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The Source of Γ-semiprimeness on Γ-semigroups

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Abstract. Let S be a Γ-semigroup with zero. We define the S_S^Γ subset of S as $S_S^\Gamma = \{a \in S \mid a\Gamma S\Gamma a = (0)\}$. This set is called the source of Γ-semiprimeness of S. In this study, we examined some properties of S_S^Γ set and defined $|S_S^\Gamma|$-idempotent, $|S_S^\Gamma|$-regular and $|S_S^\Gamma|$-reduced Γ-semigroups. We then obtained some results for these newly defined semigroups.

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1. Introduction

The literature contains numerous pioneering studies on Γ-semigroups, and the fundamental definitions have been provided based on these seminal works [2–6]. In this study, novel Γ-semigroup structures were defined, and several of their properties were presented. Let S and Γ be two nonempty sets. In [5], S is called a Γ-semigroup, for all $\alpha, \beta \in \Gamma$ and for all $a, b, c \in S$ if

$$a\alpha b \in S$$

and

$$(a\alpha b)\beta c = a\alpha(b\beta c).$$

A semigroup can be considered a Γ-semigroup when the following conditions are met. Let S be an arbitrary semigroup. Let 1 be a symbol that is not an element of the semigroup S. Let us extend the binary operation defined on S to $S \cup \{1\}$ by defining $1 \cdot 1 = 1$ and $1 \cdot a = a \cdot 1$, for all $a \in S$. The set $S \cup \{1\}$ is a semigroup with an identity element 1. For $\Gamma = \{1\}$, if $a \cdot b = a \cdot 1 \cdot b$, then semigroup S becomes a Γ-semigroup.

Since every semigroup is a Γ-semigroup, the concept of a Γ-semigroup was defined as a generalization of a semigroup. Many properties of semigroups are generalized to the Γ-semigroup. In line with these studies, we can generalize the theorems of [1]. The aim of this study is to generalize the study further and to obtain new results by using the definition of semiprimeness in Γ-semigroups in the sense of the study done in the [1].

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2. Preliminaries

This section provides the definitions needed in this study.

Definition 2.1 ([6]). A nonempty subset A of Γ-semigroup S is called a Γ-subsemigroup of S if $a\gamma b \in A$, for all $a, b \in A$ and for all $\gamma \in \Gamma$.

Definition 2.2 ([4]). The Γ-semigroup S is called a commutative Γ-semigroup if $a\gamma b = b\gamma a$, for all $a, b \in S$ and for all $\gamma \in \Gamma$.

Definition 2.3 ([5]). Let S be a Γ-semigroup. If there exists $1 \in S$ such that $1\gamma s = s$ (left identity) and $s\gamma 1 = s$ (right identity), for all $s \in S$ and $\gamma \in \Gamma$, then S is called a Γ-monoid.

Definition 2.4 ([4]). The element 0 of Γ-semigroup S is called a zero element if $0\alpha s = 0$ (left zero) and $s\alpha 0 = 0$ (right zero), for all $s \in S$ and for all $\alpha \in \Gamma$.

Remark 2.5. Throughout this study, S will be taken as a Γ-semigroup with zero elements.

Definition 2.6 ([4]). Let S be a Γ-semigroup and $e \in S$. If $e\alpha e = e$, for $\alpha \in \Gamma$, then the element $e \in S$ is called an α-idempotent element. If all the elements of S are α-idempotent, for some $\alpha \in \Gamma$, then S is called an idempotent Γ-semigroup.

Definition 2.7 ([4]). The element $x \in S$ is called a regular element if $x = x\alpha a\beta x$, for some $a \in S$ and $\alpha, \beta \in \Gamma$, i.e $x \in x\Gamma S\Gamma x$. If all elements of S are regular, then S is called a regular Γ-semigroup.

Definition 2.8 ([2]). The element $a \in S$ is called a nilpotent element if, for any $\gamma \in \Gamma$ and for some $n \in \mathbb{Z}^+$, $n = n(\gamma, a)$ $\ni (a\gamma)^{n-1}a = 0$. A Γ-semigroup S is called reduced if it has no nonzero nilpotent elements.

Definition 2.9 ([4]). Let A be a nonempty subset of S. Then, A is called a Γ-ideal if $s\alpha a \in A$ (left Γ-ideal) and $a\alpha s \in A$ (right Γ-ideal), for $s \in S$, $a \in A$ and $\alpha \in \Gamma$.

