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Investigating the Laplace Transform Method's Efficiency to a Simple Engineering Problem

Huseyin Demir[1], Inci Cilingir Sungu[2,*], Ibrahim Keleş[3]

[1]*Department of Software Engineering, Faculty of Engineering, Samsun University, 55080, Samsun, Turkiye.*
[2]*Department of Education of Mathematics and Sciences, Faculty of Education, Ondokuz Mayis University, 55139, Samsun, Turkiye.*
[3]*Department of Mechanical Engineering, Faculty of Engineering, Samsun University, 55080, Samsun, Turkiye.*

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Abstract. In this study, it is aimed to solve the differential equation that forms a simple engineering system and transform it into the Laplace domain, and then to investigate the effectiveness of the method used to compare the solutions with the exact solutions. For this purpose, first, the solutions of a given test function with analytical and numerical Laplace inverse transform methods (Durbin, Stehfest and Talbot) are given comparatively. Although the values obtained from these three methods overlap with each other but it is observed that the Talbot inverse transform method is more suitable than the other two methods due to its lower calculation time requirement. In addition, Talbot's method and analytical solutions to engineering problems related to the vibratory mechanical system, heat conduction problem and a single matrix block in a fractured reservoir non-isothermal gravity drainage are numerically compared. It is understood that the Talbot inverse transform method is quite effective, and this is evident from the consistency of the numerical results and analytical results of the study. The findings show that the proposed method is very suitable and the method is easy to implement without much difficulty for solving a simple engineering problem.

2020 AMS Classification: 65M70, 80M22, 65D15

Keywords: Laplace transform, numerical inversion, test function, Talbot method.

1. Introduction

Keeping the problem that arises in an engineering problem under control and the solution of the problem may vary depending on the boundary conditions to which the system depends. This is accomplished, for instance, by controlling the speed of the electric motor running on a crane so that the object being raised by the crane can be lifted at the same speed, regardless of how heavy or light it is. As in this example, to control the system that creates the problem, a mathematical model of the system must first be created. Most engineering problems can be expressed using differential equations. To solve the resulting differential equation, the equation model must be transformed into a more suitable form. One of the mathematical methods used to transform this into the appropriate form is the Laplace transform. Laplace transform concepts are frequently applied in science and technology fields such as solid body mechanics, electrical analysis, communication engineering, control engineering, linear system analysis, statistical optics, and quantum physics.

*Corresponding Author
Email addresses: huseyin.demir@samsun.edu.tr (H. Demir), incicilingir@gmail.com (İ. Çilingir Süngü), ibrahim.keles@samsun.edu.tr (İ. Keleş)

With the developing technology, composite, functional grade materials have begun to be used rather than classical materials in the materials used in the field of engineering. The Laplace transformation method is often successfully applied, especially in the analysis of dynamic problems posed by these materials. The accuracy and efficiency of the results obtained with the transformation depend on the transformation from Laplace space to time space. In this approach, Laplace transform is implemented to the differential equation to eliminate the time parameter in the dynamic problem. If the discrete Laplace transform is applied, the dynamic behavior of the object or the problem at a desired point can be determined without finding out the behavior at other points. If classical integration formulas are used, to find the dynamic behavior at a certain point, it is also necessary to find the behavior at other points. The set of linear algebraic equations obtained by applying the Laplace transform to time-dependent differential equations can be easily solved with the help of numerical methods. Closed-form inversion may not be possible except for certain loading situations, and many methods are recommended in the literature for numerical Laplace inversion [1, 2, 5, 7, 11, 14, 15, 17, 18].

With its use, the Talbot method (TM) has begun to be applied efficiently and successfully in many scientific fields [9,13]. Unlike other methods applied in the literature, TM should be careful in the selection of parameters to be efficient in the method based on the effectiveness of the Bromwich contour. Thus, TM becomes advantageous by automatically selecting the most suitable contour parameters. TM processes the data perfectly and does not have much impact on the outcome. Digital inverse Laplace transform techniques will also be more useful when the utilization of TM and computational resources improves along with computing capacity.

In this study, the solutions of the differential equations that emerged by mathematical modeling of engineering problems in Laplace space and their solutions in time space were obtained by using his method. These numerical solutions were compared with exact values. In the proposed method, Laplace transform was applied to the differential equation leading the performance of the mechanical model, and thus, the solutions of the linear algebraic equation/equation set achieved by eliminating the time parameter were get for a few Laplace parameters. The TM inverse transform [15] method was used to move from the solutions obtained in Laplace space to the time space. In this study, it has been shown in comparison with various numerical methods in the literature that by applying time-dependent differential equations using the Laplace transform and then applying the TM inverse transform method, very effective and accurate results can be attained.

