$, $\overline{D}(\xi_j)_x$ is an abelian ideal of $D(\xi_j)_x$ and hence $\overline{D}(\xi_j)$ is an abelian ideal bundle of $D(\xi_j)$. Further, $[\overline{D}(\xi_j), \sum_{i\neq j} D(\xi_i) + \sum_{i\neq k} D(\xi_i, \xi_k)] = (0)$ so that from Theorem 3.1(1), $\overline{D}(\xi_j)$ is an abelian ideal bundle of $D(\xi)$ which proves (1).

Let D_{ij} and $D_{ji}, i \neq j$ be two elements of $D(\xi_i, \xi_j)$ and $D(\xi_j, \xi_i)$ respectively. Then, $[D_{ij}, D_{ji}]_x(\xi_i)_x = (0)$ and $[D_{ij}, D_{ji}]_x(\xi_j)_x \in Z(\xi_j)_x$ for all x in B so that $[D_{ij}, D_{ji}]$ belongs to $\overline{D}(\xi_j)$. Therefore, (2) is proved. □

Theorem 3.5. *Let ξ be a nonabelian solvable Lie algebra bundle. If $D(\xi) = \mathfrak{S}+\mathfrak{R}$, then $D(\xi)$ is solvable and ξ is either characteristically solvable or the direct sum of a characteristically solvable ideal bundle and a central ideal bundle of rank 1.*

Proof. Let $\phi : U \times L \to \bigcup\limits_{x\in U} \xi_x$ be the local triviality of ξ. From Theorem 3.3, we need only prove the result when $Z(\xi) \not\subset [\xi, \xi]$.

Let L_1 and Z be subspaces of $Z(L)$ such that

$$Z(L) = L_1 \oplus Z, \quad L_1 \cap [L, L] = (0), \quad Z \subset [L, L].$$

Let L_2 be a subspace of the Lie algebra L containing $[L, L]$ such that $L = L_1 \oplus L_2$. For each $x \in B$, let $(\xi_1)_x$ and Z_x be subspaces of ξ_x such that

$$Z(\xi_x) = (\xi_1)_x \oplus Z_x, \quad (\xi_1)_x \cap [\xi_x, \xi_x] = (0), \quad Z(\xi_x) \subset [\xi_x, \xi_x].$$

Let $(\xi_2)_x$ be a subalgebra of ξ_x containing $[\xi_x, \xi_x]$ such that $\xi_x = (\xi_1)_x \oplus (\xi_2)_x$ for $x \in B$. Then by the local triviality

$$\phi' : U \times L_1 \to \bigcup_{x\in U} (\xi_1)_x, \qquad \phi'' : U \times Z \to \bigcup_{x\in U} Z_x$$

and

$$\phi''' : U \times L_2 \to \bigcup_{x\in U} (\xi_2)_x$$

$\xi_1 = \bigcup(\xi_1)_x$, $Z = \bigcup Z_x$ and $\xi_2 = \bigcup(\xi_2)_x$ form subbundles of ξ such that $Z(\xi) = \xi_1 \oplus Z$ and $\xi = \xi_1 \oplus \xi_2$. $\xi_1 \subset Z(\xi)$ and hence ξ_1 is a central ideal bundle of ξ. ξ is non abelian and $\xi_2 \supset [\xi, \xi]$ so that ξ_2 is a non-zero ideal bundle of ξ. $Z(\xi_2)_x \subset [(\xi_2)_x, (\xi_2)_x]$ and hence $Z(\xi_2) \subset [\xi_2, \xi_2]$. By hypothesis, $D(\xi) = \mathfrak{S} \oplus \mathfrak{R}$ where $\mathfrak{S}$ is a semisimple ideal bundle and $\mathfrak{R}$ is the radical bundle of $D(\xi)$. Let $D(\xi_2) = \mathfrak{S}_2 \oplus \mathfrak{R}_2$ by Levi decomposition for Lie algebra bundles. From Theorem 3.4(1), $\overline{D}(\xi_2)$ is an abelian ideal bundle of $D(\xi)$ and hence $\overline{D}(\xi_2)$ is solvable from which it follows $\overline{D}(\xi_2) \subset \mathfrak{R}_2$.