Equivalent to this definition; if $S\Gamma A \subseteq A$ (left Γ-ideal) and $A\Gamma S \subseteq A$ (right Γ-ideal), then A is called a Γ-ideal.

Definition 2.10 ([6]). Let I be a Γ-ideal of S. For any $a \in S$, if $a\Gamma a \subseteq I$ implies $a \in I$, then I is called a semiprime Γ-ideal of S.

Equivalent to this definition; for any Γ-ideal A of S, if $A\Gamma A \subseteq I$ implies $A \subseteq I$, then I is called a semiprime Γ-ideal.

Definition 2.11 ([6]). The I ideal of S is called the s-semiprime Γ-ideal if $a\gamma a \in I$ implies $a \in I$, for any $a \in S$ and $\gamma \in \Gamma$.

Definition 2.12 ([4]). A right Γ-ideal A of S is called a principal right Γ-ideal generated by a if A is a right Γ-ideal generated by $\{a\}$, for some $a \in S$. It is denoted $(a) = a\Gamma S \cup \{a\}$.

Remark 2.13 ([4]). In Γ-semigroup S, the product of A and B sets is

$$A\Gamma B = \{a\gamma b \mid a \in A,\ \ b \in B,\ \ \gamma \in \Gamma\}.$$

The Definition 2.14 is adapted from [3].

Definition 2.14. If $a\Gamma S\Gamma a = (0)$ with $a \in S$ implies $a = 0$, then S is called a semiprime Γ-semigroup.

3. Properties of the $|S_S^\Gamma|$ -semigroup

Definition 3.1. Let S be a Γ-semigroup with zero and $\emptyset \neq A \subseteq S$. The subset of S,

$$S_S^\Gamma(A) = \{a \in S \mid a\Gamma A\Gamma a = (0)\}$$

is called the source of Γ-semiprimeness of A in S. S_S^Γ will be used instead of $S_S^\Gamma(S)$. In that case, the source set of Γ-semiprimeness of S is denoted by

$$S_S^\Gamma = \{a \in S \mid a\Gamma S\Gamma a = (0)\}.$$

Some of some properties of aforesaid set as follows:

Remark 3.2. Let A be a Γ-subsemigroup of S.

(1) Let $x \in S_A^\Gamma$. Then, $x\Gamma A\Gamma x = (0)$, for $x \in A$. Since $A \subseteq S$, $x \in S_S^\Gamma(A)$. So, $S_A^\Gamma \subseteq S_S^\Gamma(A)$.

(2) Since $0 \in S^{\Gamma}_{S}(A)$, $S^{\Gamma}_{S}(A) \neq \emptyset$.
(3) Let $A \subseteq B$, for $\emptyset \neq A, B \subseteq S$ and $x \in S^{\Gamma}_{S}(B)$. Then, $x\Gamma B\Gamma x = (0)$, for $x \in S$. Since $A \subseteq B$, $x\Gamma A\Gamma x = (0)$, for $x \in S$. Therefore, $x \in S^{\Gamma}_{S}(A)$. Thus, $S^{\Gamma}_{S}(B) \subseteq S^{\Gamma}_{S}(A)$.

Proposition 3.3. *S is a Γ-semiprime semigroup if and only if $S^{\Gamma}_{S} = \{0\}$.*

Proof. $\Rightarrow$: Let S be a Γ-semiprime semigroup and $a \in S^{\Gamma}_{S}$. Then, $a\Gamma S\Gamma a = (0)$, for $a \in S$. From the hypothesis $a = 0$ is satisfied. Therefore, $S^{\Gamma}_{S} = \{0\}$.

$\Leftarrow$: Let $S^{\Gamma}_{S} = \{0\}$. Suppose that $a\Gamma S\Gamma a = (0)$, for $a \in S$. Then, $a \in S^{\Gamma}_{S}$ and $a = 0$ is satisfied. Namely, S is a Γ-semiprime semigroup. □

Lemma 3.4. *Let S be a Γ-semigroup. Then, the following are provided:*
(1) *$e\Gamma S^{\Gamma}_{S} \subseteq S^{\Gamma}_{e\Gamma S}$, for $e \in S$.*
(2) *If S is an idempotent Γ-semigroup, then $S^{\Gamma}_{S} = \{0\}$.*
(3) *If S is a regular Γ-semigroup, then $S^{\Gamma}_{S} = \{0\}$.*
(4) *If $a \in S^{\Gamma}_{S}$, then a is a nilpotent element.*

