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AN EXTENDED FRAMEWORK FOR BIHYPERBOLIC GENERALIZED TRIBONACCI NUMBERS

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Abstract. The aim of this article is to identify and analyze a new type special number system which is called bihyperbolic generalized Tribonacci numbers ($\mathcal{BGTN}$ for short). For this purpose, we give both classical and several new properties such as; recurrence relation, Binet formula, generating function, exponential generating function, summation formulae, matrix formula, and special determinant equations of $\mathcal{BGTN}$. Also, the system of $\mathcal{BGTN}$ is quite a big family and includes several type special cases with respect to initial values and r, s, t values, we give the subfamilies and special cases of it. In addition to these, we construct some numerical algorithms including recurrence relation and special two types determinant equations related to calculating the terms of this new type special number system. Then, we examine several properties by taking two special cases and including some illustrative numerical examples.

1. Introduction

Numbers and number systems are well-established fundamental and important topics in not only mathematics but also other disciplines with varied applications and benefits. In spite of their long history, numbers systems are still an interesting and important area to work for lots of researchers since there are several applications in different and several areas such as; differential geometry, engineering, robotics, graph theory, etc. There exist several types of number systems in the existing literature. A hyperbolic (perplex, split-complex) number is a number of the form $z = x + yj$ where $x, y \in \mathbb{R}$, $j^2 = 1, j \neq \pm 1$, $j \notin \mathbb{R}$ [39, 43, 61]. Also, a bihyperbolic

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number (canonical hyperbolic quaternion [10], hyperbolic four complex numbers [35]) is written as a linear combination of a pair of hyperbolic numbers. There exists a relationship between the bihyperbolic numbers and 4–dimensional pseudo-Euclidean spaces. Bihyperbolic numbers are denoted by $\mathcal{H}$ and are defined as [4, 10, 35, 37]:

$$\mathcal{H} := \{\zeta = \rho_0 + \rho_1 j_1 + \rho_2 j_2 + \rho_3 j_3 : \rho_0, \rho_1, \rho_2, \rho_3 \in \mathbb{R},\ j_1, j_2, j_3 \notin \mathbb{R}\},$$

where j_1, j_2, j_3 satisfy the multiplication rules:

$$j_1^2 = j_2^2 = j_3^2 = 1, \quad j_1 j_2 = j_2 j_1 = j_3, \quad j_1 j_3 = j_3 j_1 = j_2, \quad j_2 j_3 = j_3 j_2 = j_1. \tag{1}$$

On the other hand, several studies have been done and are ongoing on the special recurrence sequences which can have different orders. For example, Fibonacci and Lucas sequences [18,33] can be given as examples related to second-order recurrence sequences. The most general form of the second-order recurrence sequences is called as Horadam [26]. In this study, we deal with the generalization of third-order recurrence sequences which is called generalized Tribonacci sequence (or numbers). Generalized Tribonacci sequence $\{T_n(T_0, T_1, T_2; r, s, t)\}_{n\geq 0}$ (for short: $\{T_n\}_{n\geq 0}$) given by the following recurrence relation:

$$T_n = rT_{n-1} + sT_{n-2} + tT_{n-3}, \quad n \geq 3 \tag{2}$$

with the initial conditions $T_0 = a, T_1 = b, T_2 = c$ are arbitrary integers and r, s, t are real numbers [11]. Generalization of special third-order numbers was studied in [1, 12, 13, 15, 16, 19, 20, 36, 40–42, 44–54, 59, 60, 62].

Furthermore, the framework of bringing together the quaternions and special recurrence sequences is a popular and interesting concept for researchers. Real quaternions were investigated by W. R. Hamilton as an expansion of the complex numbers [23, 24] (see also Section "Conclusion"). There exist several studies with respect to combining several different types quaternions such as; split [17], generalized [29, 30, 34, 38], etc. Additionally, special recurrence sequences were examined considering quaternions, for instance, Fibonacci and Lucas real quaternions [20, 22, 25, 27], Fibonacci and Lucas generalized quaternions [2, 20], Narayana (or Fibonacci-Narayana) generalized quaternions [20]. Besides, the researchers started to examine the combining the third-order recurrence sequences and several types quaternions, such as; Padovan and Perrin quaternions [21, 28, 58], generalized Tribonacci real quaternions [11]. Also, generalized bicomplex Tribonacci quaternions were introduced in [32].

In the same manner, a great deal of researchers started to investigate the bihyperbolic numbers with several special recurrence sequences. Studies on bihyperbolic numbers, and bringing together the bihyperbolic numbers and some special recurrence numbers have been gathered speed in the existing literature. Bród et al. studied the generalization of bihyperbolic Pell numbers in [5]. Also, Bród et al. examined the one-parameter and two-parameter generalizations of the bihyperbolic

Jacobsthal numbers in [6, 7], respectively. Then, bihyperbolic numbers of the Fibonacci type and idempotent representation of them were investigated in [8]. In [9], some combinatorial properties of bihyperbolic numbers of the Fibonacci type are investigated. Azak examined some new identities related to bihyperbolic Fibonacci and Lucas numbers in [3]. Further, Fibonacci and Lucas bihypernomials [55] and certain bihypernomials with respect to Pell and Pell-Lucas numbers [56] examined.

In this study, we investigate a new type of number system which is called as bihyperbolic generalized Tribonacci numbers ($\mathcal{BGTN}$) and give some special cases with respect to the initial and r, s, t values. Then, we obtain the recurrence relation, Binet formula, generating function, exponential generating function, summation formulae, several new special properties, matrix formula, and special determinant equations related to these new types special numbers. Moreover, we establish some numerical algorithms including recurrence relation and special two types determinant equations related to calculating the terms of $\mathcal{BGTN}$. As a final part, we review the overall conclusions and give several contributions for future studies.

2. Basic Concepts

In this section, we give some background about bihyperbolic numbers and generalized Tribonacci numbers.

The addition and multiplication operations are commutative and associative on $\mathcal{H}$. $(\mathcal{H}, +, .)$ is a commutative ring [4]. Besides, a bihyperbolic number $\zeta = \rho_0 + \rho_1 j_1 + \rho_2 j_2 + \rho_3 j_3 \in \mathcal{H}$ has three conjugations, as follows:

$$\begin{cases} \overline{\zeta}^{j_1} = \rho_0 + \rho_1 j_1 - \rho_2 j_2 - \rho_3 j_3, \\ \overline{\zeta}^{j_2} = \rho_0 - \rho_1 j_1 + \rho_2 j_2 - \rho_3 j_3, \\ \overline{\zeta}^{j_3} = \rho_0 - \rho_1 j_1 - \rho_2 j_2 + \rho_3 j_3, \end{cases}$$

which are called as the principal conjugations of ζ [10].

Additionally, the characteristic equation of generalized Tribonacci numbers given in Eq. (2) is $x^3 - rx^2 - sx - t = 0$. The roots of this equation are given as follows:

$$x_1 = \frac{r}{3} + \alpha + \beta, \quad x_2 = \frac{r}{3} + \varepsilon\alpha + \varepsilon^2\beta, \quad x_3 = \frac{r}{3} + \varepsilon^2\alpha + \varepsilon\beta, \tag{3}$$

where

$$\begin{cases} \alpha = \sqrt[3]{\dfrac{r^3}{27} + \dfrac{rs}{6} + \dfrac{t}{2} + \sqrt{\mu}}, \\ \beta = \sqrt[3]{\dfrac{r^3}{27} + \dfrac{rs}{6} + \dfrac{t}{2} - \sqrt{\mu}}, \\ \varepsilon = \dfrac{-1 + i\sqrt{3}}{2}, \\ \mu = \dfrac{r^3 t}{27} - \dfrac{r^2 s^2}{108} + \dfrac{rst}{6} - \dfrac{s^3}{27} + \dfrac{t^2}{4}, \end{cases}$$

and

$$x_1 + x_2 + x_3 = r, \quad x_1x_2 + x_1x_3 + x_2x_3 = -s, \quad x_1x_2x_3 = t.$$

Providing $\mu > 0$, Eq. (2) has one real and two non-real solutions, the latter being conjugate complex. The following equation is called as Binet formula for generalized Tribonacci numbers [11]:

$$T_n = \frac{\widetilde{P}x_1^n}{(x_1 - x_2)(x_1 - x_3)} + \frac{\widetilde{R}x_2^n}{(x_2 - x_1)(x_2 - x_3)} + \frac{\widetilde{S}x_3^n}{(x_3 - x_1)(x_3 - x_2)}, \tag{4}$$

where

$$\begin{cases} \widetilde{P} = c - (x_2 + x_3)b + x_2x_3a, \\ \widetilde{R} = c - (x_1 + x_3)b + x_1x_3a, \\ \widetilde{S} = c - (x_1 + x_2)b + x_1x_2a. \end{cases} \tag{5}$$

Besides, the quite beneficial and functional method to generate T_n is applying S-matrix which is determined in [41, 59] and is a generalization of the R-matrix. The S-matrix is determined as follows (see also [31, 60]):

$$S = \begin{pmatrix} r & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

In Table 1, some special subfamilies (9 pieces) of generalized Tribonacci numbers are given with respect to r, s, t values. Additionally, Table 2 includes several members of the family of generalized Tribonacci numbers (38 pieces) regarding both initial values and r, s, t values [1, 12, 13, 15, 16, 19, 20, 36, 40–42, 44–54, 59, 60, 62].