For $x \in B$, set

$$\mathfrak{M}_x = \operatorname{span}\{(D_1)_x, D(\xi_1, \xi_2)_x, D(\xi_2, \xi_1)_x, (\mathfrak{R}_2)_x\}$$

where D_1 is the identity derivation of ξ_1. If I is the ideal $span\{D_1, D(L_1, L_2), D(L_2, L_1), R_2\}$ and $\psi : U \times D(L) \to \bigcup_{x \in U} D(\xi_x)$ is the local triviality of $D(\xi)$, then by the morphism $\psi|_{\mathfrak{M}} : U \times I \to \bigcup_{x \in U} \mathfrak{M}_x$,$\mathfrak{M} = \bigcup_{x \in B} \mathfrak{M}_x$ is an ideal bundle of $D(\xi)$. By Theorem 3.1 and Theorem 3.4, $\mathfrak{M}_x^{(i)} \subset (\mathfrak{R}_2)_x + (\overline{D}(\xi_2) + D(\xi_1, \xi_2) + D(\xi_2, \xi_1))_x$. Since $\mathfrak{R}_2$ is solvable, $\mathfrak{R}_2^{(k)} = 0$ for some k so that $\mathfrak{M}_x^{(k)} \subset (\overline{D}(\xi_2) + D(\xi_1, \xi_2) + D(\xi_2, \xi_1))_x$. Using Theorem 3.4 , $\mathfrak{M}_x^{(k+1)} \subset \overline{D}(\xi_2)_x$ and hence $\mathfrak{M}_x^{(k+2)} = (0)$. Hence $\mathfrak{M}$ is a solvable ideal bundle of ξ. Therefore, $\mathfrak{M} \subset \mathfrak{R}$. Since $\mathfrak{S}$ is a unique maximal semisimple subalgebra bundle of $D(\xi)$, $\mathfrak{S}$ contains $\mathfrak{S}_2$. $[\mathfrak{R}_2, \mathfrak{S}_2]$ is a solvable semisimple ideal bundle of $D(\xi)$ and hence $[\mathfrak{R}_2, \mathfrak{S}_2] = (0)$. This shows that $D(\xi_2) = \mathfrak{S}_2 \oplus \mathfrak{R}_2$. Therefore, ξ_2 is characteristically solvable by Theorem 3.3. For any $D \in \mathfrak{M}$, $D\xi_x \in (D_1)_x + D(\xi_1, \xi_2)_x + D(\xi_2, \xi_1)_x + (\mathfrak{R}_2)_x \subset (D_1)_x + D(\xi_1, \xi_2)_x + D(\xi_2, \xi_1)_x + D(\xi_2)_x$. Therefore, $\mathfrak{M} \subset (D_1) + D(\xi_1, \xi_2) + D(\xi_2, \xi_1) + D(\xi_2)$. Also for any $D \in (D_1) + D(\xi_1, \xi_2) + D(\xi_2, \xi_1) + D(\xi_2)$, by construction of $\mathfrak{M}$, $D \in \mathfrak{M}$. Hence, $\mathfrak{M} = (D_1) + D(\xi_1, \xi_2) + D(\xi_2, \xi_1) + D(\xi_2)$.

We assert that $rank(\xi_1) = 1$. If $dim(\xi_1)_x > 1$, then by [9] $D(\xi_1)_x = (\mathfrak{S}_1)_x + (D_1)_x$ where $(\mathfrak{S}_1)_x$ is a non-zero semisimple ideal of $D(\xi_1)_x$. Therefore, $D(\xi)_x = (\mathfrak{S}_1)_x + \mathfrak{M}_x$ and $[(\mathfrak{S}_1)_x, \mathfrak{M}_x] = (0)$.

