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RESEARCH ARTICLE

The Translation Surfaces on Statistical Manifolds with Normal Distribution

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(Dedicated to Professor Bang-Yen CHEN on the occasion of his 80th birthday)

ABSTRACT

In this paper, we will investigate translation surfaces on statistical manifolds. Statistical manifolds are mathematical structures that describe the geometric properties of statistical models. We will focus on minimal statistical translation surfaces and then classify statistical translation surfaces of null sectional curvature in three-dimensional hyperbolic statistical manifolds.

Keywords: *Statistical manifolds, statsitical submanifolds, normal distribution, translation surfaces, hyperbolic models.*
AMS Subject Classification (2020): *Primary: 53A35 ; Secondary: 53B20.*

1. Introduction

Our life is fully equipped with various data. In 2006, the British mathematician C.Humby introduced the phrase that "Data is the new oil" [31]. According to Wikipedia [30], M.Palmer expanded on Humby's quote by saying, like oil, "data is is valuable, but if unrefined it cannot really be used" (see [30], [31] and [49]) . Statistics is a science that helps us to make decisions from these quantitative data sets. Therefore, accurate statistical models have to be put forward. A statistical model is defined as

$$S = S(x, \theta) = \{p(x, \theta) : \int_{\chi} p(x, \theta) = 1, p(x, \theta) > 0, \theta = (\theta_1, ..., \theta_n) \in E \subset^n\}$$

where $p(x, \theta)$ is the probability density function of the data set x and E is called parameters space. Let us consider one-to-one mapping

$$\phi : S(x, \theta) \to E \subset^n ;\ p(x, \theta) \to \theta$$

such that $Rank\phi = dimE$.Thus $S(x, \theta)$ is parametrized by (E, ϕ) [3] .

On the other hand, if we want to compare between different parameter values $\theta = (\theta_1, ..., \theta_n)$ and $\hat{\theta} = (\hat{\theta}_1, ..., \hat{\theta}_n)$ on the same statistical model $S(x, \theta)$ then we need to a metric. One of these possibilities is the Fisher information matrix which is used to determine whether the difference in parameter values is statistically significant.

The notion of Fisher Information Matrix was first independently introduced by H. Jeffreys [33] and C.R. Rao [50]. Since then, it has become one of the cornerstones of mathematical statistics. The Fisher information matrix on a statistical model is defined following

$$g^F_{ij}(\theta) = \int_{x\in\chi} \frac{\partial \log p(x, \theta)}{\partial\theta_i} \frac{\partial \log p(x, \theta)}{\partial\theta_j} p(x, \theta)dx$$

where χ is sample space. The Fisher information matrix on any statistical model is symmetric, positive definite and non-degenerate [3]. These properties give us a chance to define a Riemannian metric which can be written locally as $g = \sum_{i,j=1}^{n} g^F_{ij}(\theta)d\theta_i d\theta j$ on statistical models [3].

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For a constant α, the Christoffel symbols of a torsion-free α -connection ∇^α on $S(x,\theta)$ can be defined as follows

$$\Gamma^{\alpha}_{ij,k} = \int_{x\in\chi}\{\frac{\partial^2 \log p(x,\theta)}{\partial\theta_i\partial\theta_j} + \frac{1-\alpha}{2}(\frac{\partial \log p(x,\theta)}{\partial\theta_j})(\frac{\partial \log p(x,\theta)}{\partial\theta_j})\} \times(\frac{\partial \log p(x,\theta)}{\partial\theta_k})p(x,\theta))dx$$

Moreover, the following equation holds

$$\partial\theta_i g^F(\partial\theta_j,\partial\theta_k) = g^F(\nabla^{\alpha}_{\partial\theta_i}\partial\theta_j,\partial\theta_k) + g^F(\partial\theta_j,\nabla^{(-\alpha)}_{\partial\theta_i}\partial\theta_k)$$

The α-connection ∇^α is called Amari-Chentsov α-connection in information geometry. Hence triplet (∇^α, $\nabla^{(-\alpha)}$, g^F) is called α-statistical structure on $S(x,\theta)$. When $\alpha = 0$, statistical structure returns Riemannian structure, which is called trivial statistical structure.

One of the frequently encountered statistical model is the normal (Gaussian) distribution. A normal distribution is defined as

$$N(x,\mu,\sigma^2) = \{\theta = (\theta_1,\theta_2) = (\mu,\sigma) \in \mathbb{R}\times\mathbb{R}^+ : p(x,\theta) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\{-\frac{(x-\mu)^2}{2\sigma^2}\}, x\in\mathbb{R}\} \tag{1.1}$$

,where μ is the mean and σ^2 is the variance. For the normal distribution, the Fisher information metric and Amari-Chentsov α -connection [23] can be expressed as

$$g^F = \frac{d\theta_1^2 + 2d\theta_2^2}{\theta_2^2} \text{ with } \theta_1 = \mu \text{ and } \theta_2 = \sigma, \tag{1.2}$$

$$\nabla^{\alpha}_{\partial\theta_1}\partial\theta_1 = \frac{1-\alpha}{2\theta_2}\partial\theta_2, \nabla^{\alpha}_{\partial\theta_1}\partial\theta_2 = \nabla^{\alpha}_{\partial\theta_2}\partial\theta_1 = -\frac{1+\alpha}{\theta_2}\partial\theta_1, \nabla^{\alpha}_{\partial\theta_2}\partial\theta_2 = -\frac{1+\alpha}{2\theta_2}\partial\theta_2.$$

In differential geometry, an (M,g) Riemannian manifold equipped with such structures is referred to as a statistical manifold [37]. In other words, a statistical manifold can be thought of as a Riemannian manifold where each point corresponds to a probability distribution. This concept connects information geometry and (affine) differential geometry [3] , [37], [45].

In 1989, Vos [66] pioneered the investigation of the geometry of submanifolds within statistical manifolds, introducing the Gauss-Weingarten formulas, as well as the Gauss and Codazzi equations. Furuhata [20] investigated the study of hypersurfaces in statistical manifolds. He gave elementary properties of hypersurfaces in statistical manifolds of constant curvature as a first step of the statistical submanifold theory.

Later on, this theory found applications in various areas of geometry as almost-contact geometry,[1], [22], [35], [47] , Hermitian-Kaehler geometry, [2], [21], [42], [43]. [55], [64], almost Norden manifolds [51], almost paracontact geometry [13], submersions [11], [36], [60] [61] [62] and Hessian geometry [29].

In recent years, during literature reviews on statistical submanifolds, it has been observed that studies have focused on Chen inequalities ([5], [6], [15], [53]),Wintgen inequalities ([8], [25], [44]) and inequalities involving the normalized δ-Casorati curvatures ([12], [16], [17], [26], [38], [54]).

In geometry, the problem of finding surfaces with the least area among all surfaces having the same boundary is an important research topic. These surfaces are called minimal surfaces. They appear as surfaces with zero mean curvature in differential geometry. Some well-known examples are catenoid, helicoid, Enneper surfaces, and Scherk's surface in Euclidean space with three dimensions.

Scherk's surface, called after H. Scherk , is defined by

$$z(x,y) = \log(\cos x) - \log(\cos y)$$

It is said to be translation surfaces in three dimensional Euclidean space $\mathbb{R}^3$ if it is given by an immersion

$$\phi : D\subset\mathbb{R}^2 \to \mathbb{R}^3, \phi(x,y) = (x,y,f(x)+g(y))$$

where each of f and g are real valued smooth functions depending on one variable [18].

If we may attenion, Scherk's surface can be parametrized as

$$\phi(x,y) = (x,y,f(x)+g(y)),$$

where $f(x) = \log(\cos x)$ and $g(y) = -\log(\cos y)$. So Scherk's surface is a typical example of the translation surface in $\mathbb{R}^3$.

In 1991, Dillen et al.[18] generalized this result to higher-dimensional Euclidean space. Minimal translation surfaces have been studied in various geometric spaces [4], [7], [9], [32], [39], [40], [63].

A Darboux surface in $\mathbb{R}^3$ can be parametrized as

$$\psi(s,t) = R(t)\alpha(s) + \beta(t),$$

where α, β are space curves and R is an orthogonal matrix. If R is specifically the identity matrix, we obtain translation surfaces [27]. T.Hasanis and R.Lopez [27] proved that cylindirical surfaces are the only translation surfaces with constant Gaussian curvature in $\mathbb{R}^3$. Then they classified all surfaces with constant Gaussian curvature in $\mathbb{R}^3$that can be expressed by an implicit equation of type $f(x) + g(y) + h(z) = 0$ which are called separable surfaces, where f, g and h are real functions of one variable [28]. This classification is generalized for $n > 3$ by M.E. Aydın, R. Lopez and G-E. Vilcu [10].

