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On Solutions of Some Diophantine Equations

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Highlights

- This paper focuses on positive integer solutions of some Diophantine equations.
- The solutions to the Diophantine equations are related to generalized Fibonacci and Lucas numbers.
- k and p are taken from prime numbers in these equations.

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Abstract

Let $\boldsymbol{k}$ and $\boldsymbol{p}$ be prime numbers with $\boldsymbol{p > 2}$. In this note, we find all positive integer solutions to the Diophantine equations

$$\boldsymbol{x^2 - kxy - y^2 = \pm k},$$
$$\boldsymbol{x^2 - (k^2+4)y^2 = \pm 4k},$$
$$\boldsymbol{x^2 - 2pxy - y^2 = \pm 2p},$$
$$\boldsymbol{x^2 - kxy - y^2 = \pm k(k^2+4)}$$

in terms of generalized Fibonacci and Lucas numbers.

1. INTRODUCTION

Let k and s be two nonzero integers with $k^2 + 4s > 0$. The sequence of generalized Fibonacci numbers $(U_n(k,s))$ is defined by the recurrence relation

$$U_{n+1}(k,s) = kU_n(k,s) + sU_{n-1}(k,s)$$

with the initial conditions $U_0(k,s) = 0$ and $U_1(k,s) = 1$ for $n \geq 1$. Let $(V_n(k,s))$ be a sequence of generalized Lucas numbers given by the recurrence

$$V_{n+1}(k,s) = kV_n(k,s) + sV_{n-1}(k,s)$$

with the initial conditions $V_0(k,s) = 2$ and $V_1(k,s) = k$ for $n \geq 1$. When we choose $k = s = 1$, these sequences are the well known Fibonacci and Lucas sequences, respectively. The characteristic equation

$$x^2 - kx - s = 0$$

has roots $\alpha = \frac{k+\sqrt{k^2+4s}}{2}$ and $\beta = \frac{k-\sqrt{k^2+4s}}{2}$. It is easily seen that $\alpha + \beta = k$, $\alpha - \beta = \sqrt{k^2+4s}$, $\alpha\beta = -s$. Binet formulas for these numbers are $U_n(k,s) = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ and $V_n(k,s) = \alpha^n + \beta^n$.

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Diophantine equations whose solutions are generalized Fibonacci and Lucas numbers has been of interest to mathematicians. Solutions to the Diophantine equation

$$x^2 - kxy \pm y^2 = \pm 1$$

can be found in [1]. Moreover, solutions to the Diophantine equations

$x^2 - (k^2 \pm 4)y^2 = \pm 4$ and $x^2 - (k^2 + 1)y^2 = \pm 1$

can be seen in [2]. For $k > 3$, Keskin and Duman [3] have dealt with the Diophantine equations

$$x^2 - kxy + y^2 = \pm(k \pm 2),$$

$$x^2 - (k^2 - 4)y^2 = \pm 4(k \pm 2),$$

$$x^2 - kxy + y^2 = \pm(k^2 - 4)(k \pm 2),$$

$$x^2 - (k^2 + 2)xy + y^2 = -k^2,$$

$$x^2 - (k^2 \pm 2)xy + y^2 = k^2,$$

$$x^2 + 4xy - [(k^2 - 2)y]^2 = 4k^2.$$

Let k and p be prime numbers with $p > 2$. Motivated by these works, we solve the Diophantine equations

$$x^2 - kxy - y^2 = \pm k,$$

$$x^2 - (k^2 + 4)y^2 = \pm 4k,$$

$$x^2 - 2pxy - y^2 = \pm 2p,$$

$$x^2 - kxy - y^2 = \pm k(k^2 + 4).$$

2. PRELIMINARIES

From now on, we will take $U_n = U_n(k, 1)$ and $V_n = V_n(k, 1)$ unless stated otherwise. Let $k \geq 1$ be an integer. Then, the following properties hold:

$$U_n^2 - kU_nU_{n-1} - U_{n-1}^2 = (-1)^{n-1} \tag{2.1}$$

$$V_n^2 - kV_nV_{n-1} - V_{n-1}^2 = (-1)^n(k^2 + 4) \tag{2.2}$$

$$V_n^2 - (k^2 + 4)U_n^2 = 4(-1)^n \quad . \tag{2.3}$$

We have given above only the properties of the generalized Fibonacci and Lucas numbers that we will use in the proof of our main results. More information about these sequences can be found in [4]. The following lemmas are given in [2].