Let D_{11} be any element of $(\mathfrak{S}_1)_x$. Then $[D_{21}, D_{11}] = (0)$ for any element D_{21} of $D(\xi_2, \xi_1)_x$ so that $D_{21}D_{11} = (0)$. $(\xi_1)_x$ is abelian and by Theorem 3.1(2),

$$D(\xi_2, \xi_1)_x(\xi_2)_x = (\xi_1)_x.$$

Therefore, $D_{11} = 0$ whence $(\mathfrak{S}_1)_x = (0)$ which is a contradiction. Therefore, $dim(\xi_1)_x = 1$. This gives that rank of ξ_1 is 1. Thus $D(\xi_1) = (D_1)$. Therefore, $D(\xi) = \mathfrak{M}$ from which it can be concluded that $D(\xi)$ is solvable. This proves the theorem. □

From [1] $Herm(E) = \bigcup_{x \in B} Herm(E_x)$ forms a vector bundle where $Herm(E_x)$ is the vector space of all Hermitian forms on E_x.

For $x \in B$ and $\forall f, g \in Herm(E_x)$ define $[f, g] = fog - gof$. Then $Herm(E_x)$ becomes a Lie algebra with this product structure. For every $x \in B$ and a neighbourhood U of x in B let $\phi : p^{-1}(U) \to U \times \xi_x$ be the local trivialization of ξ.

Define $Herm\phi : U \times Herm(E_x) \to (Hermp)^{-1}$ by $Herm\phi(s, T) = \phi_s oTo(\phi_s)^{-1}$.

We observe that $Herm\phi$ is a Lie bundle isomorphism. Hence $Herm(E)$ is a Lie algebra bundle.

Definition 3.6. A Hermitian metric on a Lie algebra bundle is a section $h : E \to HermE$ such that $h(x)$ is positive definite for all $x \in B$. A Lie bundle with a specified Hermitian metric is called a Hermitian Lie bundle.

Proposition 3.7. *Let $\xi = (E, p, B)$ be a Lie algebra bundle. η be a subbundle of ξ and h be the Hermitian metric on ξ. Then there exists a subbundle η' of ξ such that $\xi = \eta \oplus \eta'$.*

Proof. For all $x \in B$ consider the orthogonal projection $P_x : \xi_x \to \eta_x$ given by the Hermitian metric. Define $P : \xi \to \eta$ such that P on ξ_x is the projection P_x for all $x \in B$.

We claim that P is continuous. As the problem is local in nature, we assume that ξ is trivial. Then there exists sections $s_1, s_2, \ldots, s_n$ of ξ which forms a basis on each fibre. For any $v \in \xi_x$, $v = \sum h_x(v, s_i(x))s_i(x)$. P is continuous since h is continuous. Therefore, P is a projection operator on ξ. If $\eta_x^\perp$ is the Lie subalgebra of ξ_x which is orthogonal to η_x under the metric h, then $\eta^\perp = \cup \eta_x^\perp$ is the kernel of P and hence is a subbundle of ξ. Therefore, $\xi = \eta \oplus \eta^\perp$. □

Proposition 3.8. *Let ξ be any subbundle of the Lie algebra bundle $End(\xi)$. If $C(\xi) \subset I(\xi)^*$, then $Z(\xi) = 0$ or $\xi = [\xi, \xi]$.*

Proof. By Levi decomposition, $\xi = \mathfrak{S} + \mathfrak{R}$ where $\mathfrak{S}$ is a semisimple subbundle and $\mathfrak{R}$ is the radical bundle of ξ. Further, since ξ^* is algebraic, $\xi^* = \mathfrak{S} + \mathfrak{R}^*$. Suppose $Z(\xi) \neq 0$ and $\xi \neq [\xi, \xi]$. We have,

$$\begin{aligned} [\xi, \xi] &= [\xi, \mathfrak{S} + \mathfrak{R}] \\ &= [\xi, \mathfrak{S}] + [\xi, \mathfrak{R}] \\ &= [\mathfrak{S} + \mathfrak{R}, \mathfrak{S}] + [\xi, \mathfrak{R}] \\ &= \mathfrak{S} + [\xi, \mathfrak{R}]. \end{aligned}$$

