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The New Class $L_{p,\Phi}$ of s-Type Operators

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Abstract

In this study, the class of s-type $\ell_p(\Phi)$ operators is introduced and it is shown that $L_{p,\Phi}$ is a quasi-Banach operator ideal. Also, some other classes are defined by using approximation, Gelfand, Kolmogorov, Weyl, Chang, and Hilbert number sequences. Then, some properties are examined.

1. Introduction

In this study, all natural numbers are symbolized by $\mathbb{N}$ and all non-negative real numbers are symbolized by $\mathbb{R}^+$. If the dimension of the range space of a bounded linear operator is finite, it is called a finite rank operator [1].
Throughout this study, E and F represent real or complex Banach spaces. The space of all bounded linear operators from E to F is denoted by $\mathscr{B}(E,F)$ and the space of all bounded linear operators from an arbitrary Banach space to another arbitrary Banach space is denoted by $\mathscr{B}$.
The operator ideal theory is a very important field in functional analysis. The theory of normed operator ideals first appeared in the 1950s in [2]. In functional analysis, most of the operator ideals are constructed via different scalar sequence spaces. s-number sequence is one of the most important example of this. For more information about operator ideals and $s-$numbers, we refer to [3–8]. The definition of s-numbers goes back to E. Schmidt [9] who used this concept in the theory of non-selfadjoint integral equations. In Banach spaces, there are many different possibilities of defining some equivalents for s-numbers, namely Kolmogorov numbers, Gelfand numbers, approximation numbers, etc. In the following years, Pietsch developed the concept of s-number sequence to collect all s-numbers in a single definition [10–12].
A map

$$S : K \to (s_r(K))$$

which assigns a non-negative scalar sequence to each operator, is called an s-number sequence if for all Banach spaces E, F, E_0, and F_0 the following conditions are satisfied:

(*i*) $\|K\| = s_1(K) \geq s_2(K) \geq \ldots \geq 0$, for every $K \in \mathscr{B}(E,F)$,
(*ii*) $s_{p+r-1}(L+K) \leq s_p(L) + s_r(K)$ for every $L, K \in \mathscr{B}(E,F)$ and $p, r \in \mathbb{N}$,
(*iii*) $s_r(MLK) \leq \|M\| s_r(L) \|K\|$ for some $M \in \mathscr{B}(F,F_0)$, $L \in \mathscr{B}(E,F)$ and $K \in \mathscr{B}(E_0,E)$, where E_0, F_0 are arbitrary Banach spaces,
(*iv*) If $rank(K) \leq r$, then $s_r(K) = 0$,
(*v*) $s_r(I_r) = 1$, where I_r is the identity map of r-dimensional Hilbert space l_2^r to itself [13].

$s_r(K)$ represents the r-th $s-$number of the operator K.
Pietsch defined approximation numbers, which are frequently used examples of s-number sequence, $a_r(K)$, the r-th approximation number of a bounded linear operator as

$$a_r(K) = \inf\{\, \|K-A\| : A \in \mathscr{B}(E,F),\, rank(A) < r\,\},$$

where $K \in \mathscr{B}(E,F)$ and $r \in \mathbb{N}$ [10]. Let $K \in \mathscr{B}(E,F)$ and $r \in \mathbb{N}$. *Gel′fand* numbers $(c_r(K))$, *Kolmogorov* numbers $(d_r(K))$, *Weyl* numbers $(x_r(K))$, *Chang* numbers $(y_r(K))$, *Hilbert* numbers $(h_r(K))$, are some other examples of s-number sequences. For more information about these sequences, we refer to [1].

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Some necessary properties of $s-$number sequences are given in the sequel.
Let $\mathscr{J} \in \mathscr{B}(F,F_0)$ be a metric injection. If the sequence $s=(s_r)$ satisfies $s_r(K)=s_r(\mathscr{J}K)$ for all $K\in\mathscr{B}(E,F)$ the sequence of s-number is named injective [14, p.90].

Proposition 1.1. *[14, p.90-94] The number sequences $(c_r(K))$ and $(x_r(K))$ are injective.*

Let $\mathscr{S}\in\mathscr{B}(E_0,E)$ be a metric surjection. If the sequence $s=(s_r)$ satisfies $s_r(K)=s_r(K\mathscr{S})$ for all $K\in\mathscr{B}(E,F)$ the s-number sequence is named surjective [14, p.95].