TABLE 1. A brief classification for generalized Tribonacci numbers

Name	$\{T_n\} = \{T_n(T_0, T_1, T_2; r, s, t)\}$	**Recurrence Relation**
G. Tribonacci (usual)	$\{\mathscr{A}_n\} = \{T_n(T_0, T_1, T_2; 1, 1, 1)\}$	$\mathscr{A}_n = \mathscr{A}_{n-1} + \mathscr{A}_{n-2} + \mathscr{A}_{n-3}$
G. Padovan	$\{\mathscr{G}_n\} = \{T_n(T_0, T_1, T_2; 0, 1, 1)\}$	$\mathscr{G}_n = \mathscr{G}_{n-2} + \mathscr{G}_{n-3}$
G. Pell-Padovan	$\{\mathscr{M}_n\} = \{T_n(T_0, T_1, T_2; 0, 2, 1)\}$	$\mathscr{M}_n = 2\mathscr{M}_{n-2} + \mathscr{M}_{n-3}$
G. T. Pell	$\{\mathscr{S}_n\} = \{T_n(T_0, T_1, T_2; 2, 1, 1)\}$	$\mathscr{S}_n = 2\mathscr{S}_{n-1} + \mathscr{S}_{n-2} + \mathscr{S}_{n-3}$
G. T. Jacobsthal	$\{\mathscr{X}_n\} = \{T_n(T_0, T_1, T_2; 1, 1, 2)\}$	$\mathscr{X}_n = \mathscr{X}_{n-1} + \mathscr{X}_{n-2} + 2\mathscr{X}_{n-3}$
G. Jacobsthal-Padovan	$\{\boldsymbol{\chi}_n\} = \{T_n(T_0, T_1, T_2; 0, 1, 2)\}$	$\boldsymbol{\chi}_n = \boldsymbol{\chi}_{n-2} + 2\boldsymbol{\chi}_{n-3}$
G. Narayana	$\{\boldsymbol{\vartheta}_n\} = \{T_n(T_0, T_1, T_2; 1, 0, 1)\}$	$\boldsymbol{\vartheta}_n = \boldsymbol{\vartheta}_{n-1} + \boldsymbol{\vartheta}_{n-3}$
G. 3-primes	$\{\boldsymbol{\kappa}_n\} = \{T_n(T_0, T_1, T_2; 2, 3, 5)\}$	$\boldsymbol{\kappa}_n = 2\boldsymbol{\kappa}_{n-1} + 3\boldsymbol{\kappa}_{n-1} + 5\boldsymbol{\kappa}_{n-3}$
G. Reverse 3-primes	$\{\boldsymbol{\nabla}_n\} = \{T_n(T_0, T_1, T_2; 5, 3, 2)\}$	$\boldsymbol{\nabla}_n = 5\boldsymbol{\nabla}_{n-1} + 3\boldsymbol{\nabla}_{n-1} + 2\boldsymbol{\nabla}_{n-3}$

*G.: Generalized, T.: Third Order

Table 2. Some special cases of generalized Tribonacci numbers

Name	$\boldsymbol{\{T_n\} = \{T_n(T_0, T_1, T_2; r, s, t)\}}$	**Recurrence Relation**
Tribonacci	$\{A_n\} = \{T_n(0,1,1;1,1,1)\}$	$A_n = A_{n-1} + A_{n-2} + A_{n-3}$
Tribonacci-Lucas	$\{B_n\} = \{T_n(3,1,3;1,1,1)\}$	$B_n = B_{n-1} + B_{n-2} + B_{n-3}$
Tribonacci-Perrin	$\{C_n\} = \{T_n(3,0,2;1,1,1)\}$	$C_n = C_{n-1} + C_{n-2} + C_{n-3}$
M. Tribonacci	$\{D_n\} = \{T_n(1,1,1;1,1,1)\}$	$D_n = D_{n-1} + D_{n-2} + D_{n-3}$
M. Tribonacci-Lucas	$\{E_n\} = \{T_n(4,4,10;1,1,1)\}$	$E_n = E_{n-1} + E_{n-2} + E_{n-3}$
A. Tribonacci-Lucas	$\{F_n\} = \{T_n(4,2,0;1,1,1)\}$	$F_n = F_{n-1} + F_{n-2} + F_{n-3}$
Padovan (Cordonnier)	$\{G_n\} = \{T_n(1,1,1;0,1,1)\}$	$G_n = G_{n-2} + G_{n-3}$
Perrin	$\{H_n\} = \{T_n(3,0,2;0,1,1)\}$	$H_n = H_{n-2} + H_{n-3}$
Van der Laan	$\{I_n\} = \{T_n(1,0,1;0,1,1)\}$	$I_n = I_{n-2} + I_{n-3}$
Padovan-Perrin	$\{J_n\} = \{T_n(0,0,1;0,1,1)\}$	$J_n = J_{n-2} + J_{n-3}$
M. Padovan	$\{K_n\} = \{T_n(3,1,3;0,1,1)\}$	$K_n = K_{n-2} + K_{n-3}$
A. Padovan	$\{L_n\} = \{T_n(0,1,0;0,1,1)\}$	$L_n = L_{n-2} + L_{n-3}$
Pell-Padovan	$\{M_n\} = \{T_n(1,1,1;0,2,1)\}$	$M_n = 2M_{n-2} + M_{n-3}$
Pell-Perrin	$\{N_n\} = \{T_n(3,0,2;0,2,1)\}$	$N_n = 2N_{n-2} + N_{n-3}$
T. Fibonacci-Pell	$\{O_n\} = \{T_n(1,0,2;0,2,1)\}$	$O_n = 2O_{n-2} + O_{n-3}$
T. Lucas-Pell	$\{P_n\} = \{T_n(3,0,4;0,2,1)\}$	$P_n = 2P_{n-2} + P_{n-3}$
A. Pell-Padovan	$\{R_n\} = \{T_n(0,1,0;0,2,1)\}$	$R_n = 2R_{n-2} + R_{n-3}$
T. Pell	$\{S_n\} = \{T_n(0,1,2;2,1,1)\}$	$S_n = 2S_{n-1} + S_{n-2} + S_{n-3}$
T. Pell-Lucas	$\{U_n\} = \{T_n(3,2,6;2,1,1)\}$	$U_n = 2U_{n-1} + U_{n-2} + U_{n-3}$
T. modified Pell	$\{V_n\} = \{T_n(0,1,1;2,1,1)\}$	$V_n = 2V_{n-1} + V_{n-2} + V_{n-3}$
T. Pell-Perrin	$\{W_n\} = \{T_n(3,0,2;2,1,1)\}$	$W_n = 2W_{n-1} + W_{n-2} + W_{n-3}$
T. Jacobsthal	$\{X_n\} = \{T_n(0,1,1;1,1,2)\}$	$X_n = X_{n-1} + X_{n-2} + 2X_{n-3}$
T. Jacobsthal-Lucas	$\{Y_n\} = \{T_n(2,1,5;1,1,2)\}$	$Y_n = Y_{n-1} + Y_{n-2} + 2Y_{n-3}$
M. T. Jacobsthal	$\{Z_n\} = \{T_n(3,1,3;1,1,2)\}$	$Z_n = Z_{n-1} + Z_{n-2} + 2Z_{n-3}$
T. Jacobsthal-Perrin	$\{\Gamma_n\} = \{T_n(3,0,2;1,1,2)\}$	$\Gamma_n = \Gamma_{n-1} + \Gamma_{n-2} + 2\Gamma_{n-3}$
Jacobsthal-Padovan	$\{\chi_n\} = \{T_n(1,1,1;0,1,2)\}$	$\chi_n = \chi_{n-2} + 2\chi_{n-3}$
Jacobsthal-Perrin	$\{\Delta_n\} = \{T_n(3,0,2;0,1,2)\}$	$\Delta_n = \Delta_{n-2} + 2\Delta_{n-3}$
A. Jacobsthal-Padovan	$\{\omega_n\} = \{T_n(0,1,0;0,1,2)\}$	$\omega_n = \omega_{n-2} + 2\omega_{n-3}$
M. Jacobsthal-Padovan	$\{\Omega_n\} = \{T_n(3,1,3;0,1,2)\}$	$\Omega_n = \Omega_{n-2} + 2\Omega_{n-3}$
Narayana	$\{\vartheta_n\} = \{T_n(0,1,1;1,0,1)\}$	$\vartheta_n = \vartheta_{n-1} + \vartheta_{n-3}$
Narayana-Lucas	$\{\tau_n\} = \{T_n(3,1,1;1,0,1)\}$	$\tau_n = \tau_{n-1} + \tau_{n-3}$
Narayana-Perrin	$\{\sigma_n\} = \{T_n(3,0,2;1,0,1)\}$	$\sigma_n = \sigma_{n-1} + \sigma_{n-3}$
3-primes	$\{\kappa_n\} = \{T_n(0,1,2;2,3,5)\}$	$\kappa_n = 2\kappa_{n-1} + 3\kappa_{n-2} + 5\kappa_{n-3}$
Lucas 3-primes	$\{\theta_n\} = \{T_n(3,2,10;2,3,5)\}$	$\theta_n = 2\theta_{n-1} + 3\theta_{n-2} + 5\theta_{n-3}$
M. 3-primes	$\{\gamma_n\} = \{T_n(0,1,1;2,3,5)\}$	$\gamma_n = 2\gamma_{n-1} + 3\gamma_{n-2} + 5\gamma_{n-3}$
Reverse 3-primes	$\{\nabla_n\} = \{T_n(0,1,5;5,3,2)\}$	$\nabla_n = 5\nabla_{n-1} + 3\nabla_{n-2} + 2\nabla_{n-3}$
Reverse Lucas 3-primes	$\{\Lambda_n\} = \{T_n(3,5,31;5,3,2)\}$	$\Lambda_n = 5\Lambda_{n-1} + 3\Lambda_{n-2} + 2\Lambda_{n-3}$
Reverse M. 3-primes	$\{\phi_n\} = \{T_n(0,1,4;5,3,2)\}$	$\phi_n = 5\phi_{n-1} + 3\phi_{n-2} + 2\phi_{n-3}$