Since $\xi \neq [\xi, \xi]$, $\xi \neq \mathfrak{S} + [\xi, \mathfrak{R}]$ so that $[\xi, \mathfrak{R}] \neq \mathfrak{R}$. Since $[\xi, \xi] \neq 0$, $[\xi, \mathfrak{R}] \neq \mathfrak{R}$, by Proposition 3.7 we can choose a subbundle U of $\mathfrak{R}$ such that $\mathfrak{R} = U \oplus [\xi, \mathfrak{R}]$. Then $\xi = \mathfrak{S} \oplus U \oplus [\xi, \mathfrak{R}]$. Define a non-zero morphism D on ξ as, for $x \in B$,

$$DU_x \subset Z(\xi_x) \quad \text{and} \quad D(\mathfrak{S} + [\xi, R(\xi)]) = 0.$$

Then D is a central derivation on ξ. Given $C(\xi) \subset I(\xi)^*$ and hence $C(\xi) \subset I(\mathfrak{S} + \mathfrak{R}^*)$ so that D is in $I(\mathfrak{S} + \mathfrak{R}^*)$ which implies that there exists a section s with $s(x) \in \xi_x = \mathfrak{S}_x + \mathfrak{R}_x^*$ such that $D(u) = [u, v_0]$ where $v_0 = p_x + r_x \in \mathfrak{S}_x + \mathfrak{R}_x^*$. Therefore, $D(u) = [u, p_x] + [u, r_x]$. Since $D\mathfrak{S} = 0$, $D(u) = ad_{r_x}(u)$. This implies

$D(U_x) = ad_{r_x}(U_x) = [r_x, U_x] \subset [R_x^*, U_x] \subset [R_x^*, R_x] = 0$. Hence $D(U_x) = 0$ which shows that D is a zero morphism which is a contradiction. Therefore, $Z(\xi) = 0$ or $\xi = [\xi, \xi]$. □

Let $\xi = (E, p, B)$ be a Lie algebra bundle. Set $Z_1(\xi_x) = \{e \in \xi_x : [e, \xi_x] \subseteq Z(\xi_x)\}$ for all $x \in B$. Then $Z_1(\xi_x)$ is a subalgebra of ξ_x, $x \in B$. Define $Z_1(\xi) = \bigcup Z_1(\xi_x)$. Then $Z_1(\xi)$ is a subbundle of ξ.

Proposition 3.9. *Let $Z(\xi)$ be the center of ξ. Then $I(\xi) \cap C(\xi) = I(Z_1(\xi))$.*

Proof. Let $D \in I(\xi) \cap C(\xi)$. Then $D : \xi \to \xi$ is an inner derivation. Therefore, there exists a section s of ξ such that $D(u) = [u, s(x)]$ for all u in ξ_x and x in B. Let $x \in B$. Then $[u, s(x)] = D(u) \in Z(\xi_x)$ for all $u \in \xi_x$ which implies $s(x) \in Z_1(\xi_x)$. Therefore, $s : \xi \to Z_1(\xi)$ is a section of $Z_1(\xi)$. Hence $D \in I(Z_1(\xi))$.
On the other hand, let $D \in I(Z_1(\xi))$. Clearly $D \in I(\xi)$. Also, $D(u) = [u, s(x)]$ for some section s of $Z_1(\xi)$ and $s(x) \in Z_1(\xi_x)$ so that $D(u) = [u, s(x)] \in Z(\xi_x)$ which shows that $D \in C(\xi)$. Hence the proof is completed. □

Theorem 3.10. *$I(\xi) \subset C(\xi)$ if and only if $\xi^3 = 0$.*

Proof. $I(\xi) \subset C(\xi)$ gives $I\xi_x \subset Z(\xi_x)\ \forall x \in B$. Then for any I in $I(\xi)$, $[\xi_x, I(\xi_x)] = 0$ for all $x \in B$.
Now,

$$\begin{aligned} I\xi_x &= \{[u, s(x)] \mid \forall u \in \xi_x \text{ and s is a section of } \xi \text{ such that } s(x) = v_0 \in \xi_x\} \\ &= \{[u, v_0] \mid \forall u \in \xi_x \text{ and for some } v_0 \in \xi_x\}. \end{aligned}$$

Therefore, $I\xi_x = [\xi_x, v_0]$ for some $v_0 \in \xi_x$ so that $I\xi_x = \xi_x^2$.
Let $v_0 \in \xi_x$. Then $Y = \{x\}$ is a closed subspace of B. Define a section $s : Y \to E/Y$ as $s(x) = v_0$. Then the section can be extended to E by [1]. Therefore, $\forall\ v_0 \in\ \xi_x$, there exists a section s of ξ such that $s(x) = v_0$.