2. The Talbot Inverse Laplace Method

Scientists have successfully utilized the TM to many different fields and different problems, and based on a simple algorism to convert from Laplace space to time space [6, 10, 16].

Its inverse transformation, also referred to as the Bromwich contour integral,

$$L^{-1}\left\{\bar{f}(s)\right\} = f(t) = \frac{1}{2}\int_{\sigma-i\infty}^{\sigma+i\infty} \bar{f}(s)\, e^{st} ds, \tag{2.1}$$

where σ is the Laplace transformation abscissa of convergence [3]. The inverse Laplace problem is the reconstruction of $f(t)$ from known values of $\bar{f}(s)$. Since s is negative and large in the Talbot method, it reduces the e^{st} term in Equation (2.1), allowing the Bromwich contour integral to converge faster. The Bromwich contour is parametrized as $s(\theta) = r\theta(\cot(\theta) + i)$, where $0 \leq \theta \leq \pi$, and as a rule of thumb $r = \frac{2M}{5t_{\max}}$, where $t_{\max}$ is the upper limit time coupled with the vector of times (typically just the time demanded), M is the integer order of calculation (number of terms), and r is the abscissa for s (otherwise calculated using the rule of thumb $2M/5$) after applying a single-parameter "fixed" Talbot technique [17]. The fixed Talbot method is

$$f(t, Z) = \frac{r}{Z}\left[\frac{\bar{f}(r)}{2}e^{rt} + \sum_{k=1}^{Z-1} \mathrm{Re}\left\{e^{ts(\theta_k)}\bar{f}[s(\theta_k)][1 + i\zeta(\theta_k)]\right\}\right],$$

where $\zeta(\theta_k) = \theta_k + [\theta_k \cot(\theta_k) - 1]\cot(\theta_k)$ and $\theta_k = k\pi/N$ [17]. Although $\bar{f}(s)$ doesn't depend on t, the free parameter r depends on $t_{\max}$. Step change $\bar{f}_t(s)$ for non-zero time becomes very large as $s \to -\infty$, since $\mathcal{L}[H(t-\tau)] = e^{-\tau s/s}$, where $H(t-\tau)$ is the Heaviside step function centered on time τ. We can distort the Bromwich contour into a parabola around the negative real axis if $\bar{f}(s)$ is analytic in the region between the Bromwich and the distorted Talbot contours [15].

3. Computational Details and Accuracy

The Stehfest, Durbin, and Talbot methods were applied to solve the test function. The results were displayed in Table 1 and compared to the exact solutions.

The Laplace Transform and its inverse of the test function are given below:

$$F(s) = \frac{2s+3}{(s+1)(s+2)}, \quad f(t) = e^{-2t} + e^{-t}$$

For a real function $f(t)$, its expression for $f(t) = 0$, where $t < 0$ is:

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)dt.$$

If the inverse transformation is:

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{st} F(s)ds,$$

where s is the Laplace parameter and a is an arbitrary real number ($s = a + iw$) [3].

The table values were obtained from Stehfest, Durbin, Talbot, respectively, through the codes used in the Mathematica package program (Table 1). Here, it was observed that the Stehfest method deviated more from the analytical result in some equations containing trigonometric expressions, and the engineering problem was solved with the use of TM, which provides more precise results. Detailed information on how Stehfest, Durbin methods are used can be found in references [5, 14, 15, 17].

Since it is difficult to obtain analytical solutions of functions with complex structures, some numerical methods have been developed that will provide approximate solutions, provide precision and convenience. Some of these are Fourier series approaches. In this context, Fourier series were used for the first time by Dubner and Abate [4].

Table 1. Comparison of test function results

Time	Exact	Durbin Method	Stehfest Method	Talbot Method (TM)
1	0.5032150000	0.5018127567	0.5032162433	0.5032147244
2	0.1536510000	0.1531914299	0.1536454844	0.1536509221
3	0.0522658000	0.0521094730	0.0522598720	0.0522658205
4	0.0186511000	0.0185953146	0.0186573364	0.0186511015
5	0.0067833500	0.0067630610	0.0067882222	0.0067833469
6	0.0024849000	0.0024774677	0.0024848200	0.0024848964
7	0.0009127130	0.0009099831	0.0009129027	0.0009127135
8	0.0003355750	0.0003345713	0.0003396854	0.0003355752
9	0.0001234250	0.0001230558	0.0001308812	0.0001234250
10	0.0000454020	0.0000452662	0.0000537505	0.0000454020

Table 1 demonstrates how efficiently the Talbot Method provides relative errors and is compatible with the final result when compared to other methods.