Proposition 1.2. *[14, p.95] The number sequences $(d_r(K))$ and $(y_r(K))$ are surjective.*

Proposition 1.3. *[14, p.115] Let $K\in\mathscr{B}(E,F)$. Then, the following inequalities hold:*

(i) $h_r(K)\le x_r(K)\le c_r(K)\le a_r(K)$,
(ii) $h_r(K)\le y_r(K)\le d_r(K)\le a_r(K)$.

Lemma 1.4. *[11] Let $S,K\in\mathscr{B}(E,F)$, then $|s_r(K)-s_r(S)|\le\|K-S\|$ for $r=1,2,\ldots$.*

Let the space of all real-valued sequences be denoted by ω. Then, a sequence space is any vector subspace of ω.
Maddox defined the linear space $l(p)$ as follows in [15]:

$$l(p)=\left\{x\in\omega:\sum_{n=1}^{\infty}|x_n|^{p_n}<\infty\right\},$$

where(p_n) is a bounded sequence of strictly positive real numbers.
If an operator $K\in\mathscr{B}(E,F)$ satisfies $\sum\limits_{n=1}^{\infty}(a_n(K))^p<\infty$ for $0<p<\infty$, K is defined as an l_p- type operator in [10] by Pietsch. Afterward *ces-p* type operators which is a new class obtained via Cesaro sequence space are introduced by Constantin [16]. Later on, Tita in [17], proved that the class of l_p type operators and *ces-p* type operators coincide.
In this paper φ denotes the Euler function. For every $u\in\mathbb{N}$ with $u\ge1$, $\varphi(u)$ is the number of positive integers less than u which are coprime with u and $\varphi(1)=1$. If $p_1^{a_1}p_2^{a_2}...p_r^{a_r}$ is the prime factorization of a natural number $u>1$ then

$$\varphi(u)=u(1-\frac{1}{p_1})(1-\frac{1}{p_2})...(1-\frac{1}{p_r}).$$

Also, the equality

$$u=\sum_{t|u}\varphi(t)$$

holds for every $u\in\mathbb{N}$ and $\varphi(u_1u_2)=\varphi(u_1)\varphi(u_2)$, where $u_1,u_2\in\mathbb{N}$ are coprime [18].
In [19] the sequence space $\ell_p(\Phi)$ is defined as:

$$\ell_p(\Phi)=\left\{x=(x_n)\in\omega:\sum_n\left|\frac{1}{n}\sum_{t|n}\varphi(t)x_t\right|^p<\infty\right\}\quad(1\le p<\infty).$$

Let E^*, the dual of E, be the set of continuous linear functionals on E. The map $x^{'}\otimes y:E\to F$ is defined as

$$\left(x^{'}\otimes y\right)(x)=x^{'}(x)y$$

where $x\in E$, $x^{'}\in E^*$ and $y\in F$.
A subcollectionϑ of $\mathscr{B}$ is said to be an operator ideal if for each component $\vartheta(E,F)=\vartheta\cap\mathscr{B}(E,F)$ the following conditions are hold:

(i) if $x^{'}\in E^*$, $y\in F$, then $x^{'}\otimes y\in\vartheta(E,F)$,
(ii) if $L,K\in\vartheta(E,F)$, then $L+K\in\vartheta(E,F)$,
(iii) if $L\in\vartheta(E,F)$, $K\in\mathscr{B}(E_0,E)$ and $M\in\mathscr{B}(F,F_0)$, then $MLK\in\vartheta(E_0,F_0)$ [12].

Let ϑ be an operator ideal and $\rho:\vartheta\to\mathbb{R}^+$ be a function on ϑ. Then, if the following conditions are hold:

(i) if $x^{'}\in E^*$, $y\in F$, then $\rho\left(x^{'}\otimes y\right)=\left\|x^{'}\right\|\|y\|$;
(ii) if $\exists\mathscr{C}\ge1$ such that $\rho(L+K)\le\mathscr{C}[\rho(L)+\rho(K)]$;
(iii) if $L\in\vartheta(E,F)$, $K\in\mathscr{B}(E_0,E)$ and $M\in\mathscr{B}(F,F_0)$, then $\rho(MLK)\le\|M\|\rho(L)\|K\|$,