Proof. (1) Let $e\Gamma a \subseteq e\Gamma S^{\Gamma}_{S}$, for $a \in S^{\Gamma}_{S}$. Since $a \in S^{\Gamma}_{S}$, $a\Gamma S\Gamma a = (0)$, for $a \in S$. From here

$$e\Gamma a\Gamma(e\Gamma S)\Gamma e\Gamma a \subseteq e\Gamma(a\Gamma S\Gamma a) = e\Gamma(0) = (0).$$

This gives $e\Gamma a \subseteq S^{\Gamma}_{e\Gamma S}$. So, $e\Gamma S^{\Gamma}_{S} \subseteq S^{\Gamma}_{e\Gamma S}$.
(2) If $a \in S^{\Gamma}_{S}$, then $a\Gamma S\Gamma a = (0)$. Thus, $a\alpha a\alpha a = (0)$, for $a \in S$ and $\alpha \in \Gamma$. Since a is an α-idempotent element,

$$(0) = a\alpha a\alpha a = a\alpha a = a$$

is satisfied. Accordingly, $S^{\Gamma}_{S} = \{0\}$.
(3) Let S be a regular Γ-semigroup and $a \in S^{\Gamma}_{S}$. Then, $a\Gamma S\Gamma a = (0)$, for $a \in S$. Therefore, $a\alpha x\beta a = 0$, for $x \in S$ and $\alpha, \beta \in \Gamma$. Since a is a regular element, $a = 0$. Then, $S^{\Gamma}_{S} = \{0\}$.
(4) Let $a \in S^{\Gamma}_{S}$. Then, $a\Gamma a\Gamma a = (0)$, for $a \in S$. Hence, $0 = a\gamma a\gamma a = (a\gamma)^2 a$, for $\gamma \in \Gamma$. Thus, a is a nilpotent element.

□

From Lemma 3.4, the following results are obtained.

Corollary 3.5. *Let S be a Γ-semigroup. Then, the following are provided:*
(1) *There are no nonzero α-idempotent elements in the set S^{Γ}_{S}.*
(2) *There are no nonzero regular elements in the set S^{Γ}_{S}.*
(3) *Every element of S^{Γ}_{S} is a nilpotent element.*

Definition 3.6. Let S be a Γ-semigroup with zero and $S \neq S^{\Gamma}_{S}$. Then,
(1) If every element of $S - S^{\Gamma}_{S}$ is an α-idempotent element, for some $\alpha \in \Gamma$, then S is called a $|S^{\Gamma}_{S}|$-idempotent Γ-semigroup.
(2) If every element of $S - S^{\Gamma}_{S}$ is a regular element, then S is called a $|S^{\Gamma}_{S}|$-regular Γ-semigroup.
(3) If $S - S^{\Gamma}_{S}$ has no nilpotent elements, then S is called a $|S^{\Gamma}_{S}|$-reduced Γ-semigroup.

Remark 3.7. The following can be obtained from Definition 3.6.
(1) If $S = \{0\}$, then $S^{\Gamma}_{S} = \{0\} = S$. Since $S - S^{\Gamma}_{S} = \emptyset$, this is meaningless for the Γ-semigroup S with zero. Similarly, in the case of $S = S^{\Gamma}_{S}$, $S - S^{\Gamma}_{S} = \emptyset$.
(2) (a) If a is a β-idempotent element, then $a\beta a = a$, for $\beta \in \Gamma$. Since $a\beta(a\beta a) = a\beta a = a$, a is a regular element.
(b) Let $a \in S$ be a nilpotent element and assume that a is a γ-idempotent element. Then $a\gamma a = a$, for $\gamma \in \Gamma$. Since a is a nilpotent element, $(a\gamma)^{n-1}a = 0$. From here,

$$0 = (a\gamma)^{n-1}a = (a\gamma)^{n-2}(a\gamma a) = (a\gamma)^{n-2}a$$

is obtained. This contradicts the fact that $(a\gamma)^{n-2}a \neq 0$. Accordingly, a is not a γ-idempotent element. Based on the obtained properties, the following results have been achieved.

If S is a $|S^{\Gamma}_{S}| -$ idempotent $\Gamma -$ semigroup, then S is a $|S^{\Gamma}_{S}| -$ regular $\Gamma -$ semigroup.