4. Laplace Transform in the Analysis and Modeling of Engineering Systems

4.1. **Laplace Transform in Vibrating Mechanical System.** A mechanical model of a single-degree-of-freedom vehicle suspension system under the influence of sinusoidal load can be modeled using Newton and Hooke's law and the differential equation governing the system.

The differential equation of the system given in Figure 1 is expressed below:

$$\frac{d^2x}{dy^2} + 6\frac{dx}{dt} + 25x = 4\sin(wt). \tag{4.1}$$

First, let's write the Laplace transform of equation (4.1):

$$(s^2 + 6s + 25)\mathcal{L}[x(t)] = sx(0) + x'(0) + 6x(0) + \frac{4w}{s^2 + w^2} \tag{4.2}$$

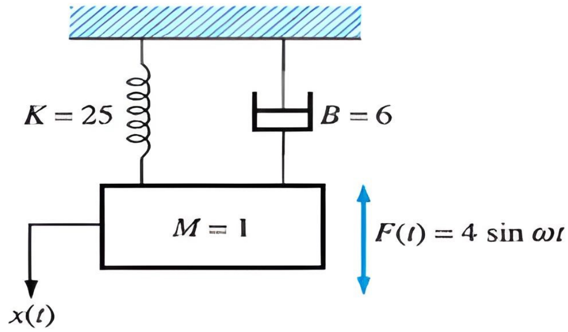

Figure 1. A simple mechanical model

Here, s is the Laplace parameter, $x(0)$, and $x'(0)$ indicate the initial displacement and velocity, respectively. If Equation (4.2) is arranged by taking the initial conditions into consideration, the expression of the displacement in Laplace space is obtained (with $w = 2$):

$$\mathcal{L}[x(t)] = \bar{x}(s) = \frac{8}{(s^2 + 4)(s^2 + 6s + 25)}. \tag{4.3}$$

Equation (4.3) is solved analytically and is presented graphically in Figure 2. When Figure 2 is examined, it is seen that all solutions obtained for various Laplace parameters and time increment amounts overlap with the analytical values.

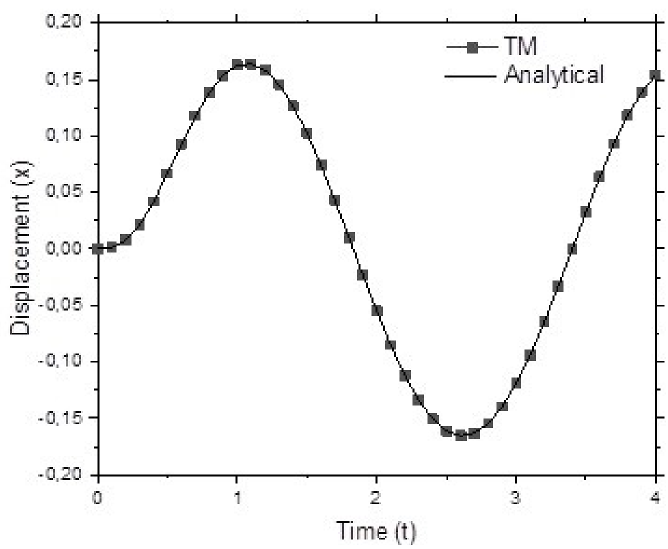

Figure 2. Variation of system displacement over time

4.2. **Laplace Transform in Heat Conduction Problem.** The problem of one-dimensional heat conduction in a solid body is one of the well-known problems, especially in chemical and mechanical engineering. The partial differential equation governing the heat conduction of a solid of dimensionless length 2, which is suddenly brought to zero in both directions and initially has unit dimensionless temperature, can be expressed as follows [8]:

$$\frac{d^2\theta}{d\zeta^2} = \frac{d\theta}{d\tau} \tag{4.4}$$

and boundary conditions:

$$\theta(\zeta, 0) = 1, \tag{4.5}$$
$$\theta(-1, \tau) = 0, \tag{4.6}$$
$$\theta(1, \tau) = 0, \tag{4.7}$$

where θ is temperature, ζ is length, and τ is time, all in dimensionless form.

The equation expressing the one-dimensional heat conduction problem in Laplace space after the necessary simplification at the end of Laplace transformation using Equations (4.4) and (4.5), (4.6), (4.7) is given below:

$$\theta(s) = \frac{1}{s}\left[1 - \frac{\cosh(\sqrt{s}\zeta)}{\cosh(\sqrt{s})}\right].$$

This equation is numerically inverted by the TM inversion method, and the obtained results and analytical results are given in Figures 3-4 for comparison.