Now let us consider upper half space hyperbolic model $\tilde{H}^{n+1} = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} > 0\}$ with Riemannian metric $\tilde{g}_{\tilde{H}^{n+1}} = \frac{dx_1^2 + \cdots + dx_{n+1}^2}{x_{n+1}^2}$. By Koszul formula, we have Levi-Civita connection of $\tilde{H}^3$ as follows:

$$\begin{array}{lll} \nabla^{LC}_{\partial_x}\partial_x = \frac{1}{z}\partial_z, & \nabla^{LC}_{\partial_y}\partial_x = 0, & \nabla^{LC}_{\partial_z}\partial_x = -\frac{1}{z}\partial_x, \\ \nabla^{LC}_{\partial_x}\partial_y = 0, & \nabla^{LC}_{\partial_y}\partial_y = \frac{1}{z}\partial_z, & \nabla^{LC}_{\partial_z}\partial_y = -\frac{1}{z}\partial_y, \\ \nabla^{LC}_{\partial_x}\partial_z = -\frac{1}{z}\partial_x, & \nabla^{LC}_{\partial_y}\partial_z = -\frac{1}{z}\partial_y, & \nabla^{LC}_{\partial_z}\partial_z = -\frac{1}{z}\partial_z. \end{array} \tag{1.3}$$

R.Lopez,[39], extends the concept of translation surfaces to half space model the three-dimensional hyperbolic space, that is

$$\tilde{H}^3 = \{(x, y, z) \in \mathbb{R}^3 : z > 0\}$$

endowed with a metric tensor

$$\tilde{g}_{\tilde{H}^3} = \frac{dx^2 + dy^2 + dz^2}{z^2} \tag{1.4}$$

and establishes that the only minimal translation surfaces in this space are the geodesic hyperplanes.

On the other hand, using equations quations (1.1) and (1.2),one can observe that there is a nice similarity between normal distribution statistical model ($N(\theta = (\mu, \sigma^2), g^F$) and upper half space hyperbolic model ($\tilde{H}^2$, $\tilde{g}_{\tilde{H}^2}$) [23].

The aim of this study is to find out what kinds of surfaces are minimal translational surfaces and flat translation surfaces in statistical hyperbolic spaces, motivated by the articles [28] and [39].

2. Statistical Manifolds

Let $\tilde{\nabla}$ be a torsion-free affine connection on a Riemannian manifold $(\tilde{M}, \tilde{g})$. A triple $(\tilde{M}, \tilde{g}, \tilde{\nabla})$ is called a statistical manifold if $\tilde{\nabla}\tilde{g}$ is symmetric i.e.

$$(\tilde{\nabla}_X \tilde{g})(Y, Z) = (\tilde{\nabla}_Z \tilde{g})(X, Y) = (\tilde{\nabla}_Y \tilde{g})(Z, X), \quad \forall X, Y, Z \in \Gamma(TM)$$

[3]. The first typical example that comes to mind is $(\tilde{M}, \tilde{g}, \tilde{\nabla}^{LC})$, where $\tilde{\nabla}^{LC}$ is a Levi-Civita connection on $(\tilde{M}, \tilde{g})$. The difference $(1, 2)$ type tensor field $\tilde{\mathcal{K}}$ between $\tilde{\nabla}$ and $\tilde{\nabla}^{LC}$ connections is given by

$$\bar{\mathcal{K}} = \tilde{\nabla} - \tilde{\nabla}^{LC}. \tag{2.1}$$

Due to $\tilde{\nabla}$ and $\tilde{\nabla}^{LC}$ being torsion-free it can be seen that

$$\tilde{\mathcal{K}}_X Y = \tilde{\mathcal{K}}_Y X, \quad \tilde{g}(\tilde{\mathcal{K}}_X Y, Z) = \tilde{g}(Y, \tilde{\mathcal{K}}_X Z) \tag{2.2}$$

for any $X, Y, Z \in \Gamma(T\tilde{M})$.

If an affine connection $\tilde{\nabla}^*$ on $\tilde{M}$ satisfies following formula

$$Z\tilde{g}(X, Y) = \tilde{g}(\tilde{\nabla}_Z X, Y) + \tilde{g}(X, \tilde{\nabla}^*_Z Y), \tag{2.3}$$

where $X, Y, Z \in \Gamma(T\tilde{M})$,then $\tilde{\nabla}^*$ is called conjugate connection or dual connection of $\tilde{\nabla}$ with respect to $\tilde{g}$. U.Simon ([46], [59]) introduces an excellent survey for notion conjugate connection. In the triple $(\tilde{g}, \tilde{\nabla}, \tilde{\nabla}^*)$ is called a dualistic structure on $\tilde{M}$. Using (2.1) and (2.3), we have

$$2\tilde{\mathcal{K}} = \tilde{\nabla} - \tilde{\nabla}^* \text{ and } \tilde{\nabla}^{LC} = \frac{\tilde{\nabla} + \tilde{\nabla}^*}{2} \tag{2.4}$$

By the way it should be stated that $\tilde{\nabla} = \tilde{\nabla}^*$ if and only if $\tilde{\nabla}$ coincides with the Levi-Civita connection $\bar{\nabla}^{LC}$. By (2.2) if $(\tilde{M}, \tilde{g}, \tilde{\nabla})$ is a statistical manifold then $(\tilde{M}, \tilde{g}, \tilde{\nabla}^*)$ is also.

Now, let us introduce the following example of a statistical manifold that will form the foundation of our work. Furthermore, for 2 dimensions, we would like to emphasize once again this example is closely related to the normal distribution.

Example 2.1 ([20, 1]). Let us consider upper half sapace $\tilde{H}^{n+1} = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} | x_{n+1} > 0\}$ with Riemannian metric $\tilde{g} = \frac{dx_1^2 + \cdots + dx_{n+1}^2}{x_{n+1}^2}$ Then one can define an affine connection $\tilde{\nabla}$ on $\tilde{H}^{n+1}$ by the following relations

$$\begin{array}{c} \tilde{\nabla}_{\partial_{x_i}} \partial_{x_j} = \frac{2\delta_{ij}}{x_{n+1}} \partial_{x_{n+1}} \\ \tilde{\nabla}_{\partial_{x_i}} \partial_{x_{n+1}} = 0 = \tilde{\nabla}_{\partial_{x_{n+1}}} \partial_{x_i} = 0 \\ \tilde{\nabla}_{\partial_{x_{n+1}}} \partial_{x_{n+1}} = \frac{1}{x_{n+1}} \partial_{x_{n+1}} \end{array} \tag{2.5}$$

Then $(\tilde{H}^{n+1} <, >, \tilde{\nabla})$ is a statistical manifold of constant sectional curvature 0.

Now we review some basic information for statistical immersion of statistical manifolds.

Lemma 2.1 ([20]). *Let $(\tilde{M}, \tilde{g}, \tilde{\nabla})$ be a statistical manifold and $\phi : M \to \tilde{M}$ be an immersion. If g and ∇ are defined as following*

$$g(\phi_* X, \phi_* Y) = \tilde{g}(X, Y) \text{ and } g(\nabla_X Y, Z) = \tilde{g}(\bar{\nabla}_X \phi_* Y, \phi_* Z)$$

for any $X, Y, Z \in \Gamma(TM)$ then (M, g, ∇) is a statistical manifold, ϕ is called statistical immersion. Moreover M is said to be statistical hypersurface if $codim(M) = 1$.

If M is statistical hypersurface of $\bar{M}$ then the Gauss and Weingarten formulas are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y)N, \;\; \tilde{\nabla}_X N = -A_N^* X + \tau(X)N, \tag{2.6}$$

$$\tilde{\nabla}_X^* Y = \nabla_X^* Y + h^*(X, Y)N, \;\; \tilde{\nabla}_X^* N = -A_N X + \tau^*(X)N \tag{2.7}$$

for $X, Y \in \Gamma(TM)$ and $N \in \Gamma(T^{\perp} M)$, where $h(X, Y)$, $h^*(X, Y)$ are symmetric bilinear forms on $\Gamma(TM)$ and called second fundamental forms. $\tau, \tau^*$are 1-forms $(1, 1)$ type tensor field A_N (A_N^*) denotes shape operators of M with respect to $\bar{\nabla}$ $(\bar{\nabla}^*)$. The second fundamental forms, the shape operators and 1-forms τ, τ^* are related by

$$h(X, Y) = g(A_N X, Y), \;\; h^*(X, Y) = g(A_N^* X, Y), \;\; \tau(X) + \tau^*(X) = 0$$

For Levi-Civita connection, we have

$$\tilde{\nabla}_X^{LC} Y = \nabla_X^{LC} Y + h^{LC}(X, Y)N, \;\; \tilde{\nabla}_X^{LC} N = -A_N^{LC} X \tag{2.8}$$

The mean curvatures of M are defined with respect to $\tilde{\nabla}$ and $\tilde{\nabla}^*$ by

$$H = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i) \text{ and } H^* = \frac{1}{n} \sum_{i=1}^{n} h^*(e_i, e_i)$$

where $\{e_1, ..., e_n\}$ is a local orthonormal frame field on M. By (2.4), we get $2h^{LC} = h + h^*$ and $2H^{LC} = H + H^*$. A statistical hypersurface M is said to be H (or H^*, H^{LC})-type minimal if $H = 0$ (or $H^* = 0, H^{LC} = 0$).