Lemma 2.1. Let $k \geq 1$ be an integer. Then, all solutions to the equation

$$x^2 - (k^2 + 4)y^2 = 4$$

in positive integers are given by $(x, y) = (V_{2n}, U_{2n})$ with $n \geq 1$.

Lemma 2.2. Let $k \geq 1$ be an integer. Then, all solutions to the equation

$$x^2 - (k^2+4)y^2 = -4$$

in positive integers are given by $(x, y) = (V_{2n-1}, U_{2n-1})$ with $n \geq 1$.

Lemma 2.3. Let $k \geq 1$ be an integer with $k \neq 2$. Then, all solutions to the equation

$$x^2 - (k^2+1)y^2 = 1$$

in positive integers are given by $(x, y) = \left(\frac{V_{2n}(2k,1)}{2}, U_{2n}(2k, 1)\right)$ with $n \geq 1$.

Lemma 2.4. Let $k \geq 1$ be an integer with $k \neq 2$. Then, all solutions to the equation

$$x^2 - (k^2+1)y^2 = -1$$

in positive integers are given by $(x, y) = \left(\frac{V_{2n-1}(2k,1)}{2}, U_{2n-1}(2k, 1)\right)$ with $n \geq 1$.

3. MAIN RESULTS

Theorem 3.1. Let k be a prime number. Then, all solutions to the equation

$$x^2 - kxy - y^2 = k \tag{3.1}$$

in positive integers are given by

$(x, y) = (U_{2n-1} + U_{2n}, U_{2n-2} + U_{2n-1})$ or $(x, y) = (U_{2n+1} - U_{2n}, U_{2n} - U_{2n-1})$ with $n \geq 1$.

Proof. Assume that $x^2 - kxy - y^2 = k$ for some positive integers x and y. So, $x^2 - y^2 > 0$. Taking $u = x + y$ and $v = x - y$, we get $4uv = 4k + k(u^2 - v^2)$. From this, we can say $k|uv$. Let $k|v$. Then we can write that $u^2 - v^2 - 4u\left(\frac{v}{k}\right) = -4$. Here, it can be easily shown that $ku - 2v > 0$. The last equation implies that $\left(u - 2\frac{v}{k}\right)^2 - (k^2+4)\left(\frac{v}{k}\right)^2 = -4$. According to Lemma 2.2, we can say $u - 2\frac{v}{k} = V_{2n-1}$ and $\frac{v}{k} = U_{2n-1}$ with $n \geq 1$. Thus, we get $u = 2U_{2n-1} + V_{2n-1}$ and $v = kU_{2n-1}$. From here, we conclude that

$$x = \frac{u+v}{2} = \frac{kU_{2n-1} + U_{2n-2} + 2U_{2n-1} + U_{2n}}{2} = \frac{U_{2n} + U_{2n} + 2U_{2n-1}}{2} = U_{2n} + U_{2n-1},$$

$$y = \frac{u-v}{2} = \frac{2U_{2n-1} + U_{2n} + U_{2n-2} - kU_{2n-1}}{2} = \frac{2U_{2n-1} + 2U_{2n-2}}{2} = U_{2n-1} + U_{2n-2}.$$