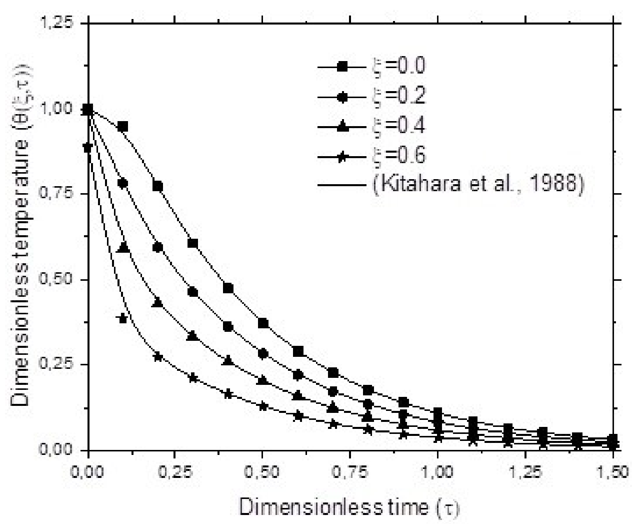

Figure 3. Comparison of exact solutions with TM numerical inversion method for different dimensionless length values

Figure 3 shows the temperature distribution in a one-dimensional solid body for different dimensionless lengths. As the dimensionless length increases, the temperature decreases. The dimensionless temperature change decreases over time in a solid body. In addition, it has been found to be compatible with the analytical results given in the literature.

Figure 4 shows the dimensionless temperature distribution throughout the solid body corresponding to different dimensionless time values. Dimensionless temperature change decreases as dimensionless time increases. In addition, it has been found to be compatible with the analytical results given in the literature.

4.3. **Laplace Transform of Single Matrix Block in Naturally Fractured Reservoirs Non-Isothermal Gravity Drainage.** An attempt has been made to present the Laplace transform TM method for non-isothermal gravity-induced drainage of a single matrix block in a plate-shaped fractured reservoir. Thermal oil production in the crack reservoir involves heating the high viscosity oil in the matrix blocks that form it and steam injection to flow the oil into the

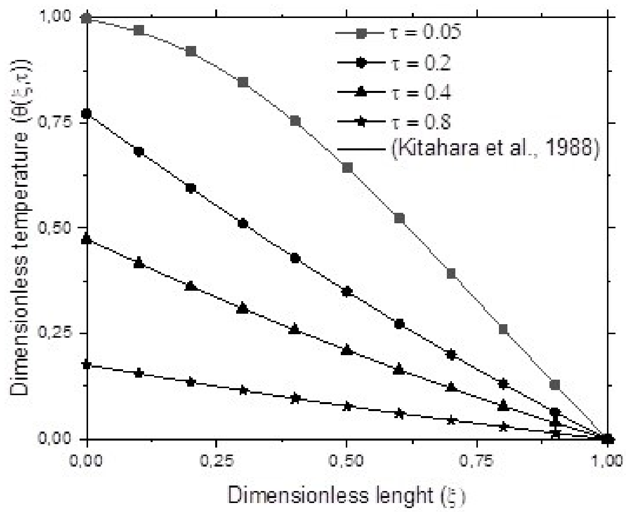

Figure 4. Comparison of exact solutions with TM numerical inversion method for different dimensionless time values

cracking system. The partial differential equation expressing the one-dimensional heat flow that defines this situation is given in Equation (4.8). The assumptions made here are given by Pooladi-Darvish et al. [12].

$$\frac{\partial^2\theta}{\partial\zeta^2} + N_{Pe}\frac{\partial\theta}{\partial\zeta} = \frac{\partial\theta}{\partial\tau} \tag{4.8}$$

and boundary conditions:

$$\begin{aligned} &\theta(\zeta, 0) = 0, \\ &\theta(0, \tau) = 1, \\ &\theta(\zeta, \tau) = 0 \text{ as } \zeta \to \infty, \end{aligned}$$

where θ is normalized temperature, ζ is dimensionless distance, N_{Pe} is the Peclet number, and τ is dimensionless time.

The Laplace domain solution is also given by the following equation:

$$\theta(\zeta, s) = \frac{1}{s}\exp\left(-\frac{N_{Pe}}{2}\zeta - \left(\sqrt{\frac{N_{Pe}}{2}}\right)^2 + s\zeta\right).$$

This equation is numerically inverted by the TM inversion method, and the obtained results and analytical results are given in Figures 5- 6 for comparison.