*M.: Modified, A.: Adjusted, T.: Third order

3. The Bihyperbolic Generalized Tribonacci Numbers

In this section, we introduce bihyperbolic generalized Tribonacci numbers ($\mathcal{BGTN}$) by taking into account several special cases with respect to r, s, t values, and initial values. Besides, we scrutinize not only classical several properties but also some new and interesting equations. Then, we support these new results with some numerical algorithms. Finally, we examine two special cases of $\mathcal{BGTN}$.

Definition 1. *The nth $\mathcal{BGTN}$ is defined as:*

$$\mathfrak{T}_n = T_n + T_{n+1}j_1 + T_{n+2}j_2 + T_{n+3}j_3, \quad n \geq 0 \tag{6}$$

with the initial values

$$\begin{cases} \mathfrak{T}_0 = a + bj_1 + cj_2 + (rc + sb + ta)j_3, \\ \mathfrak{T}_1 = b + cj_1 + (rc + sb + ta)j_2 + \left(\left(r^2 + s\right)c + (rs + t)\,b + rta\right)j_3, \\ \mathfrak{T}_2 = c + (rc + sb + ta)\,j_1 + \left(\left(r^2 + s\right)c + (rs + t)\,b + rta\right)j_2 \\ \qquad + \left(\left(r^3 + 2rs + t\right)c + \left(r^2 s + s^2 + rt\right)b + \left(r^2 t + st\right)a\right)j_3, \end{cases}$$

where the rules of j_1, j_2, j_3 are given in Eq. (1) *and T_n is the nth generalized Tribonacci number given in Eq.* (2)*.*

In the following Definition 2, we give some basic algebraic properties such as; equality, summation, subtraction, multiplication with a constant (a constant is a real number), multiplication of any two $\mathcal{BGTN}$, and also three types principal conjugations of $\mathcal{BGTN}$.

Definition 2 (Algebraic Properties)**.** *Let $\mathfrak{T}_n$ and $\mathfrak{T}_m$ be the nth and mth $\mathcal{BGTN}$, respectively. Then, the followings are defined:*

- ***Equality:***

$$\mathfrak{T}_n = \mathfrak{T}_m \Leftrightarrow \quad T_n = T_m, \quad T_{n+1} = T_{m+1}, \quad T_{n+2} = T_{m+2}, \quad T_{n+3} = T_{m+3},$$

- ***Addition/Subtraction:***

$$\mathfrak{T}_n \pm \mathfrak{T}_m = T_n \pm T_m + (T_{n+1} \pm T_{m+1})j_1 + (T_{n+2} \pm T_{m+2})j_2 + (T_{n+3} \pm T_{m+3})j_3,$$

- ***Multiplication by a scalar:***

$$\upsilon\mathfrak{T}_n = \upsilon T_n + \upsilon T_{n+1}j_1 + \upsilon T_{n+2}j_2 + \upsilon T_{n+3}j_3, \quad \upsilon \in \mathbb{R},$$

- ***Multiplication:***

$$\begin{aligned} \mathfrak{T}_n\mathfrak{T}_m = & T_nT_m + T_{n+1}T_{m+1} + T_{n+2}T_{m+2} + T_{n+3}T_{m+3} \\ & + (T_nT_{m+1} + T_{n+1}T_m + T_{n+2}T_{m+3} + T_{n+3}T_{m+2})j_1 \\ & + (T_nT_{m+2} + T_{n+1}T_{m+3} + T_{n+2}T_m + T_{n+3}T_{m+1})j_2 \\ & + (T_nT_{m+3} + T_{n+1}T_{m+2} + T_{n+2}T_{m+1} + T_{n+3}T_m)j_3, \end{aligned}$$

by using the rules in Eq. (1) *for multiplication.*

• ***Principal Conjugates:*** *Also, the following three types principal conjugations of $\mathcal{T}_n$ are defined by:*

$$\begin{cases} \overline{\mathcal{T}_n}^{j_1} = T_n + T_{n+1}j_1 - T_{n+2}j_2 - T_{n+3}j_3, \\ \overline{\mathcal{T}_n}^{j_2} = T_n - T_{n+1}j_1 + T_{n+2}j_2 - T_{n+3}j_3, \\ \overline{\mathcal{T}_n}^{j_3} = T_n - T_{n+1}j_1 - T_{n+2}j_2 + T_{n+3}j_3. \end{cases} \tag{7}$$

Now, let us give the recurrence relation of $\mathcal{BGTN}$.

Theorem 1 (Recurrence Relation)**.** *Let $\mathcal{T}_n$ be the nth $\mathcal{BGTN}$. Then, the following recurrence relation is satisfied:*

$$\mathcal{T}_n = r\mathcal{T}_{n-1} + s\mathcal{T}_{n-2} + t\mathcal{T}_{n-3}, \quad n \geq 3. \tag{8}$$

Proof. Using Eqs. (2) and (6), we complete the proof:

$$\begin{aligned} r\mathcal{T}_{n-1} + s\mathcal{T}_{n-2} + t\mathcal{T}_{n-3} &= r(T_{n-1} + T_n j_1 + T_{n+1}j_2 + T_{n+2}j_3) \\ &\quad + s(T_{n-2} + T_{n-1}j_1 + T_n j_2 + T_{n+1}j_3) \\ &\quad + t(T_{n-3} + T_{n-2}j_1 + T_{n-1}j_2 + T_n j_3) \\ &= T_n + T_{n+1}j_1 + T_{n+2}j_2 + T_{n+3}j_3 \\ &= \mathcal{T}_n. \end{aligned}$$

□

In the following, we construct a numerical algorithm (Algorithm 1) in order to calculate the *nth* term of $\mathcal{BGTN}$ based on the recurrence relation given in Eq. (8).

Algorithm 1 A numerical algorithm for finding *nth* term of $\mathcal{BGTN}$

1: Begin
2: Input $\mathcal{T}_0, \mathcal{T}_1$ and $\mathcal{T}_2$
3: Compose $\mathcal{T}_n$ with respect to Eq. (8) for every $n \geq 3$
4: Count up $\mathcal{T}_n$
5: Output $\mathcal{T}_n = T_n + T_{n+1}j_1 + T_{n+2}j_2 + T_{n+3}j_3$
6: Complete

With the same logic of Table 1 and Table 2 in Section "Basic Concepts", we can also obtain the same classifications and give special cases of $\mathcal{BGTN}$ in the following Table 3 and Table 4. The members of the $\mathcal{BGTN}$ which are written in Table 3 can be also classified and expressed in detail linked to Table 2 regarding recurrence relations and the initial values. For the sake of brevity, the small parts of them are written in Table 4 and Table 5 for readers to examine. The other members can be easily observed and examined, as well. The first three initial values for special cases written in Table 4 are given in Table 5.