Set, $Du = [u, v_0] = [u, S(x)]$ for every $u \in \xi_x$. Then D is an inner derivation of ξ. Therefore, $I(\xi) \subset C(\xi)$ giving $[\xi_x, D(\xi_x)] = 0\ \forall x \in B$. It follows that $\xi_x^3 = 0$ $\forall x \in B$ and hence $\xi^3 = 0$.

Conversely, suppose $\xi^3 = 0$. This implies $\xi^2 \subset Z$ which gives $[\xi_x^2, u] = 0\ \forall u \in \xi_x, x \in B$. Hence $[I(\xi_x), u] = 0\,\forall\, x \in B$ which shows that $I(\xi_x) \subset Z(\xi_x)$. Therefore, $I(\xi) \subset C(\xi)$. This proves the result. □

Theorem 3.11. *Assume that the center $Z(\xi_x)$ is non-zero for each $x \in B$. Then $I(\xi) = C(\xi)$ if and only if $\xi^2 = Z$ and $\operatorname{rank} Z(\xi) = 1$.*

Proof. Let $\phi : U \times Z(L) \to \bigcup_{x \in U} Z(\xi_x)$ be the local triviality of centre bundle of ξ, D be the identity derivation on $Z(\xi)$ i.e., $D(y) = y \,\forall\, y \in Z(\xi_x)$, x in B. Extend this trivially to a derivation of ξ. Then D is a central derivation of ξ. If D is an inner derivation, then there exists a section s of ξ such that $Du = [u, v_0]$, $v_0 = s(x)$, $u \in \xi_x$ and $x \in B$. If $\xi^2 \neq Z(\xi)$, then there exists $x \in B$ such that $\xi_x^2 \neq Z(\xi_x)$. Let $u \in Z(\xi_x) - \xi_x^2$. Then $Du = [u, v_0] = 0$, $v_0 = s(x)$ which is not true since $Du = u$ for all $u \in Z(\xi_x)$. Therefore, D is not an inner derivation which is a contradiction. Hence $\xi^2 = Z(\xi)$.

For any $x \in B$, $dimI(\xi_x) = dim(\xi_x/Z(\xi_x))$ and $dimC(\xi_x) = dim(\xi_x/\xi_x^2) \times dimZ(\xi_x)$ which gives $dimZ(\xi_x) = 1$ and hence $rank\, Z(\xi) = 1$.
Conversely, suppose $\xi^2 = Z$ and $rank\, Z(\xi) = 1$. Then $\xi^2 = Z$ gives $\xi^3 = 0$. By Theorem 3.10, $I(\xi) \subset C(\xi)$. By the above formulae on dimensions of $I(\xi_x)$ and $C(\xi_x)$, $dim\, I(\xi_x) = dim\, C(\xi_x)$ $\forall x \in B$. Thus $I(\xi) = C(\xi)$. □

Theorem 3.12. *$D(\xi) = C(\xi)$ if and only if ξ is abelian.*

Proof. We prove that $\xi^2 = 0$. Suppose $\xi^2 = \bigcup_{x \in B} \xi_x^2 \neq (0)$. Then we can find a $x \in B$ such that $\xi_x^2 \neq (0)$. If $Z(\xi_x) = (0)$, then $C(\xi_x) = (0)$ which gives $D(\xi_x) = (0)$. It follows that $\xi_x = 0$. Therefore, we assume $Z(\xi) \neq (0)$. Then by Theorem 3.10, $\xi^3 = 0$. Since $\xi^2 \neq 0$, for some x in B we obtain a Lie subalgebra $U_x \neq 0$ such that $\xi_x = U_x \oplus (\xi_x)^2$. Let D_x be the identity mapping of U_x, $D_x u = u$, for all $x \in U_x$. We shall extend this mapping trivially to a derivation D of ξ. Then D is a derivation of ξ which is not central - a contradiction to our supposition. Therefore, $\xi^2 = 0$. Conversely, if ξ is abelian, then $Z(\xi) = \xi$ and hence every derivation is central. □