These show that $(x, y) = (U_{2n} + U_{2n-1}, U_{2n-1} + U_{2n-2})$. Now, let $k|u$. Using the equality $4uv - k(u^2 - v^2) = 4k$, we can write $4\left(\frac{u}{k}\right)v - u^2 + v^2 = 4$ and so $\left(v + 2\frac{u}{k}\right)^2 - (k^2+4)\left(\frac{u}{k}\right)^2 = 4$. Thanks to Lemma 2.1, we can say $v + 2\frac{u}{k} = V_{2n}$ and $\frac{u}{k} = U_{2n}$ with $n \geq 1$. These equalities imply that $u = kU_{2n}$ and $v = V_{2n} - 2\frac{u}{k} = V_{2n} - 2U_{2n} = U_{2n+1} + U_{2n-1} - 2U_{2n}$. Thus, we arrive that

$$x = \frac{u+v}{2} = \frac{kU_{2n} + U_{2n+1} + U_{2n-1} - 2U_{2n}}{2} = \frac{2U_{2n+1} - 2U_{2n}}{2} = U_{2n+1} - U_{2n},$$

$$y = \frac{u-v}{2} = \frac{kU_{2n} - U_{2n+1} - U_{2n-1} + 2U_{2n}}{2} = \frac{kU_{2n} - (kU_{2n} + U_{2n-1} + U_{2n-1} - 2U_{2n})}{2} = \frac{2U_{2n} - 2U_{2n-1}}{2} = U_{2n} - U_{2n-1}.$$

Consequently, we get $(x, y) = (U_{2n+1} - U_{2n}, U_{2n} - U_{2n-1})$. Finally, considering the equality (2.1) it can be seen that these pairs are solutions to the equation (3.1).

Theorem 3.2. Let k be a prime number. Then, all solutions to the equation

$$x^2 - kxy - y^2 = -k \tag{3.2}$$

in positive integers are given by

$(x, y) = (U_{2n-1} + U_{2n-2}, U_{2n-2} + U_{2n-3})$ or $(x, y) = (U_{2n} - U_{2n-1}, U_{2n-1} - U_{2n-2})$ with $n \geq 1$.

Proof. Let (x, y) be a positive integer solution to the equation $x^2 - kxy - y^2 = -k$. Taking $u = kx + y$ and $v = x$, this equation is converted to the equation $u^2 - kuv - v^2 = k$. Theorem 3.1 tells us that the solutions to the last equation are $(u, v) = (U_{2n-1} + U_{2n}, U_{2n-2} + U_{2n-1})$ or $(u, v) = (U_{2n+1} - U_{2n}, U_{2n} - U_{2n-1})$. The first of these equalities leads to $x = v = U_{2n-1} + U_{2n-2}$ and

$$y = u - kv = U_{2n} + U_{2n-1} - kU_{2n-1} - kU_{2n-2} = U_{2n-2} + U_{2n-3}$$

and so $(x, y) = (U_{2n-1} + U_{2n-2}, U_{2n-2} + U_{2n-3})$. Considering the second of these equalities, we obtain

$x = v = U_{2n} - U_{2n-1}$ and

$$y = u - kv = U_{2n+1} - U_{2n} - kU_{2n} + kU_{2n-1} = U_{2n-1} - U_{2n-2}$$

and so $(x, y) = (U_{2n} - U_{2n-1}, U_{2n-1} - U_{2n-2})$. Moreover, taking into account the equality (2.1) it can be shown that these pairs are solutions to the equation (3.2).

Theorem 3.3. Let k be a prime number. Then, all solutions to the equation

$$x^2 - (k^2 + 4)y^2 = 4k \tag{3.3}$$

in positive integers are given by

$(x, y) = (V_{2n-1} + V_{2n-2}, U_{2n-1} + U_{2n-2})$ or $(x, y) = (V_{2n} + V_{2n-1}, U_{2n} + U_{2n-1})$ with $n \geq 1$.