Let

$$\phi : (s, t) \in D \subset \mathbb{R}^2 \to \bar{M}^3$$

be parametrization of the statistical surface M of the statistical manifold $\bar{M}^3$.The natural coefficients of the first fundamental form, expressed in terms of the basis $\{\phi_s, \phi_t\}$ associated with a parametrization $\phi(s, t)$ at point p, can be written as follows:

$$E = <\phi_s, \phi_s> , \; F = <\phi_s, \phi_t> \text{ and } G = <\phi_t, \phi_t> . \tag{2.9}$$

The unit normal vector field of M is given by

$$N = \frac{\phi_s \times \phi_t}{\| \phi_s \times \phi_t \|}. \tag{2.10}$$

By Equation (2.3) and Gauss and Weingarten formulas (2.6), (2.7) we have

$$-g(\bar{\nabla}^*_X N, Y) = \tilde{g}(\tilde{\nabla}_X Y, N) = g(A_N X, Y), \tag{2.11}$$
$$-g(\tilde{\nabla}_X N, Y) = \tilde{g}(\tilde{\nabla}^*_X Y, N) = g(A^*_N X, Y) \tag{2.12}$$

where $X, Y \in \Gamma(TM)$.

According to the basis $\{\phi_s, \phi_t\}$,the shape operator A_N and conjugate shape operator can A^*_N be expressed as follows:

$$\begin{aligned} A_N \phi_s &= h_{11}\phi_s + h_{12}\phi_t \\ A_N \phi_t &= h_{21}\phi_s + h_{22}\phi_t \end{aligned} \tag{2.13}$$

$$\begin{aligned} A^*_N \phi_s &= h^*_{11}\phi_s + h^*_{12}\phi_t \\ A^*_N \phi_t &= h^*_{21}\phi_s + h^*_{22}\phi_t \end{aligned} \tag{2.14}$$

If we take notice of Equations (2.11) and (2.13) then we readily arrive at coefficients of second fundamental form h

$$h_{11} = \frac{\begin{vmatrix} \tilde{g}(\bar{\nabla}_{\phi_s}\phi_s, N) & F \\ \tilde{g}(\bar{\nabla}_{\phi_s}\phi_t, N) & G \end{vmatrix}}{EG - F^2}, h_{22} = \frac{\begin{vmatrix} E & \tilde{g}(\bar{\nabla}_{\phi_t}\phi_s, N) \\ F & \tilde{g}(\bar{\nabla}_{\phi_t}\phi_t, N) \end{vmatrix}}{EG - F^2}$$

and

$$h_{12} = \frac{\begin{vmatrix} \tilde{g}(\tilde{\nabla}_{\phi_s}\phi_t, N) & F \\ \tilde{g}(\tilde{\nabla}_{\phi_s}\phi_s, N) & E \end{vmatrix}}{EG - F^2}, h_{21} = \frac{\begin{vmatrix} \tilde{g}(\tilde{\nabla}_{\phi_t}\phi_s, N) & F \\ \tilde{g}(\tilde{\nabla}_{\phi_t}\phi_t, N) & G \end{vmatrix}}{EG - F^2}$$

We thus see that the mean curvature H

$$H = \frac{h_{11} + h_{22}}{2} = \frac{G\tilde{g}(\tilde{\nabla}_{\phi_s}\phi_s, N) - F(\tilde{g}(\tilde{\nabla}_{\phi_s}\phi_t, N) + \tilde{g}(\tilde{\nabla}_{\phi_t}\phi_s, N)) + E\tilde{g}(\tilde{\nabla}_{\phi_t}\phi_t, N)}{2(EG - F^2)}. \tag{2.15}$$

Morover, with the help of similar calculations, we have conjugate mean curvature of H^*

$$H^* = \frac{h^*_{11} + h^*_{22}}{2} = \frac{G\tilde{g}(\tilde{\nabla}^*_{\phi_s}\phi_s, N) - F(\tilde{g}(\tilde{\nabla}^*_{\phi_s}\phi_t, N) + \tilde{g}(\tilde{\nabla}^*_{\phi_t}\phi_s, N)) + E\tilde{g}(\tilde{\nabla}^*_{\phi_t}\phi_t, N)}{2(EG - F^2)} \tag{2.16}$$

The coefficients of second fundamental form h^* is given by

$$h^*_{11} = \frac{\begin{vmatrix} \tilde{g}(\tilde{\nabla}^*_{\phi_s}\phi_s, N) & F \\ \tilde{g}(\tilde{\nabla}^*_{\phi_s}\phi_t, N) & G \end{vmatrix}}{EG - F^2}, h^*_{22} = \frac{\begin{vmatrix} E & \tilde{g}(\tilde{\nabla}^*_{\phi_t}\phi_s, N) \\ F & \tilde{g}(\tilde{\nabla}^*_{\phi_t}\phi_t, N) \end{vmatrix}}{EG - F^2},$$

and

$$h^*_{12} = \frac{\begin{vmatrix} \tilde{g}(\tilde{\nabla}^*_{\phi_s}\phi_t, N) & F \\ \tilde{g}(\tilde{\nabla}^*_{\phi_s}\phi_s, N) & E \end{vmatrix}}{EG - F^2}, h^*_{21} = \frac{\begin{vmatrix} \tilde{g}(\tilde{\nabla}^*_{\phi_t}\phi_s, N) & F \\ \tilde{g}(\tilde{\nabla}^*_{\phi_t}\phi_t, N) & G \end{vmatrix}}{EG - F^2}.$$

3. Statistical minimal translation surfaces in upperhalf-space model of hyperbolic space $\tilde{\mathbb{H}}^3$with natural statistical structure

We recall that Lopez [39] studied the translation surfaces in the upper half-space model of hyperbolic space 3 and proved that the only minimal ones are the totally geodesic. Later on, the minimal translation hypersurfaces in the upper half-space model of hyperbolic space is characterized by Seo [52].

By the above motivation, it will be interesting to investigate the statistical translation surface of the upper half-space model of statistical hyperbolic space $\tilde{H}^3$ introduced in Example1 . So, we deduce the following relations between components of the torsion free affine conections $\tilde{\nabla}$ and $\tilde{\nabla}^*$ on $\tilde{H}^3$,respectively:

$$\begin{array}{ccc} \tilde{\nabla}_{\partial_x}\partial_x = \frac{2}{z}\partial_z, & \tilde{\nabla}_{\partial_y}\partial_x = 0, & \tilde{\nabla}_{\partial_z}\partial_x = 0, \\ \tilde{\nabla}_{\partial_x}\partial_y = 0, & \tilde{\nabla}_{\partial_y}\partial_y = \frac{2}{z}\partial_z, & \tilde{\nabla}_{\partial_z}\partial_y = 0, \\ \tilde{\nabla}_{\partial_x}\partial_z = 0, & \tilde{\nabla}_{\partial_y}\partial_z = 0, & \tilde{\nabla}_{\partial_z}\partial_z = \frac{1}{z}\partial_z, \end{array} \tag{3.1}$$

and

$$\begin{array}{ccc} \tilde{\nabla}^*_{\partial_x}\partial_x = 0, & \tilde{\nabla}_{\partial_y}\partial_x = 0, & \tilde{\nabla}^*_{\partial_z}\partial_x = -\frac{2}{z}\partial_x, \\ \tilde{\nabla}^*_{\partial_x}\partial_y = 0, & \tilde{\nabla}_{\partial_y}\partial_y = 0, & \tilde{\nabla}^*_{\partial_z}\partial_y = -\frac{2}{z}\partial_y, \\ \tilde{\nabla}^*_{\partial_x}\partial_z = -\frac{2}{z}\partial_x, & \tilde{\nabla}_{\partial_y}\partial_z = -\frac{2}{z}\partial_x, & \tilde{\nabla}^*_{\partial_z}\partial_z = -\frac{3}{z}\partial_z. \end{array} \tag{3.2}$$

Definition 3.1. I) M is said to be a statistical translation surface of I type if M is parametrized by $\phi(s,t) = (s,t,f(s)+g(t))$ in the upper half-space model of statistical hyperbolic space $\tilde{H}^3$ for $f(s)+g(t) > 0$.

2) M is said to be a statistical translation surface of II type if M is parametrized by $\phi(s,t) = (s, f(s)+g(t)), t)$ in the upper half-space model of statistical hyperbolic space $\tilde{H}^3$ for all $t > 0$.

Theorem 3.1. *Let M be a statistical translation surface of I type. Then if M is an H-type-minimal surface then the functions f and g are as follows:*

$$\begin{aligned} f &= f_0 = \textit{constant}, \\ g(t) &= I^{-1}(t) + m,\ I(t) = \mp\sqrt{3}\int^t \frac{c_1^3(f_0+\tau)^3}{\sqrt{1-4c_1^6(f_0+\tau)^6}}d\tau \end{aligned}$$

where $m, c_0 \in \mathbb{R}, c_1 \in \mathbb{R}^+$ *and* $f(s)+g(t) > 0, 1-4c_1^6(f_0+\tau)^6 > 0$.