ρ is called a quasi-norm on the operator ideal ϑ [12].
For special case $\mathscr{C}=1$, ρ is a norm on the operator ideal ϑ.
If ρ is a quasi-norm on an operator ideal ϑ, it is denoted by $[\vartheta,\rho]$. Also, if every component $\vartheta(E,F)$ is complete with the quasi-norm ρ, $[\vartheta,\rho]$ is called a quasi-Banach operator ideal.
Let $[\vartheta,\rho]$ be a quasi-normed operator ideal and $\mathscr{J}\in\mathscr{B}(F,F_0)$ be a metric injection. If for every operator $K\in\mathscr{B}(E,F)$ and $\mathscr{J}K\in\vartheta(E,F_0)$ we have $K\in\vartheta(E,F)$ and $\rho(\mathscr{J}K)=\rho(K)$, $[\vartheta,\rho]$ is called an injective quasi-normed operator ideal. Furthermore, let $[\vartheta,\rho]$ be a quasi-normed operator ideal and $\mathscr{S}\in\mathscr{B}(E_0,E)$ be a metric surjection. If for every operator $K\in\mathscr{B}(E,F)$ and $K\mathscr{S}\in\vartheta(E_0,F)$ we have $K\in\vartheta(E,F)$ and $\rho(K\mathscr{S})=\rho(K)$, $[\vartheta,\rho]$ is called a surjective quasi-normed operator ideal [12].
Let K^* be the dual of K. An $s-$number sequence is called symmetric and completely symmetric if for all $K\in\mathscr{B}$, $s_r(K)\ge s_r(K^*)$ and $s_r(K)=s_r(K^*)$, respectively [12].
The dual of an operator ideal ϑ is denoted by ϑ^* and it is defined as

$$\vartheta^*(E,F)=\left\{K\in\mathscr{B}(E,F):K'\in\vartheta(F^*,E^*)\right\}$$

[12].
An operator ideal ϑ is called symmetric if $\vartheta\subset\vartheta^*$ and is called completely symmetric if $\vartheta=\vartheta^*$ [12].

2. Main Results

An operator $K \in \mathscr{B}(E,F)$ is in the class of $s-$type $\ell_p(\Phi)$ if

$$\sum_n \left(\frac{1}{n} \sum_{t|n} \varphi(t) s_t(K) \right)^p < \infty \quad (1 \le p < \infty).$$

The class of all $s-$type $\ell_p(\Phi)$ operators is denoted by $L_{p,\Phi}(E,F)$.

Theorem 2.1. *The class $L_{p,\Phi}$ is a quasi-normed operator ideal by*

$$\|K\|_{p,\Phi} = \frac{\left(\sum\limits_{n=1}^{\infty} \left(\frac{1}{n} \sum\limits_{t|n} \varphi(t) s_t(K) \right)^p \right)^{\frac{1}{p}}}{\left(\sum\limits_{n=1}^{\infty} \left(\frac{1}{n}\right)^p \right)^{\frac{1}{p}}}, \quad (1 < p < \infty).$$

Proof. In this proof we show that the class $L_{p,\Phi}$ satisfies the conditions of an operator ideal and $\|K\|_{p,\Phi}$ satisfies the conditions for a quasi-norm. Let $x' \in E^*$ and $y \in F$. Then,

$$\begin{aligned}
\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{t|n} \varphi(t) s_t(x' \otimes y) \right)^p &= \left(\sum_{t|1} \varphi(t) s_t(x' \otimes y) \right)^p + \left(\frac{1}{2} \sum_{t|2} \varphi(t) s_t(x' \otimes y) \right)^p + \left(\frac{1}{3} \sum_{t|3} \varphi(t) s_t(x' \otimes y) \right)^p + \dots \\
&= \left(\varphi(1) s_1(x' \otimes y) \right)^p + \left(\frac{1}{2} \varphi(1) s_1(x' \otimes y) \right)^p + \left(\frac{1}{3} \varphi(1) s_1(x' \otimes y) \right)^p + \dots \\
&= \left(s_1(x' \otimes y) \right)^p \left(1 + (\frac{1}{2})^p + (\frac{1}{3})^p + \dots \right) \\
&< \infty.
\end{aligned}$$