If S is a $|S^{\Gamma}_{S}| -$ idempotent $\Gamma -$ semigroup, then S is a $|S^{\Gamma}_{S}| -$ reduced $\Gamma -$ semigroup.

(3) If S is a Γ-idempotent (regular, reduced) semigroup, then S is a $|S^{\Gamma}_{S}|$-idempotent (regular, reduced) Γ-semigroup.

Proposition 3.8. *Let A be a Γ-subsemigroup of S.*
If S is a $|S^{\Gamma}_{S}|$-idempotent (regular, reduced) Γ-semigroup, then A is a $|S^{\Gamma}_{A}|$-idempotent (regular, reduced) Γ-semigroup.

Proof. Let S be a $|S^{\Gamma}_{S}|$-idempotent (regular, reduced) Γ-semigroup and $a \in A - S^{\Gamma}_{A}$. Then, $a \in A$ and $a \notin S^{\Gamma}_{A}$. Thus, $a\Gamma(A\Gamma a) \neq (0)$. Since $A \subseteq S$, $a\Gamma(S\Gamma a) \neq (0)$. This means $a \notin S^{\Gamma}_{S}$. Namely, a is a γ-idempotent element, for some $\gamma \in \Gamma$. Therefore, A is a $|S^{\Gamma}_{A}|$-idempotent (regular, reduced) Γ-semigroup. □

Proposition 3.9. *If S is a commutative $|S^{\Gamma}_{S}|$-idempotent Γ-semigroup, then $S - S^{\Gamma}_{S}$ is a Γ-subsemigroup.*

Proof. Let S be a commutative $|S^{\Gamma}_{S}|$-idempotent Γ-semigroup. In a commutative Γ-semigroup, the product of γ-idempotent elements, for some $\gamma \in \Gamma$ is a γ-idempotent element. So, for $a, b \in S - S^{\Gamma}_{S}$, $a\gamma b \in S$ is a γ-idempotent element. It is also seen that $a\gamma b \notin S^{\Gamma}_{S}$. Thus, $a\gamma b \in S - S^{\Gamma}_{S}$. Therefore, $S - S^{\Gamma}_{S}$ is a Γ-subsemigroup. □

Lemma 3.10. *Let S be a $|S^{\Gamma}_{S}|$-reduced Γ-semigroup. Then, the following are valid.*

(1) $S^{\Gamma}_{S} = \{a \in S \mid (a\Gamma)^2 a = (0)\}$.
(2) *If S is a Γ-monoid, then* $S^{\Gamma}_{S} = \{a \in S \mid a\gamma a = 0, \ \forall\ \gamma \in \Gamma\}$.

Proof. Let S be a $|S^{\Gamma}_{S}|$-reduced Γ-semigroup.

(1) Let $A = \{a \in S \mid (a\Gamma)^2 a = (0)\}$. If $a \in S^{\Gamma}_{S}$, then $a\Gamma a\Gamma a = (0)$, for $a \in S$. This means $(a\Gamma)^2 a = (0)$, for $a \in S$ and so $a \in A$. Hence, $S^{\Gamma}_{S} \subseteq A$.
On the other hand, if $a \in A$, then $(a\gamma)^2 a = (0)$, for $a \in S$ and $\gamma \in \Gamma$. Therefore, a is a nilpotent element. Thus, $a \in S^{\Gamma}_{S}$.

(2) Let $B = \{a \in S \mid a\gamma a = 0, \ \forall\ \gamma \in \Gamma\}$ and $a \in S^{\Gamma}_{S}$. Then, $a\Gamma S\Gamma a = (0)$, for $a \in S$. Since S is a Γ-monoid,
$$0 = a\gamma(1\beta a) = a\gamma a$$
for $1 \in S$ and $\gamma, \beta \in \Gamma$. Namely, $S^{\Gamma}_{S} \subseteq B$.
Conversely, if $a \in B$, then $a\gamma a = 0$, for $a \in S$. From here a is a nilpotent element. Thus, $a \in S^{\Gamma}_{S}$.