Proof. Then we get

$$\begin{aligned} \phi_s &= (1,0,f'(s)) = \partial_x + f'(s)\partial_z, \\ \phi_t &= (0,1,g'(t)) = \partial_y + g'(t)\partial_z \end{aligned}$$

So the coefficients of the first fundamental form of M are obtained by

$$\begin{aligned} E &= \tilde{g}_{\tilde{H}^3}(\phi_s,\phi_s) = \frac{1+f'(s)^2}{(f(s)+g(t))^2}, F = \tilde{g}_{\tilde{H}^3}(\phi_s,\phi_t) = \frac{f'(s)g'(t)}{(f(s)+g(t))^2}, \\ \text{and } G &= \tilde{g}_{\tilde{H}^3}(\phi_t,\phi_t) = \frac{1+g'(t)^2}{(f(s)+g(t))^2} \end{aligned}$$

For the computation mean curvature H, we need to know unit normal vector field N of M. Hence we have

$$N = \frac{\phi_s \times \phi_t}{\| \phi_s \times \phi_t \|} = -\frac{f'(s)}{W}\partial_x - \frac{g'(t)}{W}\partial_y + \frac{1}{W}\partial_z, \tag{3.3}$$

where $W = \sqrt{\| \phi_s \times \phi_t \|} = \frac{1}{f(s)+g(t)}\sqrt{f'(s)^2+g'(t)^2+1}$. On the other hand, by Equation (3.1), to compute Equation (2.15), we have to need the following quantities:

$$\begin{aligned} \tilde{\nabla}_{\phi_s}\phi_s &= (f''(s) + \frac{2}{f(s)+g(s)} + \frac{f'(s)^2}{f(s)+g(t)})\partial_z, \\ \tilde{\nabla}_{\phi_s}\phi_t &= \frac{f'(s)g'(t)}{f(s)+g(t)}\partial_z = \tilde{\nabla}_{\phi_t}\phi_s \\ \tilde{\nabla}_{\phi_t}\phi_t &= (g''(t) + \frac{2}{f(s)+g(t)} + \frac{g'(t)^2}{f(s)+g(t)})\partial_z. \end{aligned} \tag{3.4}$$

We thus have

$$\begin{aligned} \tilde{g}_{\tilde{H}^3}(\tilde{\nabla}_{\phi_s}\phi_s, N) &= \frac{1}{W(f(s)+g(t))^3}(f''(s)(f(s)+g(t)) + 2 + f'(s)^2), \\ \tilde{g}_{\tilde{H}^3}(\tilde{\nabla}_{\phi_s}\phi_t, N) &= \frac{1}{W(f(s)+g(t))^3}(f'(s)g'(t)) = \tilde{g}_{\tilde{H}^3}(\tilde{\nabla}_{\phi_t}\phi_s, N) \\ \tilde{g}_{\tilde{H}^3}(\tilde{\nabla}_{\phi_t}\phi_t, N) &= \frac{1}{W(f(s)+g(t))^3}(g''(t)((f(s)+g(t)) + 2 + g'(t)^2). \end{aligned} \tag{3.5}$$

It is evident from Equation (2.15) that $H = 0$,if and only if

$$0 = (1+g'(t)^2)(f''(s)(f(s)+g(t))+2+f'(s)^2) - 2f'(s)^2g'(t)^2 + (1+f'(s)^2)(g''(t)(f(s)+g(t))+2+g'(t)^2). \tag{3.6}$$

It follows that

$$0 = (f(s)+g(t))\left(\frac{f''(s)}{1+f'(s)^2}+\frac{g''(t)}{1+g'(t)^2}\right) + \frac{4+3f'(s)^2+3g'(t)^2}{(1+f'(s)^2)(1+g'(t)^2)} \tag{3.7}$$

Differentiating (3.7) with respect to s and then t, we get

$$f'(s)\left(\frac{g''(t)}{1+g'(t)^2}\right)' + g'(t)\left(\frac{f''(s)}{1+f'(s)^2}\right)' = 8\frac{f'(s)f''(s)g'(t)g''(t)}{(1+f'(s)^2)(1+g'(t)^2)} \tag{3.8}$$

We now have three cases:
Case1: $f'(s)g'(t) \neq 0$,
Case2: $f'(s) = 0$ or $g'(t) = 0$,
Case3: $f'(s) = 0$ and $g'(t) = 0$.
Now we discuss these cases respectively.Case1:
Then Equation (3.8) can be written as

$$\frac{1}{g'(t)}\left(\frac{g''(t)}{1+g'(t)^2}\right)' + \frac{1}{f'(s)}\left(\frac{f''(s)}{1+f'(s)^2}\right)' = 8\frac{f''(s)g''(t)}{(1+f'(s)^2)^2(1+g'(t)^2)^2} \tag{3.9}$$

We differentiate (3.9) with respect to s and t, and we conclude that

$$(f'''(s)(1+f'(s)^2) - 4f''(s)^2f'(s))(g'''(t)(1+g'(t)^2) - 4g'(t)g''(t)^2) = 0. \tag{3.10}$$

Then Equation (3.10) means that

$$f'''(s)(1+f'(s)^2) - 4f''(s)^2f'(s) = 0, \text{ or } g'''(t)(1+g'(t)^2) - 4g'(t)g''(t)^2 = 0. \tag{3.11}$$

Since there is no superiority of f over g, we can take

$$f'''(s)(1+f'(s)^2) - 4f''(s)^2f'(s) = 0.$$

If we integrate above equation we obtain $f''(s) = a(1+f'(s)^2)^2$ for some constant a. If this equality is used in Equation (3.9), it is obtained

$$\frac{1}{g'(t)}\left(\frac{g''(t)}{1+g'(t)^2}\right)' + 2af''(s) = 8a\frac{g''(t)}{(1+g'(t)^2)^2} \tag{3.12}$$

Now we discuss subcases.
Subcase1:
Let us suppose that $a = 0$.Then we obtain $f(s) = c_1s + c_0$, with $c_1, c_0 \in \mathbb{R}$. Then by (3.12) we have $g''(t) = b(1+g'(t)^2)$ for some constant b.
If $b \neq 0$, we conclude that

$$g(t) = -\frac{1}{b}\ln(\cos(bt+d_1)) + d_0, \text{with } d_1, d_0 \in \mathbb{R}. \tag{3.13}$$

Substituting (3.13) in (3.7), we get

$$0 = b(c_1s + c_0 - \frac{1}{b}\ln(\cos(bt+d_1)) + d_0 + \frac{4+3c_1^2+3\tan^2(bt+d_1)}{(1+c_1^2)(1+\tan^2(bt+d_1)} \tag{3.14}$$

So we deduce that $c_1 = 0$. Then we $f(s) = c_0$ which concludes that f is a constant function and it is a contradiction with $f'(s)g'(t) \neq 0$. Finally b cannot be different from zero.

If $b = 0$, we get $g(t) = m_1 t + m_0$,with $m_0, m_1 \in \mathbb{R}$. Then Equation (3.7) can be written as

$$\frac{4 + 3c_1^2 + 2m_1^2}{(1 + c_1^2)(1 + m_1^2)} = 0$$

This equality is not meaningful. So it raises a contradiction again.

Subcase2:

Assume now $a \neq 0$. Then by (3.7), we deduce the existence of a real number $c \in \mathbb{R}$ such that

$$2af''(s) = -b \tag{3.15}$$

$$\frac{1}{g'(t)}\left(\frac{g''(t)}{1 + g'(t)^2}\right)' - 8a\frac{g''(t)}{(1 + g'(t)^2)^2} = b \tag{3.16}$$

From (3.15), we have

$$f(s) = -\frac{b}{4a}s^2 + c_1 s + c_0, \text{with } c_1, c_0 \in \mathbb{R} \tag{3.17}$$

On the other hand we know that function f satisfies

$$f''(s) = a(1 + f'(s)^2)^2 \tag{3.18}$$

Substituting (3.17) in to (3.18) we get a polynomial equation of degree four in $f(s)$. Then all the coefficients must be zero. It means that $b = c_1 = 0$.Thus we say that f is a constant function. In this case it leads to contradiction with $f'(s)g'(t) \neq 0$.

Case2: Now let assume that $f'(s) = 0$ and $g'(t) \neq 0$. Then $f(x) = f_0$ for some constant f_0. It follows from (3.7) we have

$$0 = (c_2 + g(t))g''(t) + 4 + 3g'(t)^2 \tag{3.19}$$

We put $g'(t) = u$. Then

$$g''(t) = \frac{du}{dg}\frac{dg}{dt} = u\frac{du}{dg}$$

So (3.19) becomes

$$(f_0 + g)u\frac{du}{dg} + 3u^2 + 4 = 0$$

and its general solution is given by

$$u = \frac{dg}{dt} = \mp\frac{1}{\sqrt{3}}\frac{\sqrt{1 - 4c_1^6(f_0 + g)^6}}{c_1^3(f_0 + g)^3}, \quad c_1 \in \mathbb{R}^+.$$

Integration implies

$$\mp\frac{1}{\sqrt{3}}\int^t \frac{\sqrt{1 - 4c_1^6(f_0 + g(\tau)^6}g'(\tau)}{c_1^3(f_0 + g(\tau))^3}d\tau = t + c_0$$

Let us consider

$$I(t) = \mp\frac{1}{\sqrt{3}}\int^t \frac{\sqrt{1 - 4c_1^6(f_0 + \tau)^6}}{c_1^3(f_0 + \tau)^3}d\tau, c_1 \in \mathbb{R}^+$$

Since $f_0 + g > 0$ and $c_1 \in \mathbb{R}^+$, $I(t)$ is stricly increasing (decreasing) function with respect to $\mp\sqrt{3}$. Thus, the equation $I(g(t)) = t + c_0$ has a unique solution $g(t) = I^{-1}(t)$.