Since the operator $x' \otimes y$ has rank one, $s_n\left(x' \otimes y\right) = 0$ for $n \ge 2$. Therefore, $x' \otimes y \in L_{p,\Phi}(E,F)$.
And also,

$$\begin{aligned}
\left\| x' \otimes y \right\|_{p,\Phi} &= \frac{\left(\sum\limits_{n=1}^{\infty} \left(\frac{1}{n} \sum\limits_{t|n} \varphi(t) s_t(x' \otimes y) \right)^p \right)^{\frac{1}{p}}}{\left(\sum\limits_{n=1}^{\infty} \left(\frac{1}{n}\right)^p \right)^{\frac{1}{p}}} \\
&= \frac{\left[\left(s_1(x' \otimes y) \right)^p \left(1 + (\frac{1}{2})^p + (\frac{1}{3})^p + \dots \right) \right]^{\frac{1}{p}}}{\left(\sum\limits_{n=1}^{\infty} \left(\frac{1}{n}\right)^p \right)^{\frac{1}{p}}} \\
&= s_1(x' \otimes y) = \|x' \otimes y\| = \|x'\| \|y\|.
\end{aligned}$$

Hence $\left\| x' \otimes y \right\|_{p,\Phi} = \|x'\| \|y\|$.
Let $L, K \in L_{p,\Phi}(E,F)$. Then,

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{t|n} \varphi(t) s_t(L) \right)^p < \infty, \quad \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{t|n} \varphi(t) s_t(K) \right)^p < \infty.$$

To show that $L + K \in L_{p,\Phi}(E,F)$, let begin with

$$\begin{aligned}
\sum_{t|n} \varphi(t) s_t(L+K) &\le \sum_{t|n} \varphi(2t-1) s_{2t-1}(L+K) + \sum_{t|n} \varphi(2t) s_{2t}(L+K) \\
&\le \sum_{t|n} \big(\varphi(2t-1) + \varphi(2t) \big) s_{2t-1}(L+K) \\
&\le \sum_{t|n} \mathscr{C} \varphi(t) s_{2t-1}(L+K) \\
&\le \mathscr{C} \left(\sum_{t|n} \varphi(t) s_t(L) + \sum_{t|n} \varphi(t) s_t(K) \right)
\end{aligned}$$

since $\exists \mathscr{C} \ge 1$ which satisfies $\varphi(2t-1) + \varphi(2t) \le \mathscr{C} \varphi(t)$.

By using Minkowski inequality, we get;

$$\begin{aligned}\left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{t|n}\varphi(t)s_t(L+K)\right)^p\right)^{\frac{1}{p}} &\leq C\left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{t|n}\varphi(t)s_t(L)+\frac{1}{n}\sum_{t|n}\varphi(t)s_t(K)\right)^p\right)^{\frac{1}{p}}\\ &\leq C\left[\left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{t|n}\varphi(t)s_t(L)\right)^p\right)^{\frac{1}{p}}+\left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{t|n}\varphi(t)s_t(K)\right)^p\right)^{\frac{1}{p}}\right]\\ &<\infty.\end{aligned}$$

Hence, $L+K\in L_{p,\Phi}(E,F)$. Additionally,

$$\begin{aligned}\|L+K\|_{p,\Phi} &= \frac{\left(\sum\limits_{n=1}^{\infty}\left(\frac{1}{n}\sum\limits_{t|n}\varphi(t)s_t(L+K)\right)^p\right)^{\frac{1}{p}}}{\left(\sum\limits_{n=1}^{\infty}\left(\frac{1}{n}\right)^p\right)^{\frac{1}{p}}}\\ &\leq C\left[\frac{\left(\sum\limits_{n=1}^{\infty}\left(\frac{1}{n}\sum\limits_{t|n}\varphi(t)s_t(L)\right)^p\right)^{\frac{1}{p}}}{\left(\sum\limits_{n=1}^{\infty}\left(\frac{1}{n}\right)^p\right)^{\frac{1}{p}}}+\frac{\left(\sum\limits_{n=1}^{\infty}\left(\frac{1}{n}\sum\limits_{t|n}\varphi(t)s_t(K)\right)^p\right)^{\frac{1}{p}}}{\left(\sum\limits_{n=1}^{\infty}\left(\frac{1}{n}\right)^p\right)^{\frac{1}{p}}}\right]\\ &= C\left[\|L\|_{p,\Phi}+\|K\|_{p,\Phi}\right].\end{aligned}$$

Let $M\in\mathscr{B}(F,F_0)$, $S\in L_{p,\Phi}(E,F)$ and $K\in\mathscr{B}(E_0,E)$. Then,

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{t|n}\varphi(t)s_t(MLK)\right)^p &\leq \sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{t|n}\varphi(t)\,\|R\|\,s_t(L)\,\|K\|\right)^p\\ &\leq \|R\|^p\,\|T\|^p\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{t|n}\varphi(t)s_t(L)\right)^p<\infty.\end{aligned}$$