TABLE 3. A brief classification for special cases of $\mathcal{BGTN}$

Name	Definition	Recurrence Relation
B. G. Tribonacci (usual)	$\widehat{\mathscr{A}_n} = \mathscr{A}_n + \mathscr{A}_{n+1}j_1 + \mathscr{A}_{n+2}j_2 + \mathscr{A}_{n+3}j_3$	$\widehat{\mathscr{A}_n} = \widehat{\mathscr{A}_{n-1}} + \widehat{\mathscr{A}_{n-2}} + \widehat{\mathscr{A}_{n-3}}$
B. G. Padovan	$\widehat{\mathscr{G}_n} = \mathscr{G}_n + \mathscr{G}_{n+1}j_1 + \mathscr{G}_{n+2}j_2 + \mathscr{G}_{n+3}j_3$	$\widehat{\mathscr{G}_n} = \widehat{\mathscr{G}_{n-2}} + \widehat{\mathscr{G}_{n-3}}$
B. G. Pell-Padovan	$\widehat{\mathscr{M}_n} = \mathscr{M}_n + \mathscr{M}_{n+1}j_1 + \mathscr{M}_{n+2}j_2 + \mathscr{M}_{n+3}j_3$	$\widehat{\mathscr{M}_n} = 2\widehat{\mathscr{M}_{n-2}} + \widehat{\mathscr{M}_{n-3}}$
B. G. T. Pell	$\widehat{\mathscr{S}_n} = \mathscr{S}_n + \mathscr{S}_{n+1}j_1 + \mathscr{S}_{n+2}j_2 + \mathscr{S}_{n+3}j_3$	$\widehat{\mathscr{S}_n} = 2\widehat{\mathscr{S}_{n-1}} + \widehat{\mathscr{S}_{n-2}} + \widehat{\mathscr{S}_{n-3}}$
B. G. T. Jacobsthal	$\widehat{\mathscr{X}_n} = \mathscr{X}_n + \mathscr{X}_{n+1}j_1 + \mathscr{X}_{n+2}j_2 + \mathscr{X}_{n+3}j_3$	$\widehat{\mathscr{X}_n} = \widehat{\mathscr{X}_{n-1}} + \widehat{\mathscr{X}_{n-2}} + 2\widehat{\mathscr{X}_{n-3}}$
B. G. Jacobsthal-Padovan	$\widehat{\boldsymbol{\chi}}_n = \boldsymbol{\chi}_n + \boldsymbol{\chi}_{n+1}j_1 + \boldsymbol{\chi}_{n+2}j_2 + \boldsymbol{\chi}_{n+3}j_3$	$\widehat{\boldsymbol{\chi}}_n = \widehat{\boldsymbol{\chi}}_{n-2} + 2\widehat{\boldsymbol{\chi}}_{n-3}$
B. G. Narayana	$\widehat{\boldsymbol{\vartheta}}_n = \boldsymbol{\vartheta}_n + \boldsymbol{\vartheta}_{n+1}j_1 + \boldsymbol{\vartheta}_{n+2}j_2 + \boldsymbol{\vartheta}_{n+3}j_3$	$\widehat{\boldsymbol{\vartheta}}_n = \widehat{\boldsymbol{\vartheta}}_{n-1} + \widehat{\boldsymbol{\vartheta}}_{n-3}$
B. G. 3-primes	$\widehat{\boldsymbol{\kappa}}_n = \boldsymbol{\kappa}_n + \boldsymbol{\kappa}_{n+1}j_1 + \boldsymbol{\kappa}_{n+2}j_2 + \boldsymbol{\kappa}_{n+3}j_3$	$\widehat{\boldsymbol{\kappa}}_n = 2\widehat{\boldsymbol{\kappa}}_{n-1} + 3\widehat{\boldsymbol{\kappa}}_{n-2} + 5\widehat{\boldsymbol{\kappa}}_{n-3}$
B. G. Reverse 3-primes	$\widehat{\boldsymbol{\nabla}}_n = \boldsymbol{\nabla}_n + \boldsymbol{\nabla}_{n+1}j_1 + \boldsymbol{\nabla}_{n+2}j_2 + \boldsymbol{\nabla}_{n+3}j_3$	$\widehat{\boldsymbol{\nabla}}_n = 5\widehat{\boldsymbol{\nabla}}_{n-1} + 3\widehat{\boldsymbol{\nabla}}_{n-2} + 2\widehat{\boldsymbol{\nabla}}_{n-3}$

*B.: Bihyperbolic, G.: Generalized, T.: Third Order

TABLE 4. Some special cases of $\mathcal{BGTN}$

Name	Definition	Recurrence Relation
B. Tribonacci-Lucas	$\mathcal{B}_n = B_n + B_{n+1}j_1 + B_{n+2}j_2 + B_{n+3}j_3$	$\mathcal{B}_n = \mathcal{B}_{n-1} + \mathcal{B}_{n-2} + \mathcal{B}_{n-3}$
B. Perrin	$\mathcal{H}_n = H_n + H_{n+1}j_1 + H_{n+2}j_2 + H_{n+3}j_3$	$\mathcal{H}_n = \mathcal{H}_{n-2} + \mathcal{H}_{n-3}$
B. Pell-Padovan	$\mathcal{M}_n = M_n + M_{n+1}j_1 + M_{n+2}j_2 + M_{n+3}j_3$	$\mathcal{M}_n = 2\mathcal{M}_{n-2} + \mathcal{M}_{n-3}$
B. T. Pell	$\mathcal{S}_n = S_n + S_{n+1}j_1 + S_{n+2}j_2 + S_{n+3}j_3$	$\mathcal{S}_n = 2\mathcal{S}_{n-1} + \mathcal{S}_{n-2} + \mathcal{S}_{n-3}$
B. T. Jacobsthal	$\mathcal{X}_n = X_n + X_{n+1}j_1 + X_{n+2}j_2 + X_{n+3}j_3$	$\mathcal{X}_n = \mathcal{X}_{n-1} + \mathcal{X}_{n-2} + 2\mathcal{X}_{n-3}$
B. Jacobsthal-Padovan	$\widetilde{\chi}_n = \chi_n + \chi_{n+1}j_1 + \chi_{n+2}j_2 + \chi_{n+3}j_3$	$\widetilde{\chi}_n = \widetilde{\chi}_{n-2} + 2\widetilde{\chi}_{n-3}$
B. Narayana	$\widetilde{\vartheta}_n = \vartheta_n + \vartheta_{n+1}j_1 + \vartheta_{n+2}j_2 + \vartheta_{n+3}j_3$	$\widetilde{\vartheta}_n = \widetilde{\vartheta}_{n-1} + \widetilde{\vartheta}_{n-3}$
B. 3-primes	$\widetilde{\kappa}_n = \kappa_n + \kappa_{n+1}j_1 + \kappa_{n+2}j_2 + \kappa_{n+3}j_3$	$\widetilde{\kappa}_n = 2\widetilde{\kappa}_{n-1} + 3\widetilde{\kappa}_{n-2} + 5\widetilde{\kappa}_{n-3}$
B. reverse 3-primes	$\widetilde{\nabla}_n = \nabla_n + \nabla_{n+1}j_1 + \nabla_{n+2}j_2 + \nabla_{n+3}j_3$	$\widetilde{\nabla}_n = 5\widetilde{\nabla}_{n-1} + 3\widetilde{\nabla}_{n-2} + 2\widetilde{\nabla}_{n-3}$

*B.: Bihyperbolic, T.: Third Order

TABLE 5. Initial values of special cases

For	$n=0$	$n=1$	$n=2$
$\mathcal{B}_n$	$3+j_1+3j_2+7j_3$	$1+3j_1+7j_2+11j_3$	$3+7j_1+11j_2+21j_3$
$\mathcal{H}_n$	$3+2j_2+3j_3$	$2j_1+3j_2+2j_3$	$2+3j_1+2j_2+5j_3$
$\mathcal{M}_n$	$1+j_1+j_2+3j_3$	$1+j_1+3j_2+3j_3$	$1+3j_1+3j_2+7j_3$
$\mathcal{S}_n$	$j_1+2j_2+5j_3$	$1+2j_1+5j_2+13j_3$	$2+5j_1+13j_2+33j_3$
$\mathcal{X}_n$	$j_1+j_2+2j_3$	$1+j_1+2j_2+5j_3$	$1+2j_1+5j_2+9j_3$
$\widetilde{\chi}_n$	$1+j_1+j_2+3j_3$	$1+j_1+3j_2+3j_3$	$1+3j_1+3j_2+5j_3$
$\widetilde{\vartheta}_n$	$j_1+j_2+j_3$	$1+j_1+j_2+2j_3$	$1+j_1+2j_2+3j_3$
$\widetilde{\kappa}_n$	$j_1+2j_2+7j_3$	$1+2j_1+7j_2+25j_3$	$2+7j_1+25j_2+81j_3$
$\widetilde{\nabla}_n$	$j_1+5j_2+28j_3$	$1+5j_1+28j_2+157j_3$	$5+28j_1+157j_2+879j_3$