Case3. If $f'(s) = 0$ and $g'(t) = 0$ then (3.7) becomes $0 = 4$. So this case not occure. □

Corollary 3.1. *Let f and g be real valued smooth,non constant functions with $f + g > 0$. Then there are not a H-type-minimal statistical translation surface given with parameterization $(s, t, f(s) + g(t))$ in the upper half-space model of statistical hyperbolic space $\tilde{H}^3$.*

Theorem 3.2. *Let M be a statistical translation surface of II type. If M is an H-type-minimal surface then then the smooth functions f and g are given by*

$$\begin{aligned} f(s) &= ms+n, \\ g(t) &= \pm\sqrt{3(1+a^2)}\int \frac{c^3t^3}{\sqrt{1-4c^6t^6}}dt, \end{aligned}$$

where m, n, a, c are constants and $t > 0$.

Proof. Let

$$\begin{aligned} \phi &: (s,t) \in D \subset \mathbb{R}^2 \to \tilde{H}^3 \\ (s,t) &\to \phi(s,t) = (s, f(s)+g(t), t) \end{aligned} \tag{3.20}$$

be a parameterization of M of the statstistical manifold $\tilde{H}^3$, where $t > 0$ and. Then we get

$$\begin{aligned} \phi_s &= (1, f'(s), 0) = \partial_x + f'(s)\partial_y, \\ \phi_t &= (0, g'(t), 1) = g'(t)\partial_y + \partial_z. \end{aligned} \tag{3.21}$$

Hence, Riemannian metric tensor components reduced from $\tilde{H}^3$ to M are given by

$$\begin{aligned} E &= \tilde{g}_{\tilde{H}^3}(\phi_s,\phi_s) = \frac{1+f'(s)^2}{t^2}, F = \tilde{g}_{\tilde{H}^3}(\phi_s,\phi_t) = \frac{f'(s)g'(t)}{t^2}, \\ \text{and } G &= \tilde{g}_{\tilde{H}^3}(\phi_t,\phi_t) = \frac{1+g'(t)^2}{t^2}. \end{aligned}$$

To compute the secondfundamental form of M, we have to calculate Riemannian connection

$$\begin{aligned} \tilde{\nabla}_{\phi_s}\phi_s &= f''(s)\partial_y + \frac{2}{t}(1+f'(s)^2)\partial_z, \\ \tilde{\nabla}_{\phi_s}\phi_t &= 2\frac{f'(s)g'(t)}{t}\partial_z = \tilde{\nabla}_{\phi_t}\phi_s \\ \tilde{\nabla}_{\phi_t}\phi_t &= g''(t)\partial_y + \frac{2g'(t)^2+1}{t}\partial_z, \end{aligned} \tag{3.22}$$

and unit normal vector field

$$N = \frac{\phi_s \times \phi_t}{\| \phi_s \times \phi_t \|} = \frac{f'(s)}{W}\partial_x - \frac{1}{W}\partial_y + \frac{g'(t)}{W}\partial_z \tag{3.23}$$

where $W = \| \phi_s \times \phi_t \| = \frac{1}{t}\sqrt{1+f'(s)^2+g'(t)^2}$. So we obtain coefficients of the second fundamental form of M due to (3.22) and (3.23)

$$\begin{aligned} \tilde{g}_{\tilde{H}^3}(\tilde{\nabla}_{\phi_s}\phi_s, N) &= -\frac{1}{Wt^2}f''(s) + \frac{2}{Wt^3}(1+f'(s)^2)g'(t)), \\ \tilde{g}_{\tilde{H}^3}(\tilde{\nabla}_{\phi_s}\phi_t, N) &= \frac{2}{Wt^3}f'(s)g'(t)^2 = \tilde{g}_{\tilde{H}^3}(\tilde{\nabla}_{\phi_t}\phi_s, N) \\ \tilde{g}_{\tilde{H}^3}(\tilde{\nabla}_{\phi_t}\phi_t, N) &= -\frac{1}{Wt^2}g''(t) + \frac{1}{Wt^3}(1+2g'(t)^2)g'(t)). \end{aligned} \tag{3.24}$$

So it follows from (2.15) and (3.24) that $H = 0$ if and only if

$$\begin{aligned} 0 &= (1+g'(t)^2)(-tf''(s)+2(1+f'(s)^2)g'(t)) - 4f'(s)^2g'(t)^3 \\ &\quad +(1+f'(s)^2)(-tg''(t)+(1+2g'(t)^2)g'(t)) \end{aligned} \tag{3.25}$$

From (3.25),we immediately obtain

$$t\left(\frac{f''(s)}{1+f'(s)^2} + \frac{g''(t)}{1+g'(t)^2}\right) = \left(\frac{3+3f'(s)^2+4g'(t)^2}{(1+f'(s)^2)(1+g'(t)^2)}\right)g'(t) \tag{3.26}$$

If $g'(t) = 0$ then $g(t) = g_0$ =constant. In this case we have $f(s) = ms+n$ for some constants m, n.

Now we suppose that $g'(t) \neq 0$.Then differentiating (3.25) with respect to s, we get

$$t(\frac{f''(s)}{1+f'(s)^2})' = -8\frac{f'(s)f''(s)g'(t)^3}{(1+f'(s)^2)^2(1+g'(t)^2)} \tag{3.27}$$

Firstly we suppose that $f'(s)f''(s) = 0$. Then $f(x) = ax+\ b$ for some constants a and b. In this case Equation (3.25) can be written by

$$tg''(t) = 3g'(t) + \frac{4}{1+a^2}g'(t)^3 \tag{3.28}$$

Now we put $g'(t) = p$. Then Equation (3.28) can bewritten as

$$t\frac{dp}{dt} = 3p + \frac{4}{1+a^2}p^3.$$

Its general solution is given by

$$p = \pm\sqrt{3(1+a^2)}\frac{c^3t^3}{\sqrt{1-4c^6t^6}},$$

where c is constant. From this, we thus get

$$g(t) = \pm\sqrt{3(1+a^2)}\int\frac{c^3t^3}{\sqrt{1-4c^6t^6}}dt.$$

Now let us suppose that $f'(s)f''(s) \neq 0$. Then by (3.28) we get

$$-\frac{(1+f'(s)^2)^2}{8f'(s)f''(s)}(\frac{f''(s)}{1+f'(s)^2})' = \frac{g'(t)^3}{t(1+g'(t)^2)} = c \tag{3.29}$$

If $c = 0$ then $\frac{f''(s)}{1+f'(s)^2} = k \in \mathbb{R}$ and g is constant function. By (3.28) one sees that $k = 0$. Thus we say that $f(s) = ms + n$, $m, n \in \mathbb{R}$.This is a contradiction by our our assumption that $f'(s)f''(s) \neq 0$.If $c \neq 0$ then

$$g'(t)^3 = ct(1+g'(t)^2). \tag{3.30}$$

Differentiating (3.30) we get

$$g''(t)g'(t)(3g'(t) - 2ct) = c(1+g'(t)^2).$$

Since $c \neq 0$, we have $g''(t)g'(t)(3g'(t) - 2ct) \neq 0$. So we arrive

$$\frac{g''(t)}{1+g'(t)^2} = \frac{c}{g'(t)(3g'(t)-2ct)}. \tag{3.31}$$

Integrating first side of (3.29), we have

$$\frac{f''(s)}{1+f'(s)^2} = \frac{4c}{1+f'(s)^2} + b, b \in \mathbb{R}. \tag{3.32}$$

Substituting (3.31) and (3.32) into (3.26), we have

$$t(\frac{4c}{1+f'(s)^2} + b + \frac{c}{g'(t)(3g'(t)-2ct)}) = (\frac{3+3f'(s)^2+4g'(t)^2}{(1+f'(s)^2)(1+g'(t)^2)})g'(t). \tag{3.33}$$

On inserting (3.30) into (3.33) we obtain

$$\frac{4c}{1+f'(s)^2} + b + \frac{c}{g'(t)(3g'(t)-2ct)} = c\frac{3+3f'(s)^2+4g'(t)^2}{g'(t)^2(1+f'(s)^2)}. \tag{3.34}$$

If we rearrange (3.34), we get

$$bg'(t)^2(3g'(t)^2 - 2cg'(t)t) + cg'(t)^2 = 3c(3g'(t)^2 - 2ctg'(t)).$$

Simplfying the last equation, we have

$$3bg'(t)^3 - 2bctg'(t)^2 - 8cg'(t) + 6c^2t = 0 \tag{3.35}$$

We assume that $b = 0$. Then $g'(t) = \frac{3ct}{4}$. Substituting this equation in (3.30) we obtain following polynominal function with respect to variable t.

$$\frac{63c^2}{256}t^2 + 1 = 0 \tag{3.36}$$

This equation leads to a contradiction.

