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Research Article

On the Inverses of Lines under the Inversion in a Generalized Taxicab Circle

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Abstract
In this study, the images of lines under the inversion in a generalized taxicab circle are examined. It is observed that the image of the line not passing through the inversion center is not a generalized taxicab circle, but the closed curve. The images of the lines under the inversion mapping are investigated depending on their positions and some features related to the images are presented. Furthermore, it is concluded that the inversion in a generalized taxicab circle maps the pencil of parallel lines (except the line passing through the center) to the set of the curves passing through the inversion center.

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1. Introduction

The distance between two points can be measured in various ways according to different distance functions. Some of the most popular distance functions comprise the Euclidean distance, the maximum distance, the taxicab distance. The analytical planes equipped with distance functions such as taxicab distance, maximum distance, alpha distance are the non-Euclidean planes. These non-Euclidean geometries have been studied in different aspects by many researchers [1–6]. The taxicab distance measures the sum of the Euclidean lengths of line segments parallel to the coordinate axes between two points. Wallen redefined the taxicab distance to eliminate potentially misleading symmetry. The generalized taxicab distance between two points is found by multiplying the Euclidean lengths of line segments parallel to the coordinate axes by positive real integers, then adding the results. This is the reason why the weighted taxicab distance is also known as the generalized taxicab distance, providing an alternative approach to computing distances in non-Euclidean geometry. In recent years, metric geometry based on the generalized taxicab distance has been studied and developed. For further reading on the subject, refer to the studies [7–11].

Apollonius of Perga first introduced the inversion in the circle. In the 1830s, Steiner examined the circle inversion method in a systematic manner. Since then, many researchers have examined the inversions in circles and contributed to their advancement [12–15]. Many variations of the inversion mapping have been accomplished by using geometric objects such as ellipses, spheres, ellipsoids, parallel lines, central cones, and star-shaped sets, rather than circles to describe the inversion [13, 14, 16–18]. Furthermore, many distance functions, including the taxicab distance, the maximum distance, the Chinese Checkers distance and the α-distance, have been taken into consideration in the study of the inversion transformations [4, 19–27].

In this work, the images of lines under the inversion in the generalized taxicab circle are examined. It is observed that the image of line not passing through the inversion center is not a generalized taxicab circle , but the closed curve. The images of the lines under the inversion mapping are investigated depending on their positions and some features related to the images are presented. Furthermore, it is concluded that the inversion in the generalized taxicab circle maps the pencil of parallel lines (except the line passing the center) to the set of the curves passing the inversion center.

2. Preliminaries

In this section, some concepts used throughout this work are mentioned.

Definition 1. *The generalized taxicab distance between points $A_1 = (x_1, y_1)$ and $A_2 = (x_2, y_2)$ in the analytical plane is*

$$d_G(A,B) = a|x_2 - x_1| + b|y_2 - y_1|,$$

where the real numbers $a, b > 0$ [11].

It is seen from definition that the generalized taxicab distance between the points A_1 and A_2 is equal to the sum of positive multiples a and b of the Euclidean lengths of the sides parallel to the coordinate axes in the right triangle with the hypotenuse A_1A_2. Indeed, d_G is a family of distance functions depending on the positive numbers a and b. In the special case of $a = b = 1$, d_G is the taxicab distance. Throughout this paper, we will assume that a and b are constant values given at the beginning, unless otherwise stated.

The generalized taxicab plane is the analytical plane equipped with the generalized taxicab distance and denoted by $\mathbb{R}^2_G$. It is almost the same as the Euclidean plane except the distance function.

The classification of lines in the generalized taxicab plane, similar to [6], is as follows:

Definition 2. *Let m be the slope of the line l in the generalized taxicab plane. The line l is called the steep line, the gradual line and the separator (guide) line in the cases of $|m| > \frac{a}{b}$, $|m| < \frac{a}{b}$ and $|m| = \frac{a}{b}$, respectively. In the special cases that the line l is parallel to x-axis or y-axis, l is named as the horizontal line or the vertical line, respectively [10].*

One of the basic objects in $\mathbb{R}^2_G$ is circle. The generalized taxicab circle centered at $M = (m_1, m_2)$ with the radius r is a diamond with equation $a|x - m_1| + b|y - m_2| = r$, whose vertices are $v_1 = (m_1 + \frac{r}{a}, m_2)$, $v_2 = (m_1, m_2 + \frac{r}{b})$, $v_3 = (m_1 - \frac{r}{a}, m_2)$ and $v_4 = (m_1, m_2 - \frac{r}{b})$.