Theorem 2. *$\forall n \in \mathbb{N}$, the Binet formula for the $\mathcal{BGTN}$ is as follows:*

$$\mathcal{T}_n = \frac{\widetilde{P}x_1^n\widetilde{x}_1}{(x_1-x_2)(x_1-x_3)} + \frac{\widetilde{R}x_2^n\widetilde{x}_2}{(x_2-x_1)(x_2-x_3)} + \frac{\widetilde{S}x_3^n\widetilde{x}_3}{(x_3-x_1)(x_3-x_2)},$$

where

$$\begin{cases} \widetilde{x}_1 = 1 + x_1j_1 + x_1^2j_2 + x_1^3j_3, \\ \widetilde{x}_2 = 1 + x_2j_1 + x_2^2j_2 + x_2^3j_3, \\ \widetilde{x}_3 = 1 + x_3j_1 + x_3^2j_2 + x_3^3j_3. \end{cases} \tag{9}$$

Here $\widetilde{P}, \widetilde{R}, \widetilde{S}$ are given in Eq. (5) *and x_1, x_2, x_3 are given in Eq.* (3)*.*

Proof. Using Eqs. (4) and (6), we manage to prove:

$$\begin{aligned} \mathcal{T}_n = & \tfrac{\widetilde{P}x_1^n}{(x_1-x_2)(x_1-x_3)} + \tfrac{\widetilde{R}x_2^n}{(x_2-x_1)(x_2-x_3)} + \tfrac{\widetilde{S}x_3^n}{(x_3-x_1)(x_3-x_2)} \\ & + \left(\tfrac{\widetilde{P}x_1^{n+1}}{(x_1-x_2)(x_1-x_3)} + \tfrac{\widetilde{R}x_2^{n+1}}{(x_2-x_1)(x_2-x_3)} + \tfrac{\widetilde{S}x_3^{n+1}}{(x_3-x_1)(x_3-x_2)}\right) j_1 \\ & + \left(\tfrac{\widetilde{P}x_1^{n+2}}{(x_1-x_2)(x_1-x_3)} + \tfrac{\widetilde{R}x_2^{n+2}}{(x_2-x_1)(x_2-x_3)} + \tfrac{\widetilde{S}x_3^{n+2}}{(x_3-x_1)(x_3-x_2)}\right) j_2 \\ & + \left(\tfrac{\widetilde{P}x_1^{n+3}}{(x_1-x_2)(x_1-x_3)} + \tfrac{\widetilde{R}x_2^{n+3}}{(x_2-x_1)(x_2-x_3)} + \tfrac{\widetilde{S}x_3^{n+3}}{(x_3-x_1)(x_3-x_2)}\right) j_3 . \end{aligned}$$

Finally, we reach $\mathcal{T}_n = \frac{\widetilde{P}x_1^n\widetilde{x}_1}{(x_1-x_2)(x_1-x_3)} + \frac{\widetilde{R}x_2^n\widetilde{x}_2}{(x_2-x_1)(x_2-x_3)} + \frac{\widetilde{S}x_3^n\widetilde{x}_3}{(x_3-x_1)(x_3-x_2)}$. □

Theorem 3. *The generating function of $\mathcal{BGTN}$ is as follows:*

$$\sum_{n=0}^{\infty} \mathcal{T}_n x^n = \frac{\mathcal{T}_0 + (\mathcal{T}_1 - r\mathcal{T}_0)\,x + (\mathcal{T}_2 - r\mathcal{T}_1 - s\mathcal{T}_0)\,x^2}{1 - rx - sx^2 - tx^3}. \tag{10}$$

Proof. Let the following function

$$G(x) = \sum_{n=0}^{\infty} \mathcal{T}_n x^n = \mathcal{T}_0 + \mathcal{T}_1 x + \mathcal{T}_2 x^2 + \ldots + \mathcal{T}_n x^n + \ldots$$

be generating function of $\mathcal{T}_n$. Then, if both sides of this equation are multiplied by rx, sx^2, tx^3, the followings are obtained:

$$rxG(x) = r\mathcal{T}_0 x + r\mathcal{T}_1 x^2 + r\mathcal{T}_2 x^3 + \ldots + r\mathcal{T}_n x^{n+1} + \ldots$$
$$sx^2G(x) = s\mathcal{T}_0 x^2 + s\mathcal{T}_1 x^3 + s\mathcal{T}_2 x^4 + \ldots + s\mathcal{T}_n x^{n+2} + \ldots$$
$$tx^3G(x) = t\mathcal{T}_0 x^3 + t\mathcal{T}_1 x^4 + t\mathcal{T}_2 x^5 + \ldots + t\mathcal{T}_n x^{n+3} + \ldots$$

Then, by using Eq. (8), we get:

$$(1 - rx - sx^2 - tx^3)G(x) = \mathcal{T}_0 + (\mathcal{T}_1 - r\mathcal{T}_0)x + (\mathcal{T}_2 - r\mathcal{T}_1 - s\mathcal{T}_0)x^2.$$

Consequently, we obtain Eq. (10). □

Theorem 4. *The exponential generating function of $\mathcal{BGTN}$ is as follows:*

$$\sum_{n=0}^{\infty} \mathcal{T}_n \frac{y^n}{n!} = \frac{\widetilde{P}\widetilde{x}_1 e^{x_1 y}}{(x_1 - x_2)(x_1 - x_3)} + \frac{\widetilde{R}\widetilde{x}_2 e^{x_2 y}}{(x_2 - x_1)(x_2 - x_3)} + \frac{\widetilde{S}\widetilde{x}_3 e^{x_3 y}}{(x_3 - x_1)(x_3 - x_2)}$$

(see $\widetilde{x}_1, \widetilde{x}_2$ and $\widetilde{x}_3$ in Eq. (9)).

Proof. By using Eq. (2), we get:

$$\begin{aligned}
&\sum_{n=0}^{\infty} \mathcal{T}_n \frac{y^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\frac{\widetilde{P} x_1^n \widetilde{x}_1}{(x_1 - x_2)(x_1 - x_3)} + \frac{\widetilde{R} x_2^n \widetilde{x}_2}{(x_2 - x_1)(x_2 - x_3)} + \frac{\widetilde{S} x_3^n \widetilde{x}_3}{(x_3 - x_1)(x_3 - x_2)} \right) \frac{y^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{\widetilde{P} x_1^n \widetilde{x}_1}{(x_1 - x_2)(x_1 - x_3)} \frac{y^n}{n!} + \sum_{n=0}^{\infty} \frac{\widetilde{R} x_2^n \widetilde{x}_2}{(x_2 - x_1)(x_2 - x_3)} \frac{y^n}{n!} \\
&\quad + \sum_{n=0}^{\infty} \frac{\widetilde{S} x_3^n \widetilde{x}_3}{(x_3 - x_1)(x_3 - x_2)} \frac{y^n}{n!} \\
&= \frac{\widetilde{P}\widetilde{x}_1}{(x_1 - x_2)(x_1 - x_3)} \sum_{n=0}^{\infty} \frac{(x_1 y)^n}{n!} + \frac{\widetilde{R}\widetilde{x}_2}{(x_2 - x_1)(x_2 - x_3)} \sum_{n=0}^{\infty} \frac{(x_2 y)^n}{n!} \\
&\quad + \frac{\widetilde{S}\widetilde{x}_3}{(x_3 - x_1)(x_3 - x_2)} \sum_{n=0}^{\infty} \frac{(x_3 y)^n}{n!} \\
&= \frac{\widetilde{P}\widetilde{x}_1 e^{x_1 y}}{(x_1 - x_2)(x_1 - x_3)} + \frac{\widetilde{R}\widetilde{x}_2 e^{x_2 y}}{(x_2 - x_1)(x_2 - x_3)} + \frac{\widetilde{S}\widetilde{x}_3 e^{x_3 y}}{(x_3 - x_1)(x_3 - x_2)}.
\end{aligned}$$

The proof is completed. □

Thanks to the study [52], we can get the summation formulae for $\mathcal{BGTN}$ in the following theorem. The proof is omitted due to the fact that it can be completed with mathematical induction, easily.