Now we suppose that $b \neq 0$.If we substitute (3.30) in to (3.35) and set $X = g'(t)$ we find

$$5btX^2 + 8X - 6ct + 3bt = 0. \tag{3.37}$$

When we multiply (3.30) by $-2b$ and obtained equation add to (3.30), we obtain

$$bX^3 - 8cX + 2bct + 6c^2t = 0. \tag{3.38}$$

Now after multiplying (3.37) by X and (3.38) by $-5t$ and then subtracting the equations obtained from each other, we get

$$8X^2 + t(34c + 3b)X - 5ct^2(2b + 6c) = 0. \tag{3.39}$$

By (3.37) and (3.39) we obtain

$$X = \frac{-48c + 24b + 4b^2ct^2 + 6bc^2t^2}{5bt^2(34c + 3b) - 64}t$$

If we substitute this value of X into equation (3.37) we get a 7th-degree polynomial equation dependent on the variable t as follows

$$\begin{aligned} 0 \quad = \quad & 5b(6bc^2t^2 + 4b^2ct^2 + 24b - 48c)^2t^3 \\ & +8(6bc^2t^2 + 4b^2ct^2 + 24b - 48c)((34c + 3b)bt^2 - 64)t \\ & (-6c + 3b)((34c + 3b)bt^2 - 64)t. \end{aligned}$$

From this equation, we deduce that b and c equal to 0 from the last equation. This contradicts our assumption of $g'(t) \neq 0$ and $b \neq 0$. So we completed the proof. □

4. Statistical graph translation surfaces with nullsectional curvature in upper half-space model of hyperbolic space $\tilde{\mathbf{H}}^3$with natural statistical structure

For a statistical manifold $(\tilde{M}, \tilde{g}, \tilde{\nabla})$, let $\tilde{R}$ and $\tilde{R}^*$ be cuvature tensor fields with respect to dual connections $\tilde{\nabla}$ and $\tilde{\nabla}^*$.Then the following identies hold (see [48])

$$\begin{aligned} \tilde{R}(X,Y) + \tilde{R}^*(X,Y) &= 2\tilde{R}^{LC}(X,Y) + 2\,\tilde{Q}(X,Y) \\ \tilde{g}(\tilde{R}(X,Y)Z,W) &= -\tilde{g}(Z,\tilde{R}^*(X,Y)W), \end{aligned} \tag{4.1}$$

where $\tilde{Q}(X,Y) = [\tilde{\mathcal{K}}_X, \tilde{\mathcal{K}}_Y]$.

A statistical manifold $(\tilde{M}, \tilde{g}, \tilde{\nabla})$ said to be constant curvature $\varepsilon \in \mathbb{R}$ if

$$\tilde{g}(\tilde{R}(X,Y)Z,W) = \varepsilon(\tilde{g}(Y,Z)\tilde{g}(X,W) - \tilde{g}(Y,W)\tilde{g}(X,Z))$$

where $X, Y, Z, W \in \Gamma(T\tilde{M})$ [59]. If the statistical manifold $(\tilde{M}, \tilde{g}, \tilde{\nabla})$ is flat, then it is called Hessian manifold [29]. Since the tensor field $\tilde{g}(\tilde{R}(X,Y)Z,W)$ is conjugate skew symmetric for a statistical manifold on M, the new statistical curvature tensor field $\tilde{S}$ is defined by

$$\tilde{S}(X,Y,Z,W) = \frac{1}{2}(\tilde{g}(\tilde{R}(X,Y)Z,W) + \tilde{g}(\tilde{R}^*(X,Y)Z,W))$$

in [8]. Then

$$2\tilde{S}(X,Y,Z,W) = \tilde{g}(\tilde{R}^{LC}(X,Y)Z,W) + \tilde{g}(\tilde{Q}(X,Y)Z,W)$$

Let Π be tangent plane to $\tilde{M}$ at p and $\{X,Y\}$ be basis of Π. The the statistical sectional curvature is defined by

$$\tilde{K}(\Pi) = \frac{\tilde{g}(\tilde{S}(X,Y)Y,X)}{\tilde{g}(X,X)\tilde{g}(Y,Y) - \tilde{g}(X,Y)^2}.$$

Using (2.5) one can observe that $(\tilde{H}^{n+1}, \tilde{g}, \tilde{\nabla})$ is a Hessian manifold.

In [66],Vos obtained the Gauss equations for statistical submanifolds with respect to the dual connections. Theese equations are given for statistiacal hypersurface as follows:

Proposition 4.1 ([66]). *Let* (M, g, ∇)*be statistical hypersurface of statistical manifold* $(\tilde{M}, \tilde{g}, \tilde{\nabla})$ *with* $R(R^*)$ *and* $\tilde{R}$ $(\tilde{R}^*)$ *their Riemannian curvature tensors for* $\nabla(\ \nabla^*)$ *and* $\tilde{\nabla}(\ \tilde{\nabla}^*)$*, respectively. Then*

$$\begin{aligned} \tilde{g}(\tilde{R}(X,Y)Z,W) &= g(R(X,Y)Z,W) + g(A_N X, Z)g(A_N^* Y, W) \\ &\quad - g(A_N^* X, W)g(A_N Y, Z), \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} \tilde{g}(\tilde{R}^*(X,Y)Z,W) &= g(R^*(X,Y)Z,W) + g(A_N^* X, Z)g(A_N Y, W) \\ &\quad - g(A_N X, W)g(A_N^* Y, Z) \end{aligned} \tag{4.3}$$

for all $X, Y, Z, W \in \Gamma(TM)$.

Since $(\tilde{H}^{n+1}, \tilde{g}, \tilde{\nabla})$ is a Hessian manifold $\tilde{R}$ and $\tilde{R}^*$ are zero. Now we assume that (M, g, ∇) is a statistical surface of $(\tilde{H}^{n+1}, \tilde{g}, \tilde{\nabla})$. By 2.11and 2.12 , 4.2and 4.3 can be given by

$$g(R(X,Y)Z,W) = \tilde{g}(\tilde{\nabla}_X^* W, N)\tilde{g}(\tilde{\nabla}_Y Z, N) - \tilde{g}(\tilde{\nabla}_X Z, N)\tilde{g}(\bar{\nabla}_Y^* W, N), \tag{4.4}$$

and

$$g(R^*(X,Y)Z,W) = \tilde{g}(\tilde{\nabla}_X W, N)\tilde{g}(\tilde{\nabla}_Y^* Z, N) - \tilde{g}(\tilde{\nabla}_X^* Z, N)\tilde{g}(\tilde{\nabla}_Y W, N) \tag{4.5}$$

for all $X, Y, Z, W \in \Gamma(TM)$.

Theorem 4.1. *There are no statistical translation surfaces of I type with null statistical sectional curvature.*

Proof. By (3.21) and (3.2), we have

$$\begin{aligned} \tilde{\nabla}^*_{\phi_s}\phi_s &= -4\frac{f'(s)}{f(s)+g(s)}\partial_x + \frac{f''(s)(f(s)+g(t)) - 3f'(s)^2}{f(s)+g(s)}\partial_z, \\ \tilde{\nabla}^*_{\phi_s}\phi_t &= -\frac{2g'(t)}{f(s)+g(t)}\partial_x - \frac{2f'(s)}{f(s)+g(t)}\partial_y - \frac{3f'(s)g'(t)}{f(s)+g(t)}\partial_z \\ \tilde{\nabla}^*_{\phi_t}\phi_t &= -4\frac{g'(t)}{f(s)+g(s)}\partial_y + \frac{g''(t)(f(s)+g(t)) - 3g'(t)^2}{f(s)+g(s)}\partial_z. \end{aligned} \tag{4.6}$$

So we get

$$\begin{aligned} \tilde{g}(\tilde{\nabla}^*_{\phi_s}\phi_s, N) &= \frac{f''(s)(f(s)+g(t)) + f'(s)^2}{W(f(s)+g(t))^3}, \\ \tilde{g}(\tilde{\nabla}^*_{\phi_s}\phi_t, N) &= \frac{f'(s)g'(t)}{W(f(s)+g(t))^3}, \\ \tilde{g}(\tilde{\nabla}^*_{\phi_t}\phi_t, N) &= \frac{g''(t)(f(s)+g(t)) + g'(t)^2}{W(f(s)+g(t))^3}. \end{aligned} \tag{4.7}$$

By (4.4) and (4.5), if M is null statistical sectional curvaturecurvature we conclude that

$$\begin{aligned} 0 &= \tilde{g}_{\tilde{H}^3}(\tilde{\nabla}^*_{\phi_s}\phi_s, N)\tilde{g}_{\tilde{H}^3}(\tilde{\nabla}_{\phi_t}\phi_t, N) - \tilde{g}_{\tilde{H}^3}(\tilde{\nabla}_{\phi_s}\phi_t, N)\tilde{g}_{\tilde{H}^3}(\bar{\nabla}^*_{\phi_s}\phi_t, N) \\ &\quad + \tilde{g}_{\tilde{H}^3}(\tilde{\nabla}_{\phi_s}\phi_s, N)\tilde{g}_{\tilde{H}^3}(\tilde{\nabla}^*_{\phi_t}\phi_t, N) - \tilde{g}_{\tilde{H}^3}(\bar{\nabla}^*_{\phi_s}\phi_t, N)\tilde{g}_{\tilde{H}^3}(\tilde{\nabla}_{\phi_s}\phi_t, N) \end{aligned} \tag{4.8}$$

If we use (3.5), (4.6) in equation (4.5) then we readily arrive at the following equation.

$$\begin{aligned} 0 &= f''(s)g''(t)(f(s)+g(t))^2 + [f''(s)(1+g'(t)^2) + g''(t)(1+f'(s)^2)](f(s)+g(t)) \\ &\quad + f'(s)^2 + g'(t)^2. \end{aligned} \tag{4.9}$$

Differentiating the equation (4.9) with respect to s, we get

$$\begin{aligned} 0 &= f'''(s)g''(t)(f(s)+g(t))^2 + [f'''(s)(1+g'(t)^2) + 4f'(s)f''(s)g''(t)](f(s)+g(t)) \\ &\quad + [f''(s)(1+g'(t)^2) + g''(t)(1+f'(s)^2)]f'(s) + 2f'(s)f''(s). \end{aligned} \tag{4.10}$$

Differentiate the equation (4.9) with respect to t, we have

$$f''(s)g'''(t)(f(s)+g(t))^2+[g'''(t)(1+f'(s)^2)+4g'(t)g''(t)f''(s)](f(s)+g(t)) \\ +[f''(s)(1+g'(t)^2)+g''(t)(1+f'(s)^2)]g'(t)+2g'(t)g''(s). \tag{4.11}$$

Now, we can consider the equations (4.9), (4.10) and (4.11) as a quadratic polynomial equation on $f(s)+g(t)$.