Every Euclidean translation preserves the generalized taxicab distance. So, it is an isometry in $\mathbb{R}^2_G$. Reflections in the coordinate axes and the separator lines through the origin are isometries in the generalized taxicab plane. The set of axes of isometric reflections is

$$\{x = 0, y = 0, y = \frac{a}{b}x, y = -\frac{a}{b}x\}.$$

Also, the set of isometric reflections in matrix form is

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & \frac{b}{a} \\ \frac{a}{b} & 0 \end{bmatrix}, \begin{bmatrix} 0 & -\frac{b}{a} \\ -\frac{a}{b} & 0 \end{bmatrix} \right\}.$$

And the set of rotations about the origin that preserve the generalized taxicab distance in matrix form is (see [7, 8]):

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -\frac{b}{a} \\ \frac{a}{b} & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & \frac{b}{a} \\ -\frac{a}{b} & 0 \end{bmatrix} \right\}.$$

The inversion in the generalized taxicab circle has been defined as an analog of the inversion in the Euclidean circle [21]. It is well-known that the inversion in the generalized taxicab circle $\mathscr{C}$ centered O with the radius r maps the point X to the inverse point X' located along the ray OX such that the product of the generalized taxicab distances from the center O to the point X and the point X' is equal to r^2. On the other hand, while inversion leaves the points on circle $\mathscr{C}$ fixed, it maps points close to the center O to points far from the center O and conversely. Also, it is not defined at the inversion center O and there is no point whose image is the inversion center O. Thus, by adding only one point O_∞ called the "ideal point" or "point at infinity" to the generalized taxicab plane, the inversion is defined at the inversion center O and becomes a one-to-one map as follows:

Definition 3. *Let $\mathscr{C}$ be the generalized taxicab circle centered at the point O with the radius r and O_∞ be the ideal point added to the generalized taxicab plane. The inversion in the generalized taxicab circle $\mathscr{C}$ is the mapping $I_\mathscr{C}$ of $\mathbb{R}^2_G \cup \{O_\infty\}$ to $\mathbb{R}^2_G \cup \{O_\infty\}$ defined by $I_\mathscr{C}(O) = O_\infty$, $I_\mathscr{C}(O_\infty) = O$ and $I_\mathscr{C}(X) = X'$ for any point X different from O, O_∞ where the point X' lies on the ray OX and $d_G(O,X).d_G(O,X') = r^2$. The point X' is called the inverse or image of the point X under the inversion $I_\mathscr{C}$. Here, $\mathscr{C}$ is termed the inversion circle; O and r are the center and the radius of the inversion $I_\mathscr{C}$, respectively.*

The following theorem states that one of the basic properties of the inversion in the generalized taxicab circle is proved.

Theorem 4. *Except for the inversion center, any point inside of the inversion circle is transformed to a point outside of it under the inversion with respect to the generalized taxicab circle, and conversely [21].*

When the inversion in the generalized taxicab circle is applied to a point (except for the inversion center), the relation between the point and its inverse in terms of the inversion center and inversion radius is expressed as follows:

Proposition 5. *Let $I_{\mathscr{C}}$ be the inversion in the generalized taxicab circle $\mathscr{C}$ centered at the point O with the radius r. If the point X' is the inverse of the point X under the inversion $I_{\mathscr{C}}$ where $X \neq O$, then the following equality between the points X and X' holds ([21]):*

$$X' - O = \frac{r^2}{(d_G(O,X))^2}(X - O).$$

In the case that the inversion center O is the origin, it is obvious from Proposition 5 that the equality $X = \lambda X'$ is valid, where $\lambda = \frac{r^2}{(d_G(O,X))^2}$ and $X \neq O$.

By substituting the coordinates of the points in Proposition 5, the coordinates of the inverse point are expressed in the following proposition.