Theorem 5. *$\forall m \in \mathbb{N}$, the following summation formulae for $\mathcal{BGTN}$ are satisfied:*

$$\text{(i)}\ \sum_{n=0}^{m} \mathfrak{T}_n = \frac{\begin{array}{l}\mathfrak{T}_{m+3} + (1-r)\mathfrak{T}_{m+2} + (1-r-s)\mathfrak{T}_{m+1} - \mathfrak{T}_2 + (r-1)\mathfrak{T}_1 \\ +(r+s-1)\mathfrak{T}_0\end{array}}{r+s+t-1},$$

$$\text{(ii)}\ \sum_{n=0}^{m} \mathfrak{T}_{2n} = \frac{\begin{array}{l}(1-s)\mathfrak{T}_{2m+2} + (t+rs)\mathfrak{T}_{2m+1} + (t^2+rt)\mathfrak{T}_{2m} + (s-1)\mathfrak{T}_2 \\ +(-t-rs)\mathfrak{T}_1 + (r^2-s^2+rt+2s-1)\mathfrak{T}_0\end{array}}{(r+s+t-1)(r-s+t+1)},$$

$$\text{(iii)}\ \sum_{n=0}^{m} \mathfrak{T}_{2n+1} = \frac{\begin{array}{l}(r+t)\mathfrak{T}_{2m+2} + (s-s^2+t^2+rt)\mathfrak{T}_{2m+1} + (t-st)\mathfrak{T}_{2m} \\ +(-r-t)\mathfrak{T}_2 + (-1+s+r^2+rt)\mathfrak{T}_1 + (-t+st)\mathfrak{T}_0\end{array}}{(r-s+t+1)(r+s+t-1)},$$

where denominators are not equal to zero.

Particular Case 1. *If $s = 1$, we can get the following summation formulae for special cases of part (ii) and (iii) of the previous Theorem 5:*

$$\text{(i)}\ \sum_{n=0}^{m} \mathfrak{T}_{2n} = \frac{\mathfrak{T}_{2m+1} + t\mathfrak{T}_{2m} - \mathfrak{T}_1 + r\mathfrak{T}_0}{r+t},$$

$$\text{(ii)}\ \sum_{n=0}^{m} \mathfrak{T}_{2n+1} = \frac{\mathfrak{T}_{2m+2} + t\mathfrak{T}_{2m+1} - \mathfrak{T}_2 + r\mathfrak{T}_1}{r+t},$$

where denominators are not equal to zero.

Thanks to the study [11], we get the following Theorem 6:

Theorem 6. *$\forall m \in \mathbb{N}$, the following summation property holds for $\mathcal{BGTN}$:*

$$\sum_{n=0}^{m} \mathfrak{T}_n = \frac{\mathfrak{T}_{m+2} + (1-r)\mathfrak{T}_{m+1} + t\mathfrak{T}_m + \eta}{\delta},$$

where

$$\begin{cases} \delta = r+s+t-1, \\ \lambda = (r+s-1)a + (r-1)b - c, \\ \eta = \lambda + (\lambda - \delta a)j_1 + (\lambda - \delta(a+b))j_2 + (\lambda - \delta(a+b+c))j_3. \end{cases}$$

Proof. Using Eq. (6) and utilizing the Lemma 2.3 on page 6 in the study [11], then we can complete the proof:

$$\begin{aligned}
\sum_{n=0}^{m}\mathfrak{T}_n =& \sum_{n=0}^{m}\left(T_n + T_{n+1}j_1 + T_{n+2}j_2 + T_{n+3}j_3\right)\\
=& \sum_{n=0}^{m}T_n + \sum_{n=0}^{m}T_{n+1}j_1 + \sum_{n=0}^{m}T_{n+2}j_2 + \sum_{n=0}^{m}T_{n+3}j_3\\
=& \frac{1}{\delta}\begin{bmatrix} T_{m+2} + (1-r)T_{m+1} + tT_m + \lambda \\ + \left(T_{m+3} + (1-r)T_{m+2} + tT_{m+1} + \lambda - \delta a\right)j_1 \\ + \left(T_{m+4} + (1-r)T_{m+3} + tT_{m+2} + \lambda - \delta\left(a+b\right)\right)j_2 \\ + \left(T_{m+5} + (1-r)T_{m+4} + tT_{m+3} + \lambda - \delta\left(a+b+c\right)\right)j_3 \end{bmatrix}\\
=& \frac{\mathfrak{T}_{m+2} + (1-r)\mathfrak{T}_{m+1} + t\mathfrak{T}_m + \eta}{\delta}.
\end{aligned}$$

We get the desired result. □

Theorem 7. $\forall n \in \mathbb{N}$, *the following properties are satisfied:*

(i) $\mathfrak{T}_n + \overline{\mathfrak{T}}_n^{\,j_1} = 2(T_n + T_{n+1}j_1)$,
(ii) $\mathfrak{T}_n + \overline{\mathfrak{T}}_n^{\,j_2} = 2(T_n + T_{n+2}j_2)$,
(iii) $\mathfrak{T}_n + \overline{\mathfrak{T}}_n^{\,j_3} = 2(T_n + T_{n+3}j_2)$.

Proof. (i) Using Eqs. (6) and (7), the proof is completed as:

$$\begin{aligned}
\mathfrak{T}_n + \overline{\mathfrak{T}}_n^{\,j_1} =& T_n + T_{n+1}j_1 + T_{n+2}j_2 + T_{n+3}j_3 + T_n + T_{n+1}j_1 - T_{n+2}j_2 - T_{n+3}j_3\\
=& 2(T_n + T_{n+1}j_1).
\end{aligned}$$

By the same way, the other parts can be obtained. □

Theorem 8. $\forall n \in \mathbb{N}$, *the following property holds:*

$$\mathfrak{T}_n - \mathfrak{T}_{n+1}j_1 - \mathfrak{T}_{n+2}j_2 - \mathfrak{T}_{n+3}j_3 = T_n - T_{n+2} - T_{n+4} + T_{n+6} - 2\mathfrak{T}_{n+3}j_3.$$

Proof. Using Eqs. (6) and (1), we have:

$$\begin{aligned}
\mathfrak{T}_n - \mathfrak{T}_{n+1}j_1 - \mathfrak{T}_{n+2}j_2 - \mathfrak{T}_{n+3}j_3 =& T_n + T_{n+1}j_1 + T_{n+2}j_2 + T_{n+3}j_3\\
&- (T_{n+1} + T_{n+2}j_1 + T_{n+3}j_2 + T_{n+4}j_3)j_1\\
&- (T_{n+2} + T_{n+3}j_1 + T_{n+4}j_2 + T_{n+5}j_3)j_2\\
&- (T_{n+3} + T_{n+4}j_1 + T_{n+5}j_2 + T_{n+6}j_3)j_3\\
=& T_n - T_{n+2} - T_{n+4} + T_{n+6} - 2\mathfrak{T}_{n+3}j_3.
\end{aligned}$$

Hence, this proof is completed. □

Theorem 9. $\forall n \in \mathbb{Z}^{+}$, *the following is obtained:*

$$\begin{pmatrix} \mathcal{T}_{n+2} \\ \mathcal{T}_{n+1} \\ \mathcal{T}_{n} \end{pmatrix} = \begin{pmatrix} r & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{n} \begin{pmatrix} \mathcal{T}_{2} \\ \mathcal{T}_{1} \\ \mathcal{T}_{0} \end{pmatrix}.$$

Proof. The proof can be conducted by mathematical induction, therefore we omit it. □

By inspiring the study [32], we present the following determinant equation for $\mathcal{BGTN}$ which enables a different way to find the *nth* term.

Theorem 10. $\forall n \in \mathbb{N}$, *the following equation holds:*

$$\mathcal{T}_n = \begin{vmatrix} \mathcal{T}_0 & -1 & 0 & 0 & 0 & \dots & 0 & 0 \\ \mathcal{T}_1 & 0 & -1 & 0 & 0 & \dots & 0 & 0 \\ \mathcal{T}_2 & 0 & 0 & -1 & 0 & \dots & 0 & 0 \\ 0 & t & s & r & -1 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ddots & r & -1 \\ 0 & 0 & 0 & 0 & 0 & \ddots & s & r \end{vmatrix}_{(n+1)\times(n+1)}. \tag{11}$$

Proof. It can be proved by using Eq. (8) and Theorem 5 on page 5 in [32]. □

In the following, we construct a numerical algorithm (Algorithm 2) with respect to the determinant equation given by Theorem 10.

Algorithm 2 A numerical algorithm for finding *nth* term of $\mathcal{BGTN}$

1: Begin
2: Input $\mathcal{T}_0, \mathcal{T}_1$ and $\mathcal{T}_2$
3: Form $\mathcal{T}_n$ with respect to Eq. (11)
4: Compute $\mathcal{T}_n$
5: Output $\mathcal{T}_n = T_n + T_{n+1}j_1 + T_{n+2}j_2 + T_{n+3}j_3$
6: Complete

Also, thanks to the study [16] and [14], we get the other method which can be examined in Theorem 11, in order to calculate the *nth* terms of $\mathcal{BGTN}$.