So $MLK\in L_{p,\Phi}(E_0,F_0)$. Furthermore,

$$\begin{aligned}\|MLK\|_{p,\Phi} &= \frac{\left(\sum\limits_{n=1}^{\infty}\left(\frac{1}{n}\sum\limits_{t|n}\varphi(t)s_t(MLK)\right)^p\right)^{\frac{1}{p}}}{\left(\sum\limits_{n=1}^{\infty}\left(\frac{1}{n}\right)^p\right)^{\frac{1}{p}}}\\ &\leq \|R\|\,\|K\|\,\frac{\left(\sum\limits_{n=1}^{\infty}\left(\frac{1}{n}\sum\limits_{t|n}\varphi(t)s_t(L)\right)^p\right)^{\frac{1}{p}}}{\left(\sum\limits_{n=1}^{\infty}\left(\frac{1}{n}\right)^p\right)^{\frac{1}{p}}} = \|R\|\,\|\,\|L\|_{p,\Phi}\,\|K\|\,.\end{aligned}$$

Therefore, $L_{p,\Phi}(E,F)$ is an operator ideal, and $\|K\|_{p,\Phi}$ is a quasi-norm on this operator ideal. □

Theorem 2.2. *Let $1<p<\infty$. $\left[L_{p,\Phi}(E,F),\|K\|_{p,\Phi}\right]$ be a quasi-Banach operator ideal.*

Proof. Let E,F be any two Banach spaces and $1\leq p<\infty$. The following inequality holds

$$\|K\|_{p,\Phi}=\left[\frac{\sum\limits_{n=1}^{\infty}\left(\frac{1}{n}\sum\limits_{t|n}\varphi(t)s_t(K)\right)^p}{\sum\limits_{n=1}^{\infty}\left(\frac{1}{n}\right)^p}\right]^{\frac{1}{p}}\geq\|K\|=s_1(K)$$

for $K\in L_{p,\Phi}(E,F)$.
Let (K_m) be a Cauchy sequence in $L_{p,\Phi}(E,F)$. Then, for every $\varepsilon>0$, there exists $n_0\in\mathbb{N}$ such that

$$\|K_m-K_l\|_{p,\Phi}<\varepsilon \tag{2.1}$$

for all $m,l\geq n_0$. It follows that

$$\|K_m-K_l\|\leq\|K_m-K_l\|_{p,\Phi}<\varepsilon.$$

Then, (K_m) is a Cauchy sequence in $\mathscr{B}(E,F)$. $\mathscr{B}(E,F)$ is a Banach space since F is a Banach space. Therefore, $\|K_m - K\| \to 0$ as $m \to \infty$ for $K \in \mathscr{B}(E,F)$. Now we show that $\|K_m - K\|_{p,\Phi} \to 0$ as $m \to \infty$ for $K \in L_{p,\Phi}(E,F)$.
The operators $K_l - K_m$, $K - K_m$ are in the class $\mathscr{B}(E,F)$ for $K_m, K_l, K \in \mathscr{B}(E,F)$. Then,

$$|s_n(K_l - K_m) - s_n(K - K_m)| \leq \|K_l - K_m - (K - K_m)\| = \|K_l - K\|.$$

Since $K_l \to T$ as $l \to \infty$ that is $\|K_l - K\| < \varepsilon$ we obtain

$$s_n(K_l - K_m) \to s_n(K - K_m) \text{ as } l \to \infty. \tag{2.2}$$

It follows from (2.1) that the statement

$$\|K_m - K_l\|_{p,\Phi} = \left[\frac{\sum\limits_{n=1}^{\infty}\left(\frac{1}{n}\sum\limits_{t|n}\varphi(t)s_t(K_m - K_l)\right)^p}{\sum\limits_{n=1}^{\infty}\left(\frac{1}{n}\right)^p}\right]^{\frac{1}{p}} < \varepsilon$$

valid for all $m, l \geq n_0$. From (2.2) the following inequality is obtained.

$$\left[\frac{\sum\limits_{n=1}^{\infty}\left(\frac{1}{n}\sum\limits_{t|n}\varphi(t)s_t(K_m - K)\right)^p}{\sum\limits_{n=1}^{\infty}\left(\frac{1}{n}\right)^p}\right]^{\frac{1}{p}} < \varepsilon, \qquad \text{as } l \to \infty.$$

Hence we have

$$\|K_m - K\|_{p,\Phi} < \varepsilon, \text{ for all } m \geq n_0.$$

Finally, we show that $K \in L_{p,\Phi}(E,F)$.