Proof. Assume that positive integers x and y satisfy the equation (3.3). So, $x^2 - k^2y^2 = 4k + 4y^2$. From the last equation, it is seen that x and ky are the same parity. Taking $u = \frac{x+ky}{2}$ and $v = y$, we obtain $u^2 - kuv - v^2 = k$. According to Theorem 3.1, we can say that $(u, v) = (U_{2n-1} + U_{2n}, U_{2n-2} + U_{2n-1})$ or $(u, v) = (U_{2n+1} - U_{2n}, U_{2n} - U_{2n-1})$ are all solutions to the last equation. Firstly, let $\frac{x+ky}{2} = U_{2n} + U_{2n-1}$. Since

$$x = 2U_{2n} - kU_{2n-1} + 2U_{2n-1} - kU_{2n-2} = U_{2n} + U_{2n-1} + U_{2n-2} + U_{2n-3} = V_{2n-1} + V_{2n-2}$$

and $y = v = U_{2n-1} + U_{2n-2}$, we can deduce that $(x, y) = (V_{2n-1} + V_{2n-2}, U_{2n-1} + U_{2n-2})$. Secondly, put $\frac{x+ky}{2} = U_{2n+1} - U_{2n}$ and $v = U_{2n} - U_{2n-1}$. Therefore, we can write

$$x = U_{2n+1} + U_{2n+1} - kU_{2n} - U_{2n} - U_{2n} + kU_{2n-1} = U_{2n+1} + U_{2n-1} - U_{2n} - U_{2n-2} = V_{2n} - V_{2n-1}$$

$y = v = U_{2n} - U_{2n-1}$ and so we find that $(x, y) = (V_{2n} - V_{2n-1}, U_{2n} - U_{2n-1})$. The equalities (2.1) and (2.3) show that $(V_{2n-1} + V_{2n-2}, U_{2n-1} + U_{2n-2})$ and $(V_{2n} - V_{2n-1}, U_{2n} - U_{2n-1})$ are solutions to the equation (3.3).

Since the proof of the following theorem can be derived from the previous one, its proof has been omitted.

Theorem 3.4. Let k be a prime number. Then, all solutions to the equation

$$x^2-(k^2+4)y^2=-4k \tag{3.4}$$

in positive integers are given by

$(x,y)=(V_{2n-2}+V_{2n-3},U_{2n-2}+U_{2n-3})$ or $(x,y)=(V_{2n-1}-V_{2n-2},U_{2n-1}-U_{2n-2})$ with $n\geq 1$.

Theorem 3.5. Let p be a prime number with $p>2$ and $U_n=U_n(2p,1)$. Then, all solutions to the equation

$$x^2-2pxy-y^2=2p \tag{3.5}$$

in positive integers are given by

$(x,y)=(U_{2n-1}+U_{2n},U_{2n-2}+U_{2n-1})$ or $(x,y)=(U_{2n+1}-U_{2n},U_{2n}-U_{2n-1})$ with $n\geq 1$.

Proof. Suppose that (3.5) is satisfied for some positive integers x and y. From this equation, we can conclude that x^2-y^2 is even integer. So, x and y are the same parity. Let $u=x+y$ and $v=x-y$. We have that $4uv-2p(u^2-v^2)=8p$. Choosing $u=2a$ and $v=2b$, we can write $2ab-p(a^2-b^2)=p$. This equation implies that $p|2ab$. Since p is odd, we can say that $p|ab$. Firstly, let $p|a$. Hence, we get $2\left(\frac{a}{p}\right)b-a^2+b^2=1$, which implies that $\left(b+\frac{a}{p}\right)^2-(p^2+1)\left(\frac{a}{p}\right)^2=1$. According to Lemma 2.3, there is an integer $n\geq 1$ such that $b+\frac{a}{p}=\frac{V_{2n}(2p,1)}{2}$ and $\frac{a}{p}=U_{2n}(2p,1)$. Thus, we can write $u=2a=2pU_{2n}=kU_{2n}$ and $v=2b=V_{2n}-2U_{2n}$. We find that

$$x=\frac{u+v}{2}=\frac{kU_{2n}+V_{2n}-2U_{2n}}{2}=\frac{kU_{2n}+U_{2n+1}+U_{2n-1}-2U_{2n}}{2}=\frac{U_{2n+1}+U_{2n+1}-2U_{2n}}{2}=U_{2n+1}-U_{2n},$$

$$y=\frac{u-v}{2}=\frac{kU_{2n}-U_{2n+1}+2U_{2n}}{2}=\frac{kU_{2n}-(kU_{2n}+U_{2n-1})-U_{2n-1}+2U_{2n}}{2}=\frac{2U_{2n}-2U_{2n-1}}{2}=U_{2n}-U_{2n-1}.$$