Theorem 11. *$\forall n \in \mathbb{N}$, the following equation is satisfied:*

$$\mathcal{T}_n = \begin{vmatrix} \mathcal{T}_0 & 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ r\mathcal{T}_0 - \mathcal{T}_1 & r & \dfrac{1}{\mathcal{T}_0} & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & r\mathcal{T}_1 - \mathcal{T}_2 & r & t & 0 & 0 & \dots & 0 & 0 \\ 0 & \mathcal{T}_0 & -\dfrac{s}{t} & r & t & 0 & \dots & 0 & 0 \\ 0 & 0 & \dfrac{1}{t} & -\dfrac{s}{t} & r & t & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & r & t \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & -\dfrac{s}{t} & r \end{vmatrix}_{(n+1)\times(n+1)} \tag{12}$$

where $\mathcal{T}_0 \overline{\mathcal{T}}_0^{j_1} \overline{\mathcal{T}}_0^{j_2} \overline{\mathcal{T}}_0^{j_3} \neq 0$ and $t \neq 0$.

Proof. For the sake of brevity, we also skip this proof. □

Now, let us give a numerical algorithm in the following (Algorithm 3) related to the Theorem 11.

Algorithm 3 A numerical algorithm for finding *nth* term of $\mathcal{BGTN}$

Begin
Input $\mathcal{T}_0, \mathcal{T}_1$ and $\mathcal{T}_2$
Form $\mathcal{T}_n$ according to Eq. (12)
Compute $\mathcal{T}_n$
Output $\mathcal{T}_n = T_n + T_{n+1}j_1 + T_{n+2}j_2 + T_{n+3}j_3$
Complete

According to the Theorem 1-Theorem 11, we can get the following two corollaries consisting of several features for bihyperbolic Tribonacci numbers and bihyperbolic Padovan numbers, respectively. With the same logic, these concepts are also valid for the other $\mathcal{BGTN}$ which are not need to be written here for the sake of brevity (see subfamilies in Table 3 and a small part of them in Table 4).

Corollary 1. *Let consider the nth bihyperbolic Tribonacci number $\mathcal{A}_n$ with the initial values*

$$\begin{cases} \mathcal{A}_0 = j_1 + j_2 + 2j_3, \\ \mathcal{A}_1 = 1 + j_1 + 2j_2 + 4j_3, \\ \mathcal{A}_2 = 1 + 2j_1 + 4j_2 + 7j_3. \end{cases}$$

Then the followings hold:

(i) *The recurrence relation for* $\mathcal{A}_n$ *is as:*

$$\mathcal{A}_n = \mathcal{A}_{n-1} + \mathcal{A}_{n-2} + \mathcal{A}_{n-3}, \quad n \geq 3.$$

(ii) *The Binet formula of* $\mathcal{A}_n$ *is as:*

$$\mathcal{A}_n = \frac{x_1^{n+1}\widetilde{x}_1}{(x_1 - x_2)(x_1 - x_3)} + \frac{x_2^{n+1}\widetilde{x}_2}{(x_2 - x_1)(x_2 - x_3)} + \frac{x_3^{n+1}\widetilde{x}_3}{(x_3 - x_1)(x_3 - x_2)} \cdot$$

(iii) *The generating function of* $\mathcal{A}_n$ *is as:*

$$\sum_{n=0}^{\infty} \mathcal{A}_n x^n = \frac{\mathcal{A}_0 + (\mathcal{A}_1 - \mathcal{A}_0)x + (\mathcal{A}_2 - \mathcal{A}_1 - \mathcal{A}_0)x^2}{1 - x - x^2 - x^3} \cdot$$

(iv) *The exponential generating function of* $\mathcal{A}_n$ *is as:*

$$\sum_{n=0}^{\infty} \mathcal{A}_n \frac{y^n}{n!} = \frac{x_1\widetilde{x}_1 e^{x_1 y}}{(x_1 - x_2)(x_1 - x_3)} + \frac{x_2\widetilde{x}_2 e^{x_2 y}}{(x_2 - x_1)(x_2 - x_3)} + \frac{x_3\widetilde{x}_3 e^{x_3 y}}{(x_3 - x_1)(x_3 - x_2)} \cdot$$

(v) $\forall m \in \mathbb{N}$, *the summation formulae for* $\mathcal{A}_n$ *are satisfied:*
- $\sum\limits_{n=0}^{m} \mathcal{A}_n = \frac{1}{2}(\mathcal{A}_{m+3} - \mathcal{A}_{m+1} - \mathcal{A}_2 + \mathcal{A}_0)$,
- $\sum\limits_{n=0}^{m} \mathcal{A}_{2n} = \frac{1}{2}(\mathcal{A}_{2m+1} + \mathcal{A}_{2m} - \mathcal{A}_1 + \mathcal{A}_0)$,
- $\sum\limits_{n=0}^{m} \mathcal{A}_{2n+1} = \frac{1}{2}(\mathcal{A}_{2m+2} + \mathcal{A}_{2m+1} - \mathcal{A}_2 + \mathcal{A}_1)$.

(vi) $\forall m \in \mathbb{N}$, *the following summation property holds for* $\mathcal{A}_n$*:*

$$\sum_{n=0}^{m} \mathcal{A}_n = \frac{\mathcal{A}_{m+2} + \mathcal{A}_m + (-1 - j_1 - 3j_2 - 5j_3)}{2} \cdot$$

(vii) *The following properties are derived:*
- $\mathcal{A}_n + \overline{\mathcal{A}_n}^{j_1} = 2(A_n + A_{n+1}j_1)$,
- $\mathcal{A}_n + \overline{\mathcal{A}_n}^{j_2} = 2(A_n + A_{n+2}j_2)$,
- $\mathcal{A}_n + \overline{\mathcal{A}_n}^{j_3} = 2(A_n + A_{n+3}j_2)$.

(viii) *The following property for* $\mathcal{A}_n$ *is supplied as:*

$$\mathcal{A}_n - \mathcal{A}_{n+1}j_1 - \mathcal{A}_{n+2}j_2 - \mathcal{A}_{n+3}j_3 = A_n - A_{n+2} - A_{n+4} + A_{n+6} - 2\mathcal{A}_{n+3}j_3.$$

(ix) *The following property for* $\mathcal{A}_n$ *is maintained as:*

$$\begin{pmatrix} \mathcal{A}_{n+2} \\ \mathcal{A}_{n+1} \\ \mathcal{A}_n \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} \mathcal{A}_2 \\ \mathcal{A}_1 \\ \mathcal{A}_0 \end{pmatrix}.$$

(x) *The following equation for* $\mathcal{A}_n$ *holds as:*

$$\mathcal{A}_n = \begin{vmatrix} \mathcal{A}_0 & -1 & 0 & 0 & 0 & \dots & 0 & 0 \\ \mathcal{A}_1 & 0 & -1 & 0 & 0 & \dots & 0 & 0 \\ \mathcal{A}_2 & 0 & 0 & -1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ddots & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & \ddots & 1 & 1 \end{vmatrix}_{(n+1)\times(n+1)}.$$

(xi) *Since the value of* $\mathcal{A}_0\overline{\mathcal{A}_0}^{j_1}\overline{\mathcal{A}_0}^{j_2}\overline{\mathcal{A}_0}^{j_3}$ *is zero, we cannot construct the method with respect to the determinant equation for the bihyperbolic Tribonacci numbers given in Eq.* (12) *written in the Theorem 11.*

Now, let us present an example with respect to the method given in part (x) of Corollary 1. Consider $n = 7$ and let us calculate the $7th$ term of the $\mathcal{BGTN}$:

$$\begin{vmatrix} \mathcal{A}_0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathcal{A}_1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ \mathcal{A}_2 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{vmatrix}_{8\times 8} = 24 + 44j_1 + 81j_2 + 149j_3 = \mathcal{A}_7.$$

Corollary 2. *Let consider the nth bihyperbolic Padovan number* $\mathcal{G}_n$ *with the initial values*

$$\begin{cases} \mathcal{G}_0 = 1 + j_1 + j_2 + 2j_3, \\ \mathcal{G}_1 = 1 + j_1 + 2j_2 + 2j_3, \\ \mathcal{G}_2 = 1 + 2j_1 + 2j_2 + 3j_3. \end{cases}$$

Then, the followings hold:

(i) *The recurrence relation for* $\mathcal{G}_n$ *is as:*

$$\mathcal{G}_n = \mathcal{G}_{n-2} + \mathcal{G}_{n-3}, \quad n \geq 3.$$

(ii) *The Binet formula of* $\mathcal{G}_n$ *is as:*

$$\mathcal{G}_n = \frac{(x_2-1)(x_3-1)x_1^n\widetilde{x}_1}{(x_1-x_2)(x_1-x_3)} + \frac{(x_1-1)(x_3-1)x_2^n\widetilde{x}_2}{(x_2-x_1)(x_2-x_3)} + \frac{(x_1-1)(x_2-1)x_3^n\widetilde{x}_3}{(x_3-x_1)(x_3-x_2)}.$$