$$\begin{aligned}
\sum_{t|n}\varphi(t)s_t(K) &\leq \sum_{t|n}\varphi(2t-1)s_{2t-1}(K) + \sum_{t|n}\varphi(2t)s_{2t}(K) \\
&\leq \sum_{t|n}\big(\varphi(2t-1) + \varphi(2t)\big)s_{2t-1}(K - K_m + K_m) \\
&\leq \sum_{t|n}\mathscr{C}\varphi(t)s_{2t-1}(K - K_m + K_m) \\
&\leq \mathscr{C}\left(\sum_{t|n}\varphi(t)s_t(K - K_m) + \sum_{t|n}\varphi(t)s_t(K_m)\right).
\end{aligned}$$

By using Minkowski inequality; since $K_m \in L_{p,\Phi}(E,F)$ for all m and $\|K_m - K\|_{p,\Phi} \to 0$ as $m \to \infty$, we have

$$\begin{aligned}
\left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{t|n}\varphi(t)s_t(K)\right)^p\right)^{\frac{1}{p}} &\leq \mathscr{C}\left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{t|n}\varphi(t)s_k(K - K_m) + \frac{1}{n}\sum_{t|n}\varphi(t)s_k(K_m)\right)^p\right)^{\frac{1}{p}} \\
&\leq \mathscr{C}\left[\left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{t|n}\varphi(t)s_k(K - K_m)\right)^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{t|n}\varphi(t)s_k(K_m)\right)^p\right)^{\frac{1}{p}}\right] \\
&< \infty
\end{aligned}$$

which means $K \in L_{p,\Phi}(E,F)$. □

Definition 2.3. *Let $\mu = (\mu_i(K))$ be one of the sequences $s = (s_n(K))$, $c = (c_n(K))$, $d = (d_n(K))$, $x = (x_n(K))$, $y = (y_n(K))$ and $h = (h_n(K))$. Then, the space $L_{p,\Phi}^{(\mu)}$ generated via $\mu = (\mu_i(K))$ is defined as*

$$L_{p,\Phi}^{(\mu)}(E,F) = \left\{K \in \mathscr{B}(E,F) : \sum_n\left(\frac{1}{n}\sum_{t|n}\varphi(t)\mu_t(K)\right)^p < \infty, (1 < p < \infty)\right\}.$$

The corresponding norm $\|K\|_{p,\Phi}^{(\mu)}$ for each class is defined as

$$\|K\|_{p,\Phi}^{(\mu)} = \frac{\left(\sum\limits_{n=1}^{\infty}\left(\frac{1}{n}\sum\limits_{t|n}\varphi(t)\mu_t(K)\right)^p\right)^{\frac{1}{p}}}{\left(\sum\limits_{n=1}^{\infty}\left(\frac{1}{n}\right)^p\right)^{\frac{1}{p}}}.$$

Proposition 2.4. *The inclusion $L^{(a)}_{p,\Phi} \subseteq L^{(a)}_{q,\Phi}$ holds for $1 < p \leq q < \infty$.*

Proof. Since $l_p \subseteq l_q$ for $1 < p \leq q < \infty$ we have $L^{(a)}_{p,\Phi} \subseteq L^{(a)}_{q,\Phi}$. □

Theorem 2.5. *Let $1 < p < \infty$. The quasi-Banach operator ideal $\left[L^{(s)}_{p,\Phi}, \|T\|^{(s)}_{p,\Phi}\right]$ is injective if the sequence $s_r(K)$ is injective.*

Proof. Let $1 < p < \infty$ and $K \in \mathscr{B}(E,F)$ and $\mathscr{J} \in \mathscr{B}(F,F_0)$ be any metric injection. Suppose that $\mathscr{J}K \in L^{(s)}_{p,\Phi}(E,F_0)$. Then,

$$\sum_n \left(\frac{1}{n}\sum_{t|n}\varphi(t)s_t(\mathscr{J}K)\right)^p$$

Since $s = (s_r)$ is injective, we have

$$s_r(K) = s_r(\mathscr{J}K) \text{ for all } K \in \mathscr{B}(E,F), r = 1,2,\ldots. \tag{2.3}$$

Hence, we get

$$\sum_n \left(\frac{1}{n}\sum_{t|n}\varphi(t)s_t(K)\right)^p = \sum_n \left(\frac{1}{n}\sum_{t|n}\varphi(t)s_t(\mathscr{J}K)\right)^p < \infty.$$