Let $p|b$. So, $2ab-p(a^2-b^2)=p$ gives us $2a\left(\frac{b}{p}\right)-a^2+b^2=1$ or $\left(a-\frac{b}{p}\right)^2-(p^2+1)\left(\frac{b}{p}\right)^2=-1$, where $pa-b>0$. Put $k=2p$. Lemma 2.4 tells us that $a-\frac{b}{p}=\frac{V_{2n-1}(k,1)}{2}$ and $\frac{b}{p}=U_{2n-1}(k,1)$ for $n\geq 1$. From the last equalities, we find that $b=pU_{2n-1}$ and $a=\frac{V_{2n-1}}{2}+U_{2n-1}=\frac{V_{2n-1}+2U_{2n-1}}{2}$. In these cases, we have $u=2a=V_{2n-1}+2U_{2n-1}$ and $v=2b=2pU_{2n-1}=kU_{2n-1}$. Therefore, we conclude that

$$x=\frac{u+v}{2}=\frac{U_{2n}+U_{2n-2}+2U_{2n-1}+kU_{2n-1}}{2}=\frac{2U_{2n}+2U_{2n-1}}{2}=U_{2n}+U_{2n-1},$$

$$y=\frac{u-v}{2}=\frac{U_{2n}+U_{2n-2}+2U_{2n-1}-kU_{2n-1}}{2}=\frac{2U_{2n-2}+2U_{2n-1}}{2}=U_{2n-1}+U_{2n-2}.$$

Conversely, if $(x,y)=(U_{2n-1}+U_{2n},U_{2n-2}+U_{2n-1})$ or $(x,y)=(U_{2n+1}-U_{2n},U_{2n}-U_{2n-1})$, then by using (2.1), it can be seen that the equality (3.5) is satisfied.

Since the proof is similar to the previous one, we leave the proof of the following theorem to the readers.

Theorem 3.6. Let p be prime number with $p>2$. Then all solutions to the equation

$$x^2-2pxy-y^2=-2p \tag{3.6}$$

in positive integers are given by

$(x,y)=(U_{2n}-U_{2n-1},U_{2n-1}-U_{2n-2})$ or $(x,y)=(U_{2n}+U_{2n+1},U_{2n-1}+U_{2n})$ with $n\geq 1$.

Theorem 3.7. Let k be a prime number and k^2+4 be square-free. Then, all solutions to the equation

$$x^2-kxy-y^2=-k(k^2+4) \tag{3.7}$$

in positive integers are given by

$(x,y)=(V_{2n+1}-V_{2n},V_{2n}-V_{2n-1})$ or $(x,y)=(V_{2n}+V_{2n-1},V_{2n-1}+V_{2n-2})$ with $n\geq 1$.

Proof: Assume that the equation (3.7) holds for positive integer x and y. In this case, $x-y$ is a positive integer. Taking $u=x+y$ and $v=x-y$, we conclude that $uv-k\left(\frac{u^2-v^2}{4}\right)=-k(k^2+4)$ and so $k|uv$. Since k is prime, we obtain $k|u$ or $k|v$. Firstly, let $k|u$. Thus, we find that $v^2+4v\left(\frac{u}{k}\right)-u^2=-4(k^2+4)$ i.e., $\left(v+\frac{2u}{k}\right)^2-(k^2+4)\left(\frac{u}{k}\right)^2=-4(k^2+4)$. Since k^2+4 is a square-free integer, we get $k^2+4|v+\frac{2u}{k}$. Then, we have $v+\frac{2u}{k}=(k^2+4)a$ for positive integer a. If these values are substituted in the last equation, we get $\left(\frac{u}{k}\right)^2-(k^2+4)a^2=4$. From Lemma 2.1, we can see that $\frac{u}{k}=V_{2n}$ and $a=U_{2n}$ for $n\geq 1$. These equalities imply that $u=kV_{2n}$ and