(iii) *The generating function of* $\mathcal{G}_n$ *is as:*

$$\sum_{n=0}^{\infty} \mathcal{G}_n x^n = \frac{\mathcal{G}_0 + \mathcal{G}_1 x + (\mathcal{G}_2 - \mathcal{G}_0)x^2}{1 - x^2 - x^3}.$$

(iv) *The exponential generating function of $\mathcal{G}_n$ is as:*

$$\sum_{n=0}^{\infty} \mathcal{G}_n \frac{y^n}{n!} = \frac{(x_2-1)(x_3-1)\widetilde{x}_1 e^{x_1 y}}{(x_1-x_2)(x_1-x_3)} + \frac{(x_1-1)(x_3-1)\widetilde{x}_2 e^{x_2 y}}{(x_2-x_1)(x_2-x_3)} + \frac{(x_1-1)(x_2-1)\widetilde{x}_3 e^{x_3 y}}{(x_3-x_1)(x_3-x_2)}.$$

(v) $\forall m \in \mathbb{N}$, *the summation formulae for $\mathcal{G}_n$ are satisfied:*

- $\sum_{n=0}^{m} \mathcal{G}_n = \mathcal{G}_{m+3} + \mathcal{G}_{m+2} - \mathcal{G}_2 - \mathcal{G}_1$,
- $\sum_{n=0}^{m} \mathcal{G}_{2n} = \mathcal{G}_{2m+1} + \mathcal{G}_{2m} - \mathcal{G}_1$,
- $\sum_{n=0}^{m} \mathcal{G}_{2n+1} = \mathcal{G}_{2m+2} + \mathcal{G}_{2m+1} - \mathcal{G}_2$.

(vi) $\forall m \in \mathbb{N}$, *the following summation property holds for $\mathcal{G}_n$:*

$$\sum_{n=0}^{m} \mathcal{G}_n = \mathcal{G}_{m+2} + \mathcal{G}_{m+1} + \mathcal{G}_m + (-2 - 3j_1 - 4j_2 - 5j_3).$$

(vii) *The following properties for $\mathcal{G}_n$ are derived:*

- $\mathcal{G}_n + \overline{\mathcal{G}}_n^{\,j_1} = 2(G_n + G_{n+1}j_1)$,
- $\mathcal{G}_n + \overline{\mathcal{G}}_n^{\,j_2} = 2(G_n + G_{n+2}j_2)$,
- $\mathcal{G}_n + \overline{\mathcal{G}}_n^{\,j_3} = 2(G_n + G_{n+3}j_2)$.

(viii) *The following property for $\mathcal{G}_n$ is supplied:*

$$\mathcal{G}_n - \mathcal{G}_{n+1}j_1 - \mathcal{G}_{n+2}j_2 - \mathcal{G}_{n+3}j_3 = G_n - G_{n+2} - G_{n+4} + G_{n+6} - 2\mathcal{G}_{n+3}j_3.$$

(ix) *The following property for $\mathcal{G}_n$ is maintained as:*

$$\begin{pmatrix} \mathcal{G}_{n+2} \\ \mathcal{G}_{n+1} \\ \mathcal{G}_n \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} \mathcal{G}_2 \\ \mathcal{G}_1 \\ \mathcal{G}_0 \end{pmatrix}.$$

(x) *The following equality for $\mathcal{G}_n$ holds:*

$$\mathcal{G}_n = \begin{vmatrix} \mathcal{G}_0 & -1 & 0 & 0 & 0 & \dots & 0 & 0 \\ \mathcal{G}_1 & 0 & -1 & 0 & 0 & \dots & 0 & 0 \\ \mathcal{G}_2 & 0 & 0 & -1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ddots & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & \ddots & 1 & 0 \end{vmatrix}_{(n+1)\times(n+1)}.$$

(xi) *The following equation for $\mathcal{G}_n$ is satisfied:*

$$\mathcal{G}_n = \begin{vmatrix} \mathcal{G}_0 & 1 & 0 & 0 & \dots & 0 & 0 \\ -\mathcal{G}_1 & 0 & \frac{1}{\mathcal{G}_0} & 0 & \dots & 0 & 0 \\ 0 & -\mathcal{G}_2 & 0 & 1 & \dots & 0 & 0 \\ 0 & \mathcal{G}_0 & -1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & -1 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & -1 & 0 \end{vmatrix}_{(n+1)\times(n+1)},$$

where $\mathcal{G}_0\,\overline{\mathcal{G}_0}^{j_1}\overline{\mathcal{G}_0}^{j_2}\overline{\mathcal{G}_0}^{j_3} = 5 \neq 0$.

Let us give an example with respect to the method given in part (xi) of Corollary 2. Consider $n = 3$, and let us calculate the $3th$ term of the $\mathcal{BGTN}$:

$$\begin{vmatrix} \mathcal{G}_0 & 1 & 0 & 0 \\ -\mathcal{G}_1 & 0 & \frac{1}{\mathcal{G}_0} & 0 \\ 0 & -\mathcal{G}_2 & 0 & 1 \\ 0 & \mathcal{G}_0 & -1 & 0 \end{vmatrix}_{4\times 4} = 2 + 2j_1 + 3j_2 + 4j_3 = \mathcal{G}_3.$$

4. Conclusions

In this present study, we introduce the $\mathcal{BGTN}$ by examining several well-known relations and identities. By putting this theory into literature, we have an extended framework for third-order linear recurrence sequences with bihyperbolic number components.

For future works, let us make a brief introduction associated with the topic: quaternions with $\mathcal{BGTN}$ components. Quaternions were defined by W. R. Hamilton [23, 24], and the algebra of quaternions is associative, non-commutative, and 4-dimensional Clifford algebra. Quaternions have huge significance in lots of areas such as; pure/applied mathematics, motion geometry, differential geometry, graph theory, differential equations, computer animation, robotics, and so on. A quaternion is represented by $q = q_0 + q_1 i + q_2 j + q_3 k$ where $q_0, q_1, q_2, q_3 \in \mathbb{R}$ and i, j, k are quaternionic units which satisfy:

$$i^2 = -1,\ j^2 = -1,\ k^2 = -1,\ ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j. \qquad (13)$$

Hence, the nth quaternion with $\mathcal{BGTN}$ components can be defined as:

$$\mathbb{T}_n = \mathcal{T}_n + \mathcal{T}_{n+1} i + \mathcal{T}_{n+2} j + \mathcal{T}_{n+3} k, \quad n \geq 0$$

with the initial conditions $\mathbb{T}_0, \mathbb{T}_1$ and $\mathbb{T}_2$ considering Eq. (13). As an illustration

$$\begin{aligned}
\mathbb{T}_0 =& a + bj_1 + cj_2 + (rc + sb + ta)j_3 \\
&+ \left\{ \begin{array}{l} b + cj_1 + (rc + sb + ta)\, j_2 + \left[\left(r^2 + s\right) c \right. \\ \left. + (rs + t)\, b + rta\right] j_3 \end{array} \right\} i \\
&+ \left\{ \begin{array}{l} c + (rc + sb + ta)\, j_1 \\ + \left[\left(r^2 + s\right) c + (rs + t)\, b + rta\right] j_2 \\ + \left[\begin{array}{l} \left(r^3 + 2rs + t\right) c + \left(r^2 s + s^2 + rt\right) b \\ + \left(r^2 t + st\right) a \end{array} \right] j_3 \end{array} \right\} j \\
&+ \left\{ \begin{array}{l} rc + sb + ta + \left[\left(r^2 + s\right) c + (rs + t)\, b + rta\right] j_1 \\ + \left[\begin{array}{l} \left(r^3 + 2rs + t\right) c + \left(r^2 s + s^2 + rt\right) b \\ + \left(r^2 t + st\right) a \end{array} \right] j_2 \\ + \left[\begin{array}{l} \left(r^3 t + 2str + t^2\right) a \\ + \left(r^3 s + r^2 t + 2s^2 r + 2st\right) b \\ + \left(r^4 + 3r^2 s + s^2 + 2tr\right) c \end{array} \right] j_3 \end{array} \right\} k.
\end{aligned}$$

Additionally, the recurrence relation $\mathbb{T}_n = r\mathbb{T}_{n-1} + s\mathbb{T}_{n-2} + t\mathbb{T}_{n-3},\ n \geq 3$ holds for $\mathbb{T}_n$. So, quaternions with several members of $\mathcal{BGTN}$ components can be easily understood by taking into account Table 3 and Table 4.

As an another aspect, the type of quaternion can also be changed in line with this objective, for instance generalized quaternion case. Additionally, with the guidance of the study [57], combining 3-parameter generalized quaternions (as a special generalization of 2-parameter generalized quaternions) with Tribonacci numbers and bihyperbolic number are our another forthcoming goals. We intend to examine these topics exhaustively in future works.

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