Thus $K \in L^{(s)}_{p,\Phi}(E,F)$ and we have from (2.3)

$$\|\mathscr{J}K\|_{p,\Phi} = \left[\frac{\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{t|n}\varphi(t)s_t(\mathscr{J}T)\right)^p}{\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^p}\right]^{\frac{1}{p}}$$

$$= \left[\frac{\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{t|n}\varphi(t)s_t(K)\right)^p}{\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^p}\right]^{\frac{1}{p}} = \|K\|^{(s)}_{p,\Phi}.$$

So, the operator ideal $\left[L^{(s)}_{p,\Phi}, \|K\|^{(s)}_{p,\Phi}\right]$ is injective. □

Corollary 2.6. *It is known that $(c_r(K))$ and $(x_r(K))$ are injective, therefore the quasi-Banach operator ideals $\left[L^{(c)}_{p,\Phi}, \|K\|^{(c)}_{p,\Phi}\right]$ and $\left[L^{(x)}_{p,\Phi}, \|K\|^{(x)}_{p,\Phi}\right]$ are injective [14, p.90-94].*

Theorem 2.7. *Let $1 < p < \infty$. The quasi-Banach operator ideal $\left[L^{(s)}_{p,\Phi}, \|K\|^{(s)}_{p,\Phi}\right]$ is surjective if the sequence $(s_r(K))$ is surjective.*

Proof. Let $1 < p < \infty$ and $K \in \mathscr{B}(E,F)$ and $\mathscr{S} \in \mathscr{B}(E_0,E)$ be any metric surjection. Suppose that $K\mathscr{S} \in L^{(s)}_{p,\Phi}(E_0,F)$. Then,

$$\sum_n \left(\frac{1}{n}\sum_{t|n}\varphi(t)s_t(K\mathscr{S})\right)^p < \infty.$$

Since $s = (s_r)$ is surjective, we have

$$s_r(K) = s_r(K\mathscr{S}) \text{ for all } K \in \mathscr{B}(E,F), r = 1,2,\ldots. \tag{2.4}$$

Hence, we get

$$\sum_n \left(\frac{1}{n}\sum_{t|n}\varphi(t)s_t(K)\right)^p = \sum_n \left(\frac{1}{n}\sum_{t|n}\varphi(t)s_t(K\mathscr{S})\right)^p < \infty.$$

Thus, $K \in L^{(s)}_{p,\Phi}(E,F)$ and we have from (2.4)

$$\|K\mathscr{S}\|^{(s)}_{p,\Phi} = \left[\frac{\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{t|n}\varphi(t)s_t(K\mathscr{S})\right)^p}{\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^p}\right]^{\frac{1}{p}}$$

$$= \left[\frac{\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{t|n}\varphi(t)s_t(K)\right)^p}{\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^p}\right]^{\frac{1}{p}} = \|K\|^{(s)}_{p,\Phi}.$$

Hence, the operator ideal $\left[L^{(s)}_{p,\Phi}, \|K\|^{(s)}_{p,\Phi}\right]$ is surjective. □

Corollary 2.8. *It is known that* $(d_r(K))$ *and* $(y_r(K))$ *are surjective, therefore, quasi-Banach operator ideals* $\left[L^{(d)}_{p,\Phi}, \|K\|^{(d)}_{p,\Phi}\right]$ *and* $\left[L^{(y)}_{p,\Phi}, \|K\|^{(y)}_{p,\Phi}\right]$ *are surjective [14, p.95].*

Theorem 2.9. *The inclusion relations in the sequel hold for* $1 < p < \infty$:

(i) $L^{(a)}_{p,\Phi} \subseteq L^{(c)}_{p,\Phi} \subseteq L^{(x)}_{p,\Phi} \subseteq L^{(h)}_{p,\Phi}$,

(ii) $L^{(a)}_{p,\Phi} \subseteq L^{(d)}_{p,\Phi} \subseteq L^{(y)}_{p,\Phi} \subseteq L^{(h)}_{p,\Phi}$.