□

Lemma 3.11. *Let I and J be two Γ-ideals of S. Then, the following are satisfied:*

(1) $S^{\Gamma}_{I} \cap S^{\Gamma}_{J} \subseteq S^{\Gamma}_{I\cap J}$.
(2) *If I is a left (right) Γ-ideal of S, then $S^{\Gamma}_{S}(I)$ is a right (left) Γ-ideal of S.*
(3) *If I is a Γ-ideal of S, then S^{Γ}_{I} is a Γ-ideal of S. Specially, S^{Γ}_{S} is a Γ-ideal of S.*
(4) *If I and J are Γ-ideals of S, then $S^{\Gamma}_{S}(I)\Gamma S^{\Gamma}_{S}(J) \subseteq S^{\Gamma}_{S}(I\Gamma J)$.*
(5) *If I and J are Γ-ideals of S, then $S^{\Gamma}_{I}\Gamma S^{\Gamma}_{J} \subseteq S^{\Gamma}_{I\Gamma J}$.*

Proof. Let I and J be a Γ-ideals of S.

(1) If $a \in S^{\Gamma}_{I} \cap S^{\Gamma}_{J}$, then $a \in S^{\Gamma}_{I}$ and $a \in S^{\Gamma}_{J}$. Then, $a\Gamma I\Gamma a = (0)$ and $a\Gamma J\Gamma a = (0)$. Since $I \cap J \subseteq I$,
$$a\Gamma(I \cap J)\Gamma a \subseteq a\Gamma I\Gamma a = (0)$$
Thus, $a \in S^{\Gamma}_{I\cap J}$.

(2) Let I be a left Γ-ideal of S. Then, $S\Gamma I \subseteq I$. If we take $x \in S^{\Gamma}_{S}(I)$, then $x\Gamma I\Gamma x = (0)$, for $x \in S$. From here,
$$x\Gamma S\Gamma I\Gamma x\Gamma S \subseteq x\Gamma(I\Gamma x\Gamma S) = (x\Gamma I\Gamma x)\Gamma S = (0)\Gamma S = (0)$$
is found. It means $x\Gamma S \subseteq S^{\Gamma}_{S}(I)$ and so $S^{\Gamma}_{S}(I)\Gamma S \subseteq S^{\Gamma}_{S}(I)$. Therefore, $S^{\Gamma}_{S}(I)$ is a right Γ-ideal of S.

(3) If I is a Γ-ideal of S, then $I\Gamma S \subseteq I$ and $S\Gamma I \subseteq I$. If $a \in S^{\Gamma}_{I}$, then $a\Gamma I\Gamma a = (0)$, for $a \in I$. So, we obtain
$$a\Gamma S\Gamma I\Gamma a\Gamma S \subseteq a\Gamma(I\Gamma a\Gamma S) = (a\Gamma I\Gamma a)\Gamma S = (0)\Gamma S = (0)$$
and
$$S\Gamma a\Gamma I\Gamma S\Gamma a \subseteq S\Gamma(a\Gamma I\Gamma a) = S\Gamma(0) = (0).$$
Hereby, $S\Gamma a \subseteq S^{\Gamma}_{I}$ and $a\Gamma S \subseteq S^{\Gamma}_{I}$. Since S is a Γ-ideal of S, S^{Γ}_{S} is a Γ-ideal of S.

(4) Let $a\gamma b \in S_S^\Gamma(I)\Gamma S_S^\Gamma(J)$, for $\gamma \in \Gamma$. Since $a \in S_S^\Gamma(I)$ and $b \in S_S^\Gamma(J)$, $a\Gamma I\Gamma a = (0)$ and $b\Gamma J\Gamma b = (0)$, for $a, b \in S$. Thus,

$$a\gamma b\Gamma I\Gamma J\Gamma a\gamma b \subseteq (a\Gamma b\Gamma I\Gamma a)\Gamma b \subseteq (a\Gamma I\Gamma a)\Gamma b = (0)\Gamma b = (0)$$

is obtained by using that I and J are Γ-ideals. So, $a\gamma b \in S_S^\Gamma(I\Gamma J)$.

(5) If $a\gamma b \in S_I^\Gamma \Gamma S_J^\Gamma$, for $\gamma \in \Gamma$, $a \in S_I^\Gamma$ and $b \in S_J^\Gamma$ becomes. Then, $a\Gamma I\Gamma a = (0)$ and $b\Gamma J\Gamma b = (0)$, for $a \in I$ and $b \in J$. In this case,

$$a\gamma b\Gamma I\Gamma J\Gamma a\gamma b \subseteq (a\Gamma b\Gamma I\Gamma a)\Gamma b \subseteq (a\Gamma I\Gamma a)\Gamma b = (0)\Gamma b = (0)$$

is obtained by using that I and J are Γ-ideals. Thence, $a\gamma b \in S_{I\Gamma J}^\Gamma$.