Firstly we assume that $f''(s)=0$ and $g''(t)\neq 0$. Then f is linear, that is, $f(s)=as+b$ for some constants a,b. In this situation the equation (4.9) can be written as

$$0=g''(t)(1+a^2)](as+b+g(t))+a^2 \tag{4.12}$$

Now if we again differentiate the equation (4.12) with respect to s then we get

$$0=g''(t)(1+a^2)]a$$

and so we say $g''(t)=0$ or $a=0$. Since $g''(t)\neq 0$, $g''(t)$ can not be null. If $a=0$ the equation (4.12) says that $g(t)=-b$. We conclude that $f(s)+g(t)=0$. This leads contradiction with $f(s)+g(t)>0$. So this case does not occur.

Now secondly we consider $f''(s)\neq 0$ and $g''(t)\neq 0$ case. For this case we may assume that $f'''(s)=0$ and $g'''(t)\neq 0$ without loss of generality.Then the equation (4.10) appears as follows:

$$0=4f''(s)g''(t)(f(s)+g(t))+f''(s)(1+g'(t)^2)+g''(t)(1+f'(s)^2)+2f''(s) \tag{4.13}$$

Again differentiating the equation (4.13) with respect to s and using our assumption $f'''(s)=0$, we have

$$f''(s)f'(s)g''(t)=0.$$

Since $g''(t)\neq 0$ we say that $f''(s)f'(s)=0$. In this situation, we once again return to the previously discussed $f''(s)=0$. So we deduce that $f'''(s)\neq 0$ and $g'''(t)\neq 0$.

Now we assume that there exists a common $f(s)+g(t)$ solution for (4.9), (4.10) and (4.11).

Let us eliminate the coefficients of $(f(s)+g(t))^2$ in equations (4.9), (4.10) and (4.11). By multiplying with suitable the coefficients of $(f(s)+g(t))^2$ in (4.9) between (4.11),(4.10, between 4.11) and (4.9 between 4.10) we obtain the following linear equation systems with respect to the unknown value $f(s)+g(t)$

$$\begin{aligned} 0 &= g''(t)(4f'(s)f''(s)^2-f'''(s)(1+f'(s)^2)(f(s)+g(t)) \\ &\quad +(f''(s)(1+g'(t)^2)+g''(t)(1+f'(s)^2))f''(s)f'(s) \\ &\quad +2f'(s)f''(s)^2-f'''(s)(f'(s)^2+g'(t)^2), \end{aligned} \tag{4.14}$$

$$\begin{aligned} 0 &= f''(s)(4g'(t)g''(t)^2-g'''(t)(1+g'(t)^2)(f(s)+g(t)) \\ &\quad +(f''(s)(1+g'(t)^2)+g''(t)(1+f'(s)^2))g''(s)g'(s) \\ &\quad +2g'(t)g''(t)^2-g'''(t)(g'(t)^2+f'(s)^2), \end{aligned} \tag{4.15}$$

and

$$\begin{aligned} 0=(\quad & f''(s)g'''(t)[f'''(s)(1+g'(t)^2)+4f'(s)f''(s)g''(t)] \\ & -g''(t)f'''(s)[g'''(t)(1+f'(s)^2)+4g'(t)g''(t)f''(s)])(f(s)+g(t)) \\ & +f'(s)f''(s)g'''(t)(f''(s)(1+g'(t)^2)+g''(t)(1+f'(s)^2) \\ & -g'(t)g''(t)f'''(s)(f''(s)(1+g'(t)^2)+g''(t)(1+f'(s)^2) \\ & +2(f'(s)f''(s)^2g'''(t)-g'(t)g''(t)^2f'''(s)). \end{aligned} \tag{4.16}$$

If the system is to have a unique solution with respect to $f(s)+g(t)$, the coefficients must be linearly dependent. So we have then we have

$$\frac{g''(t)(4f'(s)f''(s)^2-f'''(s)(1+f'(s)^2)}{f''(s)(4g'(t)g''(t)^2-g'''(t)(1+g'(t)^2)}=k_1\in\mathbb{R}, \tag{4.17}$$

$$\frac{g''(t)(4f'(s)f''(s)^2-f'''(s)(1+f'(s)^2)}{A}=k_2\in\mathbb{R}, \tag{4.18}$$

and

$$\frac{f''(s)(4g'(t)g''(t)^2 - g'''(t)(1+g'(t)^2)}{A} = k_3 = \frac{k_2}{k_1}, \tag{4.19}$$

where

$$\begin{aligned} A &= f''(s)g'''(t)(f'''(s)(1+g'(t)^2) + 4f'(s)f''(s)g''(t)) \\ &\quad -g''(t)f'''(s)(g'''(t)(1+f'(s)^2) + 4g'(t)g''(t)f''(s)). \end{aligned} \tag{4.20}$$

Now we suppose that k_1, k_2 are different zero. Then we obtain from (4.17) and (4.18)

$$\frac{g''(t)}{4g'(t)g''(t)^2 - g'''(t)(1+g'(t)^2)} = k_1 \frac{f''(s)}{(4f'(s)f''(s)^2 - f'''(s)(1+f'(s)^2)} = a \in \mathbb{R}.$$

So we find that

$$g'''(t)(1+g'(t)^2) = 4g'(t)g''(t)^2 + \frac{1}{a}g''(t), \tag{4.21}$$

$$f'''(s)(1+f'(s)^2) = 4f'(s)f''(s)^2 + \frac{k_1}{a}f''(s). \tag{4.22}$$

By using the last equations in (4.20) we conclude that

$$A = \frac{1}{a}f''(s)g''(t)(f'''(s) - k_1 g'''(t)).$$

Hence the equation (4.18) reach to the conclusion

$$f'''(s) = k_1 g'''(t) - k_3 = c_1 \in \mathbb{R}$$

So we say that

$$f(s) = \frac{c_1}{6}s^3 + \frac{c_2}{2}s^2 + c_3 s + c_4 \tag{4.23}$$

$$g(t) = \frac{(c_1 + k_3)}{6k_1}t^3 + d_2 t^2 + d_3 t + d_4 \tag{4.24}$$

If the obtained value of $f(s)$ is substituted into (4.22), then $f(s) = 0$ which is in contradiction with $f'''(s) \neq 0$. (The similar situation applies to g as well.)

Now we assume that

$$f'''(s)(1+f'(s)^2) - 4f'(s)f''(s)^2 = 0 \tag{4.25}$$

If we employ realation (4.25) into (4.16), we get

$$\begin{aligned} 0 &= f''(s)f'''(s)(g'''(t)(1+g'(t)^2) - 4g'(t)g''(t)^2)(f(s)+g(t)) \\ &\quad +f'(s)f''(s)g'''(t)(f''(s)(1+g'(t)^2) + g''(t)(1+f'(s)^2) \\ &\quad -g'(t)g''(t)f'''(s)(f''(s)(1+g'(t)^2) + g''(t)(1+f'(s)^2) \\ &\quad +2(f'(s)f''(s)^2 g'''(t) - g'(t)g''(t)^2 f'''(s)). \end{aligned} \tag{4.26}$$

In order to solve for $f+g$ from the equations (4.15) and (4.16), the coefficients in front of $f+g$ must be linearly dependent. So we conclude that

$$f'''(s) = c \in \mathbb{R} \text{ or } 4g'(t)g''(t)^2 - g'''(t)(1+g'(t)^2) = 0.$$

For $f'''(s) = a_3$ we find that

$$f(s) = \frac{a_3}{6}x^3 + \frac{a_2}{2}x^2 + a_1 x + a_0,$$

for some constants a_3, a_2, a_1 and a_0. Replacing this expression of $f(s)$ in (4.25), we obtain a polynomial equation on s, namely,

$$\frac{7a_3}{4}s^4 + 7a_3^2 a_2 s^3 + (3a_3^2 a_1 + 9a_3 a_2^2)s^2 + (6a_3 a_2 a_1 - 4a_2^3)s + 4a_2^2 - a_3 a_1^2 + a_3 = 0.$$