In Figure 5, the comparison of exact solutions for non-isothermal gravity drainage with the TM numerical inversion method at different dimensionless times is presented graphically, taking $N_{Pe} = 0.01$. As the dimensionless time value increases, the dimensionless temperature value increases, while it decreases along the dimensionless length. Moreover, the results agree with the exact solution given in the literature.

Comparison of exact solutions for non-isothermal gravity drainage at different Peclet numbers with $\tau = 0.8$ by TM numerical inversion method is presented in Figure 6. Here, while the Peclet numbers increase, the dimensionless temperature value decreases. The dimensionless temperature value also decreases along the dimensionless length. Moreover, the results agree with the exact solution given in the literature.

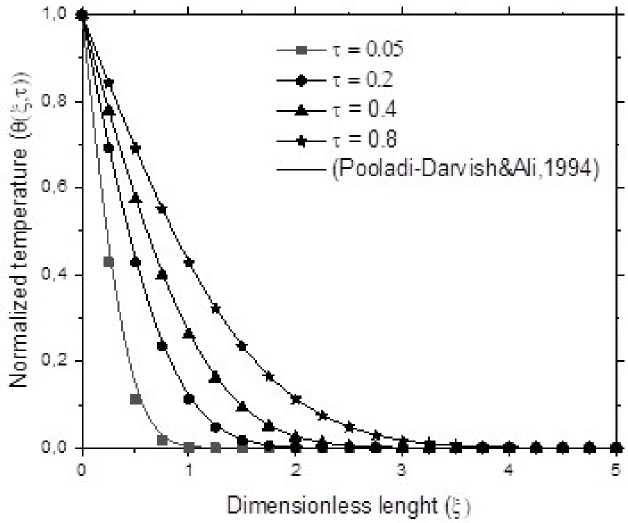

Figure 5. Comparison of exact solutions for non-isothermal gravity drainage by TM numerical inversion method at different dimensionless times with $N_{Pe} = 0.01$

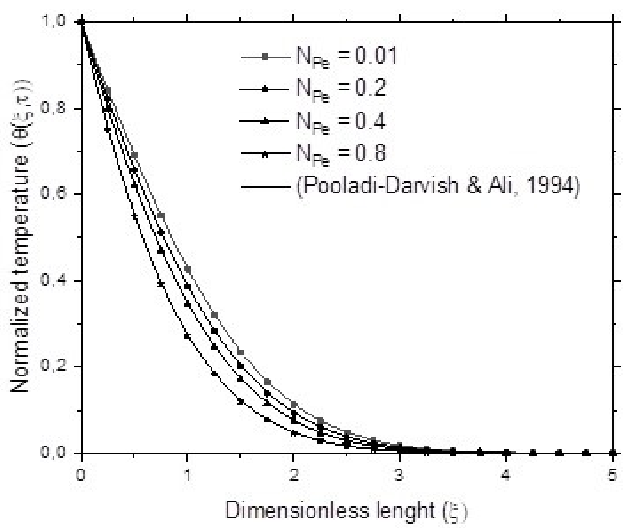

Figure 6. Comparison of exact solutions for non-isothermal gravity drainage by TM numerical inversion method at different Peclet number with $\tau = 0.8$

5. Conclusions

The equations of motion governing the engineering problem are first obtained in the time domain, and the problem is resolved numerically by using the Laplace transform to convert it to time-independent. To solve the equations found in the Laplace domain, TM is converted back into the time domain by the numerical inverse Laplace transform method. In the study, firstly, the solutions of the given test function are given analytically and with different numerical transformation methods. Then, simple problems encountered in engineering are presented in graphics using the TM numerical method. Based on the model proposed in this study, the following results were obtained:

- In the proposed method, the time parameter is eliminated by using the Laplace transform to a time-dependent differential equation that governs the behavior of engineering problems. Thus, a linear algebraic equation or set of equations that is made independent of time can be solved much more effectively and simply by numerical methods.
- From examining the graphs, it has been seen that the consistency of the solutions obtained with the TM method in time space depends on choosing the correct time increment amount. In the proposed method, results that match the exact solution can be obtained with very few parameters by using coarse time increments. Therefore, it is clearly seen that the proposed method is much more effective than other classical step-by-step integration methods.
- For example, to make a solution in Fourier space, both the system behavior and the load function must go to zero at infinity. In Laplace space, there is no such necessity.
- It can be said that TM gives accurate results with less load and progress time even in long-term research.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The authors have read and agreed to the published version of the manuscript.

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