□

Theorem 3.12. *If S is a commutative $|S_S^\Gamma|$-regular Γ-semigroup, then S is a $|S_S^\Gamma|$-reduced Γ-semigroup.*

Proof. Let S be a commutative $|S_S^\Gamma|$-regular Γ-semigroup and $a \in S - S_S^\Gamma$ be a nilpotent element. In this case, $(a\gamma)^{n-1}a = 0$, for $n \in \mathbb{Z}^+$. Since a is a regular element and S is commutative,

$$0 = (a\gamma)^{n-1}a = (a\gamma)^{n-2}a\gamma a = (a\gamma)^{n-2}a\gamma a\beta b = (a\gamma)^{n-2}a.$$

This is a contradiction. Hence, in set $S - S_S^\Gamma$ has no nilpotent elements. Thus, S is a $|S_S^\Gamma|$-reduced Γ-semigroup. □

Definition 3.13. Let S be a Γ-semigroup and $a \in S$. If

$$a\Gamma S\Gamma a \subseteq S_S^\Gamma \text{ implies that } a \in S_S^\Gamma,$$

then S is called a $|S_S^\Gamma|$-semiprime Γ-semigroup.

Lemma 3.14. *The set S_S^Γ of the source of Γ-semiprimeness is contained by every semiprime Γ-ideal of S.*

Proof. Let A be a semiprime Γ-ideal of S and $a \in S_S^\Gamma$. Then, $a\Gamma S\Gamma a = (0) \subseteq A$. Hence, $a \in A$. Thus, $S_S^\Gamma \subseteq A$. □

Proposition 3.15. *Let S_S^Γ be a semiprime Γ-ideal of S. Then, the intersection of all semiprime Γ-ideals of S is equal to S_S^Γ.*

Proof. Since the set S_S^Γ is contained by every semiprime Γ-ideal, $S_S^\Gamma \subseteq \bigcap A_i$ where A_i are semiprime Γ-ideals of S.

Conversely, since the set S_S^Γ is a semiprime Γ-ideal, $\bigcap A_i \subseteq S_S^\Gamma$. Hence, $S_S^\Gamma = \bigcap A_i$. □

Theorem 3.16. *S is a $|S_S^\Gamma|$-semiprime Γ-semigroup if and only if for any Γ-ideal A of S, $A\Gamma A \subseteq S_S^\Gamma$ implies that $A \subseteq S_S^\Gamma$.*

Proof. Let S be a $|S_S^\Gamma|$-semiprime Γ-semigroup. Suppose that $A\Gamma A \subseteq S_S^\Gamma$, for a Γ-ideal A. Then, $A\Gamma A\Gamma S\Gamma A\Gamma A = (0)$. Since

$$(A\Gamma S\Gamma A)\Gamma S\Gamma(A\Gamma S\Gamma A) \subseteq A\Gamma A\Gamma S\Gamma A\Gamma A = (0),$$

$A\Gamma S\Gamma A \subseteq S_S^\Gamma$ is satisfied. From the hypothesis, we have $A \subseteq S_S^\Gamma$.

Conversely, let $A\Gamma A \subseteq S_S^\Gamma$ implies that $A \subseteq S_S^\Gamma$, for any Γ-ideal A. If $a\Gamma S\Gamma a \subseteq S_S^\Gamma$, for $a \in S$, then

$$S\Gamma(a\Gamma S\Gamma a)\Gamma S \subseteq S\Gamma S_S^\Gamma\Gamma S \subseteq S_S^\Gamma$$

because of S_S^Γ is a Γ-ideal of S. Then we obtain,

$$(S\Gamma a\Gamma S)\Gamma(S\Gamma a\Gamma S) \subseteq S\Gamma(a\Gamma S\Gamma a)\Gamma S \subseteq S_S^\Gamma$$

and from the hypothesis, $S\Gamma a\Gamma S \subseteq S_S^\Gamma$ is provided. Now, consider the principal right Γ-ideal $(a) = a\Gamma S \cup \{a\}$. Since

$$(a)^3 = (a)\Gamma(a)\Gamma(a) \subseteq S\Gamma a\Gamma S \subseteq S_S^\Gamma$$

when

$$((a)\Gamma(a))^2 \subseteq (a)^3\Gamma(a) \subseteq S_S^\Gamma,$$

$(a)^2 = (a)\Gamma(a) \subseteq S_S^\Gamma$. From the hypothesis, $(a) \subseteq S_S^\Gamma$ and $a \in S_S^\Gamma$ is obtained. Then, S is a $|S_S^\Gamma|$-semiprime Γ-semigroup. □