This implies $a_3 = a_2 = a_1 = 0$ which leads a contradiction with $f'''(s) \neq 0$. Then we say that

$$g'''(t)(1 + g'(t)^2) - 4g'(t)g''(t)^2 = 0 \tag{4.27}$$

If we write $g'(t) = \frac{dg}{dt} = p(t)$ then

$$g''(t) = \frac{dp}{dt}, \;\; g'''(t) = \frac{d^2p}{dt^2}.$$

So the equation (4.25) is rewritten in the form

$$(1 + p^2)\frac{d^2p}{dt^2} - 4p(\frac{dp}{dt})^2 = 0 \tag{4.28}$$

We put $g''(t) = \frac{dp}{dt} = u$. Then we find

$$g'''(t) = \frac{d^2p}{dt^2} = \frac{du}{dp}\frac{dp}{dt} = u\frac{du}{dp}$$

When we set $\frac{d^2p}{dt^2} = u\ \frac{du}{dp}$in the equation (4.25) we obtain the following differential eqaution

$$(1 + p^2)u\frac{du}{dp} - 4pu^2 = 0 \tag{4.29}$$

Due to $g''(t) \neq 0$, from the equation (4.29) we have

$$(1 + p^2)\frac{du}{dp} - 4pu = 0$$

From this, we get following solution

$$u = c(1 + p^2)^2, \; c \in \mathbb{R}^+$$

and thus we conclude that

$$g''(t) = c(1 + g'(t)^2)^2, \; c \in \mathbb{R}^+.$$

It means that $g''(t) > 0$. Using the similiar solution method for the equation (4.25) ,we obtain

$$f''(s) = b(1 + f'(s)^2)^2 > 0, \; b \in \mathbb{R}^+.$$

Then from the equation (4.9) we have $f''(s) = 0$ and$g''(t) = 0$. It is a contradiction.

Now we suppose that the equations (4.9),(4.10) and (4.11) have same roots. In that case, the coefficients of these three equations must be proportional. In other words

$$\begin{aligned} \frac{f'''(s)g''(t)}{f''(s)g''(t)} &= \frac{f'''(s)(1 + g'(t)^2) + 4f'(s)f''(s)g''(t)}{f''(s)(1 + g'(t)^2) + g''(t)(1 + f'(s)^2} \\ &= \frac{[f''(s)(1 + g'(t)^2) + g''(t)(1 + f'(s)^2)]f'(s) + 2f'(s)f''(s)}{f'(s)^2 + g'(t)^2}, \end{aligned} \tag{4.30}$$

and

$$\begin{aligned} \frac{g'''(t)f''(s)}{g''(t)f''(s)} &= \frac{g'''(t)(1 + f'(s)^2) + 4g'(t)g''(t)f''(s)}{f''(s)(1 + g'(t)^2) + g''(t)(1 + f'(s)^2} \\ &= \frac{[f''(s)(1 + g'(t)^2) + g''(t)(1 + f'(s)^2)]g'(t) + 2g'(t)g''(s)}{f'(s)^2 + g'(t)^2}. \end{aligned} \tag{4.31}$$

From the equation (4.30) we get

$$f'''(s)(f''(s)(1 + g'(t)^2) + g''(t)(1 + f'(s)^2) = f''(s)(f'''(s)(1 + g'(t)^2) + 4f'(s)f''(s)g''(t))$$

After some calculations we have

$$f'''(s)(1 + f'(s)^2) = 4f'(s)f''(s)^2$$

and using the equation (4.31) we obtain

$$g'''(t)(1 + g'(t)^2) = 4g'(t)g''(t)^2.$$

In light of the discussions above, we can say that $f''(s)$ and $g''(t)$ are greater than 0. By the equation (4.9) we get a contradiction. If

$$f'''(s)(1+g'(t)^2)+4f'(s)f''(s)g''(t)=0$$

or

$$g'''(t)(1+f'(s)^2)+4g'(t)g''(t)f''(s)=0$$

then from (4.30) and (4.31) we deduce that $f'''(s)=0$ or $g'''(t)=0$. This also is a contradiction with $f'''(s)\neq 0$ or $g'''(t)\neq 0$. This discussion is also valid for

$$(f''(s)(1+g'(t)^2)+g''(t)(1+f'(s)^2))f'(s)+2f'(s)f''(s)=0$$

or

$$(f''(s)(1+g'(t)^2)+g''(t)(1+f'(s)^2))g'(t)+2g'(t)g''(s)=0$$

So we complete the proof of the theorem. □

Now, we will investigate the translation surfaces of type II with null statistical sectional curvature.

Theorem 4.2. *Let M be a statistical translation surface of II type. If M is a null statistical sectional curvature then M is parametrized as $(s, f(s)+g(t), t)$, where*
1) either $f(s)=ms+n, m, n\in\mathbb{R}$, and g is a smooth function or
2) f is a smooth function and $g(t)=at+b, a, b\in\mathbb{R}$ or $g(t)=\int e^{c-\frac{1}{2}t^2}, c\in\mathbb{R}$ or
3) $f(s)=-\frac{1}{4}c\ln(\tanh(\frac{\sqrt{2ca}(s+b)}{c})-1)-\frac{1}{4}c\ln(\tanh(\frac{\sqrt{2ca}(s+b)}{c})+1)+e,\ a, c, b, e\in\mathbb{R}$ and $g(t)$ satisfies following differential eqauation

$$(ct^3+t^2-t^4)g''(t)-cg'(t)-t=0.$$

Proof. Using (3.21) and (3.2), we have

$$\begin{aligned}\tilde{\nabla}^*_{\phi_s}\phi_s &= f''(s)\partial_y\\ \tilde{\nabla}^*_{\phi_s}\phi_t &= \frac{2}{t}f'(s)g'(t)\partial_z\\ \tilde{\nabla}^*_{\phi_t}\phi_t &= g''(t)\partial_y+\frac{1}{t}(1+2g'(t)^2)\partial_z.\end{aligned}\tag{4.32}$$

By (3.23) the coefficients of the second fundamental form of surface with repect to conjugate connection $\tilde{\nabla}^*$ are given by

$$\begin{aligned}\tilde{g}(\tilde{\nabla}^*_{\phi_s}\phi_s, N) &= -\frac{1}{t^2}f''(s),\\ \tilde{g}(\tilde{\nabla}^*_{\phi_s}\phi_t, N) &= 0,\\ \tilde{g}(\tilde{\nabla}^*_{\phi_t}\phi_t, N) &= -\frac{1}{t^2}g''(t)+\frac{4}{t^3}g'(t)\end{aligned}\tag{4.33}$$

Substituting the equations (4.7) and (3.23) in (4.8) we derive

$$f'''(s)g'(t)g''(t)(t^2-t^4)-f'''(s)g'(t)t+4f'(s)f''(s)g'(t)(g'(t)-g''(t)t^3)=0\tag{4.34}$$

If $f'''(s)=0$ then the equation (4.34) is reduced to

$$f'(s)f''(s)(g'(t)^2-g'(t)g''(t)t^3)=0$$

In this situation we say that $f'(s)f''(s)=0$ or $g'(t)(g'(t)-g''(t)t^3)=0$.

Now we suppose that $f'(s)f''(s)=0$. Then $f(s)=ms+n,\ m,\ n\in\mathbb{R}$ If $f'(s)f''(s)\neq 0$ then we have $g'(t)(g'(t)-g''(t)t^3)=0$ and it's solution

$$g(t)=at+b,\ a, b\in\mathbb{R}$$

or the smooth function $g(t)$ is represented by a Gaussian integral as follows:

$$g(t)=\int e^{c-\frac{1}{2}t^2}, c\in\mathbb{R}.$$

Now we assume that $f'''(s) \neq 0$. Dividing (4.34) by $f'''(s)$ we find

$$(t^2 - t^4)g'(t)g''(t) - tg'(t) + 4\frac{f'(s)f''(s)}{f'''(s)}g'(t)(g'(t) - g''(t)t^3) = 0 \tag{4.35}$$

Since the relation $g'(t)(g'(t) - g''(t)t^3) = 0$ has been discussed previously, we can suppose that $g'(t)(g'(t) - g''(t)t^3) \neq 0$. Thus if we divide the equation (4.35) by $g'(t)(g'(t) - g''(t)t^3)$ then we conclude that

$$\frac{(t^2 - t^4)g''(t) - t}{g'(t) - g''(t)t^3} + 4\frac{f'(s)f''(s)}{f'''(s)} = 0. \tag{4.36}$$

Since s and t are independent variables there exist a constant c such that

$$\frac{f'(s)f''(s)}{f'''(s)} = -\frac{1}{4}c, \quad \frac{(t^2 - t^4)g''(t) - t}{g'(t) - g''(t)t^3} = c.$$

If $c = 0$ then

$$f(s) = ms + n, m, n \in \mathbb{R}$$

and

$$g(t) = t\ln(t) - \ln(t^2 - 1) + c_1 t + c_0, c_1, c_0 \in \mathbb{R}.$$

Since $f'''(s) \neq 0$ this case not occur. If $c \neq 0$ then

$$\frac{1}{4}cf'''(s) + f'(s)f''(s) = 0 \tag{4.37}$$

and

$$(ct^3 + t^2 - t^4)g''(t) - cg'(t) - t = 0.$$

The solution of (4.37) is

$$f(s) = -\frac{1}{4}c\ln(\tanh(\frac{\sqrt{2ca}(s+b)}{c}) - 1) - \frac{1}{4}c\ln(\tanh(\frac{\sqrt{2ca}(s+b)}{c}) + 1) + e.$$

□

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