$$v=(k^2+4)a-2\frac{u}{k}=(k^2+4)U_{2n}-2V_{2n}.$$

Thus, by a simple calculation, the values of x and y are found to be

$$x=\frac{u+v}{2}=\frac{kV_{2n}+V_{2n+1}+V_{2n-1}-2V_{2n}}{2}=\frac{2V_{2n+1}-2V_{2n}}{2}=V_{2n+1}-V_{2n},$$

$$y=\frac{u-v}{2}=\frac{kV_{2n}-V_{2n+1}-V_{2n-1}+2V_{2n}}{2}=\frac{2V_{2n}-2V_{2n-1}}{2}=V_{2n}-V_{2n-1},$$

respectively. Secondly, let $k|v$. In this case, we have that $4u\left(\frac{v}{k}\right)-u^2+v^2=-4(k^2+4)$, it follows that $\left(u-\frac{2v}{k}\right)^2-(k^2+4)\left(\frac{v}{k}\right)^2=4(k^2+4)$. The last equation tells us that $k^2+4|\left(u-\frac{2v}{k}\right)^2$. Since k^2+4 is square-free, we can say $k^2+4|u-\frac{2v}{k}$, which leads to $u-\frac{2v}{k}=(k^2+4)a$. Here a is positive integer. Hence, we obtain $\left(\frac{v}{k}\right)^2-(k^2+4)a^2=-4$. Thanks to Lemma 2.2, we can write $\frac{v}{k}=V_{2n-1}$ and $a=U_{2n-1}$ for $n\geq 1$. Therefore, we get

$$x=\frac{u+v}{2}=\frac{V_{2n}+V_{2n-2}+2V_{2n-1}-kV_{2n-1}}{2}=\frac{2V_{2n}+2V_{2n-1}}{2}=V_{2n}+V_{2n-1},$$

$$y=\frac{u-v}{2}=\frac{V_{2n}+V_{2n-2}+V_{2n-1}-kV_{2n-1}}{2}=\frac{2V_{2n-2}+2V_{2n-1}}{2}=V_{2n-1}+V_{2n-2}.$$

On the contrary, if $(x,y)=(V_{2n+1}-V_{2n},V_{2n}-V_{2n-1})$ or $(x,y)=(V_{2n}+V_{2n-1},V_{2n-1}+V_{2n-2})$, then it can be shown that the equality (3.7) is satisfied by (2.2).

We neglect the proof of the following theorem since its proof can be derived from the previous one.

Theorem 3.8. Let k be a prime number and k^2+4 be square-free. Then, all solutions to the equation

$$x^2-kxy-y^2=k(k^2+4) \tag{3.8}$$

in positive integers are given by

$(x, y) = (V_{2n+1} + V_{2n}, V_{2n} + V_{2n-1})$ or $(x, y) = (V_{2n} - V_{2n-1}, V_{2n-1} - V_{2n-2})$ with $n \geq 1$.

4. SUGGESTIONS

In this study, we deal with some Diophantine equations. The case where k is not a prime number in these equations can be investigated separately as a new study. For this, the case where k is the product of two odd primes can be considered first. However, we believe that finding a general solution will be difficult.

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

REFERENCES

[1] Keskin, R., "Solutions of some quadratic Diophantine equations", Computer and Mathematics with Applications, 60(8): 2225-2230, (2010). DOI: 10.1016/j.camwa.2010.08.012

[2] Keskin, R., Güney, Duman, M., "Positive Integer Solutions of Some Pell Equations", Palestine Journal of Mathematics, 8: 213-226, (2019).

[3] Keskin, R., Güney, Duman, M., "Positive Integer Solutions of Some Second-Order Diophantine equations", Annale Ştiintifice ale Universitatii, "Alexandru Ioan Cuza", din Iaşi Matematica, (New Series), 3: 649-664, (2017).

[4] Ribenboim, P., "My Numbers, My Friends: Popular Lectures on Number Theory", Springer Verlag, 1-50, (2000).