Theorem 3.13. *Let ξ be the direct sum of the ideal bundles $\xi_1, \xi_2, \ldots, \xi_n$. Then $D(\xi) = I(\xi) \oplus C(\xi)$ if and only if $D(\xi_i) = I(\xi_i) \oplus C(\xi_i)$.*

Proof. Suppose $D(\xi_i) = I(\xi_i) \oplus C(\xi_i)$. From [11] $D(\xi) = \sum_{i=1}^n D(\xi_i) \oplus \sum_{i,j=1}^n D(\xi_i, \xi_j)$. For $i \neq j$, $D(\xi_i, \xi_j)\xi_i \subset Z(\xi_j)$. Therefore, $D(\xi_i, \xi_j) \subset C(\xi_i)$. It follows that $D(\xi) = \sum_{i=1}^n I(\xi_i) \oplus C(\xi_i) = \sum_{i=1}^n I(\xi_i) \oplus \sum_{i=1}^n C(\xi_i)$. Further $\sum_{i=1}^n I(\xi_i) = \sum_{i=1}^n ad(\xi_i) = ad(\xi_1 \oplus \xi_2 \oplus \ldots \oplus \xi_n) = I(\xi)$. Also, $C(\xi) = \sum_{i=1}^n C(\xi_i)$. Therefore, $D(\xi) = I(\xi) \oplus C(\xi)$.
Conversely, assume $D(\xi) = I(\xi) \oplus C(\xi)$. Any derivation D_i of ξ_i can be trivially extended to a derivation D of ξ. Then $D = D_1 + \overline{D}$ where D_1 is in $I(\xi)$ and $\overline{D}$ is in $C(\xi)$ which gives $D|\xi_i = D_i = D_1|\xi_i + \overline{D}|\xi_i$. Therefore, $\overline{D}|\xi_i = \overline{D}_i = D_i - D_1|\xi_i$ so that $\overline{D}_i \xi_i \subset \xi_i \cap Z(\xi) = Z(\xi_i)$. Hence $\overline{D}_i \in C(\xi_i)$ which gives $D(\xi_i) = I(\xi_i) \oplus C(\xi_i)$. □

Theorem 3.14. *Let ξ be a non abelian nilpotent Lie algebra bundle such that $Z(\xi)$ is not contained in $[\xi,\xi]$. Then $D(\xi)$ is not nilpotent. $D(\xi)$ contains a solvable non nilpotent ideal bundle.*

Proof. Let ξ_1 and ξ_2 be subbundles of ξ as in Theorem 3.5. Then $Z(\xi_2) \subset [\xi_2, \xi_2]$. For $x \in B$, set $\mathfrak{M}_x$=span$\{(D_1)_x, D(\xi_1,\xi_2)_x, D(\xi_2,\xi_1)_x, \overline{D}(\xi_2)_x\}$ where D_1 is the identity derivation of ξ_1. Let $\mathfrak{M} = \bigcup_{x\in B} \mathfrak{M}_{\mathfrak{x}}$. Then by Theorem 3.1, $\mathfrak{M}$ is an ideal bundle of $D(\xi)$. $\mathfrak{M}_x^{(1)} \subset \overline{D}(\xi_2)_x + D(\xi_1,\xi_2)_x + D(\xi_2,\xi_1)_x$. Then by Theorem 3.4 $\mathfrak{M}_x^{(3)} = 0$ so that $\mathfrak{M}$ is a solvable ideal bundle of $D(\xi)$. Since ξ is non abelian and nilpotent $D(\xi_1,\xi_2) \neq 0$. Since $[D_1, D(\xi_1,\xi_2)_x] = D(\xi_1,\xi_2)_x$, $\mathfrak{M}$ is not nilpotent. Thus $\mathfrak{M}$ is a solvable non nilpotent ideal bundle of $D(\xi)$. □

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