Proposition 6. *Let $I_{\mathscr{C}}$ be the inversion in the generalized taxicab circle $\mathscr{C}$ with center $O = (a_1, a_2)$ and radius r. If the points $X = (x, y)$ and $X' = (x', y')$ are a pair of inverse points with respect to the inversion $I_{\mathscr{C}}$ where $X \neq O$, then the following equalities are valid ([21]):*

$$\begin{aligned} x' &= a_1 + \frac{r^2(x - a_1)}{(a|x - a_1| + b|y - a_2|)^2}, \\ y' &= a_2 + \frac{r^2(y - a_2)}{(a|x - a_1| + b|y - a_2|)^2}. \end{aligned}$$

In the case that the inversion center is the origin, the coordinates of the inverse point follow as:

Corollary 7. *Let $I_{\mathscr{C}}$ be the inversion in the generalized taxicab circle $\mathscr{C}$ centered at the point $O = (0,0)$ with the radius r. If the points $X = (x, y)$ and $X' = (x', y')$ are a pair of inverse points with respect to the inversion $I_{\mathscr{C}}$ where $X \neq O$, then the following equality between the coordinates of X and X' holds [21]:*

$$(x', y') = \left(\frac{r^2 x}{(a|x| + b|y|)^2}, \frac{r^2 y}{(a|x| + b|y|)^2}\right).$$

Since translations are isometries in the generalized taxicab plane, no generality is lost to take the inversion center of inversion at the origin. Therefore, throughout this study, the center O is the origin unless otherwise stated.

3. Images of the lines under the inversion in a generalized taxicab circle

The inversion in a Euclidean circle leaves fixed lines passing through the inversion center, while it transforms lines not passing through the inversion center to circles passing through the inversion center. Considering the inversions in the taxicab and maximum circles, similar result is achieved for the images of the lines passing through the inversion center. On the other hand, it is noticed that the images of the lines not passing through the center have different shapes and properties depending on their positions [4, 19, 26]. It is observed that the image of line not passing through the inversion center under the inversion in taxicab and maximum circles is not a taxicab and maximum circle, respectively, but the closed curve. In this section, the images of the lines under the inversion in the generalized taxicab circle are discussed and the results are presented.

In the following theorem, the images of the lines passing through the inversion center are examined.

Theorem 8. *The inversion in the generalized taxicab circle maps the lines passing through the inversion center to themselves.*

Proof. Suppose that $I_{\mathscr{C}}$ and l are the inversion in the generalized taxicab circle $\mathscr{C}$ centered at $O = (0,0)$ with the radius r and the line with equation $a_1 x + a_2 y = 0$ where a_1, $a_2 \in \mathbb{R}$ and $a_1^2 + a_2^2 \neq 0$, respectively.

By Corollary 7, the image of the line l under the inversion $I_{\mathscr{C}}$ has equation

$$a_1 \frac{r^2 x'}{(a|x'| + b|y'|)^2} + a_2 \frac{r^2 y'}{(a|x'| + b|y'|)^2} = 0$$

and since

$$a_1x' + a_2y' = 0,$$

the proof is completed. ■

The following theorem shows that the images of the lines not passing through the inversion center under the inversion in the generalized taxicab circle have different shapes and properties depending on their positions.

Theorem 9. *The inversion in the generalized taxicab circle maps a line not passing through the inversion center to the closed curve passing through the inversion center.*

Proof. Suppose that $I_{\mathscr{C}}$ and l are the inversion in the generalized taxicab circle $\mathscr{C}$ centered at $O = (0,0)$ with the radius r and the line with equation $a_1x + a_2y + a_3 = 0$ where $a_1,\ a_2,\ a_3 \in \mathbb{R}$ and $a_1^2 + a_2^2 \neq 0$, $a_3 \neq 0$, respectively. By Corollary 7, the image of the line l under the inversion $I_{\mathscr{C}}$ has equation

$$a_1r^2x' + a_2r^2y' + a_3(a\left|x'\right| + b\left|y'\right|)^2 = 0. \tag{1}$$

Equation (1) will be examined according to the classification in Definition 2. Firstly, assume that the line l is the gradual line. Then $\left|\frac{a_1}{a_2}\right| < \frac{a}{b}$ where $a_2 \neq 0$.