Corollary 3.17. *If S_S^Γ is a s-semiprime Γ-ideal, then S is a $|S_S^\Gamma|$-semiprime Γ-semigroup.*

Definition 3.18. For $|S_S^\Gamma|$-semiprime Γ-semigroup S the following definitions are equivalent:

(1) If S_S^Γ is a semiprime Γ-ideal of S, then S is called a $|S_S^\Gamma|$-semiprime Γ-semigroup.
(2) For any Γ-ideal A of S, if $A\Gamma A \subseteq S_S^\Gamma$ implies that $A \subseteq S_S^\Gamma$, then S is called a $|S_S^\Gamma|$-semiprime Γ-semigroup.
(3) If S_S^Γ is a s-semiprime Γ-ideal of S, then S is called a $|S_S^\Gamma|$-semiprime Γ-semigroup.

Theorem 3.19. *If S is a regular Γ-semigroup, then S is a $|S_S^\Gamma|$-semiprime Γ-semigroup.*

Proof. Let S be a regular Γ-semigroup and $a\Gamma S\Gamma a \subseteq S_S^\Gamma$, for $a \in S$. Since $a \in a\Gamma S\Gamma a \subseteq S_S^\Gamma$, $a \in S_S^\Gamma$ is provided. Thus, S is a $|S_S^\Gamma|$-semiprime Γ-semigroup. □

Corollary 3.20. *If S is an idempotent Γ-semigroup, then S is a $|S_S^\Gamma|$-semiprime Γ-semigroup.*

Proof. If S is an idempotent Γ-semigroup, then S is a regular Γ-semigroup. In this way S is a $|S_S^\Gamma|$-semiprime Γ-semigroup. □

Example 3.21. Define a mapping $S \times \Gamma \times S \to S$ such that $a\gamma b = ab$, for all $a, b \in S$ and $\gamma \in \Gamma$.

.	0	a	b	c
0	0	0	0	0
a	0	a	b	0
b	0	b	b	0
c	0	0	0	0

Then, S is a Γ-semigroup. From the table,

$$S_S^\Gamma = \{0, c\}$$

and

$$S - S_S^\Gamma = \{a, b\}.$$

Since $a\gamma a = a$ and $b\gamma b = b$, S is a $|S_S^\Gamma|$–idempotent Γ-semigroup. Besides, $a = a\gamma a\gamma a$ and $b = b\gamma b\gamma b$ are regular elements. Accordingly, S is a $|S_S^\Gamma|$– regular Γ-semigroup. Otherwise, a and b are not nilpotent elements. Thus, S is a $|S_S^\Gamma|$– reduced Γ-semigroup.

Example 3.22. Define a mapping $S \times \Gamma \times S \to S$ such that $a\gamma b = ab$, for all $a, b \in S$ and $\gamma \in \Gamma$.

.	0	1	a	b	c
0	0	0	0	0	0
1	0	1	a	b	c
a	0	a	a	0	0
b	0	b	0	0	0
c	0	c	0	0	0

With the binary operation defined, S is a Γ-monoid. Using the table,

$$S_S^\Gamma = \{0, b, c\}$$

and

$$S - S_S^\Gamma = \{a, 1\}.$$

Since $a\gamma a = a$ and $1\gamma 1 = 1$, S is a $|S_S^\Gamma|$–idempotent Γ-semigroup. Moreover, since $a = a\gamma a\gamma a$ and $1 = 1\gamma 1\gamma 1$, S is a $|S_S^\Gamma|$–regular Γ-semigroup. Here a and 1 are not nilpotent elements. Thus, S is a $|S_S^\Gamma|$– reduced Γ-semigroup.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

Didem Yeşil posed the problem. The relevant results and development of the idea were made by Didem Yeşil and Rasie Mekera. The article was written by Rasie Mekera. The results were checked by Didem Karalarlıoğlu Camcı and various adjustments were made. In other words, Camcı 20%, Mekera 35%, and Yeşil 45% contributed to this study.

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