Proof. Let $K \in L^{(a)}_{p,\Phi}$. Then,

$$\sum_n \left(\frac{1}{n}\sum_{t|n} \varphi(t) s_t(K)\right)^p < \infty$$

where $1 < p < \infty$. And from Proposition 1.3, we have;

$$\begin{aligned}
\sum_n \left(\frac{1}{n}\sum_{t|n} \varphi(t) h_k(K)\right)^p &\leq \sum_n \left(\frac{1}{n}\sum_{t|n} \varphi(t) x_k(K)\right)^p \\
&\leq \sum_n \left(\frac{1}{n}\sum_{t|n} \varphi(t) c_k(K)\right)^p \\
&\leq \sum_n \left(\frac{1}{n}\sum_{t|n} \varphi(t) a_k(K)\right)^p \\
&< \infty
\end{aligned}$$

and

$$\begin{aligned}
\sum_n \left(\frac{1}{n}\sum_{t|n} \varphi(t) h_k(K)\right)^p &\leq \sum_n \left(\frac{1}{n}\sum_{t|n} \varphi(t) y_k(K)\right)^p \\
&\leq \sum_n \left(\frac{1}{n}\sum_{t|n} \varphi(t) d_k(K)\right)^p \\
&\leq \sum_n \left(\frac{1}{n}\sum_{t|n} \varphi(t) a_k(K)\right)^p \\
&< \infty.
\end{aligned}$$

Thus, the proof is completed. □

Theorem 2.10. *For* $1 < p < \infty$, $L^{(a)}_{p,\Phi}$ *is a symmetric operator ideal, and* $L^{(h)}_{p,\Phi}$ *is a completely symmetric operator ideal.*

Proof. Let $1 < p < \infty$.
Firstly, we show that $L^{(a)}_{p,\Phi}$ is symmetric in other words $L^{(a)}_{p,\Phi} \subseteq \left(L^{(a)}_{p,\Phi}\right)^*$ holds. Let $K \in L^{(a)}_{p,\Phi}$. Then,

$$\sum_n \left(\frac{1}{n}\sum_{t|n} \varphi(t) a_k(K)\right)^p < \infty.$$

It follows from [12, p.152] $a_r(K^*) \leq a_r(K)$ for $K \in \mathscr{B}$. Hence, we get

$$\sum_n \left(\frac{1}{n}\sum_{t|n} \varphi(t) a_k(K^*)\right)^p \leq \sum_n \left(\frac{1}{n}\sum_{t|n} \varphi(t) a_k(K)\right)^p < \infty.$$

Therefore, $K \in \left(L^{(a)}_{p,\Phi}\right)^*$. Thus, $L^{(a)}_{p,\Phi}$ is symmetric.
Let show that the equation $L^{(h)}_{p,\Phi} = \left(L^{(h)}_{p,\Phi}\right)^*$ is satisfied. It follows from [14, p.97] that $h_r(K^*) = h_r(K)$ for $K \in \mathscr{B}$. Then, we can write

$$\sum_{n=1}^{\infty} \left(u_n \sum_{k=1}^{n} \sum_{j\in E_k} v_j h_j(K^*)\right)^p = \sum_{n=1}^{\infty} \left(u_n \sum_{k=1}^{n} \sum_{j\in E_k} v_j h_j(K)\right)^p \text{ and}$$

$$\sum_n \left(\frac{1}{n}\sum_{t|n} \varphi(t) h_k(K^*)\right)^p = \sum_n \left(\frac{1}{n}\sum_{t|n} \varphi(t) h_k(K)\right)^p.$$

Hence, $L^{(h)}_{p,\Phi}$ is completely symmetric. □

Theorem 2.11. *Let* $1 < p < \infty$. $L^{(c)}_{p,\Phi} = \left(L^{(d)}_{p,\Phi}\right)^*$ *and* $L^{(d)}_{p,\Phi} \subseteq \left(L^{(c)}_{p,\Phi}\right)^*$ *holds. Also, for any compact operators* $L^{(d)}_{p,\Phi} = \left(L^{(c)}_{p,\Phi}\right)^*$ *holds.*

Proof. Let $1 < p < \infty$. For $T \in \mathscr{B}$ it is known from [14] that $c_r(K) = d_r(K^*)$ and $c_r(T^*) \leq d_r(K)$. Also, when K is a compact operator, the equality $c_r(K^*) = d_r(K)$ holds. So the proof is complete. □

Theorem 2.12. $L^{(x)}_{p,\Phi} = \left(L^{(y)}_{p,\Phi}\right)^*$ *and* $L^{(y)}_{p,\Phi} = \left(L^{(x)}_{p,\Phi}\right)^*$ *hold.*

Proof. Let $1 < p < \infty$. For $K \in \mathscr{B}$ we have from [14] that $x_n(K) = y_n(K^*)$ and $y_n(K) = x_n(K^*)$. Thus the proof is clear. □

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