In the case of $x'.y' \geq 0$, the locus of points satisfying equation (1) is a parabola arc that its vertex is

$$T_1 = \left(-\frac{a_2(b-ma)(a^2+b^2+a(a+mb))}{4a_3(a^2+b^2)^2(a+mb)}r^2, -\frac{a_2(b-ma)(m(a^2+b^2)+b(a+mb))}{4a_3(a^2+b^2)^2(a+mb)}r^2\right),$$

its directrix and its axis of symmetry are the line

$$bx' - ay' + \frac{a_2(1+m^2)r^2}{4a_3(a+mb)} = 0,$$

and the line

$$ax' + by' + \frac{a_2(b-ma)r^2}{2a_3(a^2+b^2)} = 0,$$

where $m = -\frac{a_1}{a_2}$, respectively.

In the case of $x'.y' < 0$, the locus of points satisfying equation (1) is a parabola arc that its vertex is

$$T_2 = \left(\frac{a_2(b+ma)(2a^2-abm+b^2))}{4a_3(a^2+b^2)^2(a-mb)}r^2, \frac{a_2(b+ma)(a^2m-ab+2b^2m)}{4a_3(a^2+b^2)^2(a-mb)}r^2\right),$$

its directrix and its axis of symmetry are the line

$$bx' + ay' - \frac{a_2(1+m^2)r^2}{4a_3(a-mb)} = 0,$$

and the line

$$-ax' + by' + \frac{a_2(b+ma)r^2}{2a_3(a^2+b^2)} = 0,$$

where $m = -\frac{a_1}{a_2}$, respectively. Thus, the image of the line l is the closed curve consisting the union of two parabola arcs determined by the sets

$$\left\{(x',y') : a_1r^2x' + a_2r^2y' + a_3(ax'+by')^2 = 0, x'.y' \geq 0\right\}$$

and

$$\left\{(x',y') : a_1r^2x' + a_2r^2y' + a_3(ax'-by')^2 = 0, x'.y' \leq 0\right\}$$

such that the both of them pass through O and the line a_1x+a_2y is tangent to them at O, (Figure 1). It is seen that the axes of symmetry of the parabolas containing the parabola arcs in the image of the line l under $I_{\mathscr{C}}$ are separator lines whose slopes are opposite signs. In the case that the line l is the steep line, it is similar.

In the case of $a_1=0$ or $a_2=0$, the line l is parallel to the coordinate axis. Suppose that $a_2=0$. Then, the image of the line l under the inversion $I_{\mathscr{C}}$ has equation

$$a_1r^2x'+a_3(a\left|x'\right|+b\left|y'\right|)^2=0 \tag{2}$$

where $a_1\neq 0$. In the case of $x'.y'\geq 0$, equation (2) signifies a parabola arc that its vertex is

$$T_1=\left(-\frac{a_1a^2}{4a_3(a^2+b^2)^2}r^2,-\frac{a_1a(2b^2+a^2)}{4a_3b(a^2+b^2)^2}r^2\right),$$

its directrix and its axis of symmetry are the line

$$bx'-ay'-\frac{a_1r^2}{4a_3b}=0,$$

and the line

$$ax'+by'+\frac{a_1ar^2}{2a_3(a^2+b^2)}=0,$$

respectively.

In the case of $x'.y'\leq 0$, equation (2) signifies a parabola arc that its vertex is

$$T_2=\left(-\frac{a_1a^2}{4a_3(a^2+b^2)^2}r^2,\frac{a_1a(2b^2+a^2)}{4a_3b(a^2+b^2)^2}r^2\right),$$

its directrix and its axis of symmetry are the line

$$bx'+ay'-\frac{a_1r^2}{4a_3b}=0,$$

and the line

$$-ax'+by'-\frac{a_1ar^2}{2a_3(a^2+b^2)}=0,$$

respectively. Thus, it is seen that if the line l is parallel to the coordinate axis, the symmetry axes of the parabola arcs on image are separator lines whose slopes are opposite signs. And, the image of the line l is the closed curve consisting the union of two parabola arcs expressed by the sets

$$\left\{(x',y'):a_1r^2x'+a_3(ax'+by')^2=0,\ x'.y'\geq 0\right\}$$

and

$$\left\{(x',y'):a_1r^2x'+a_3(ax'-by')^2=0,\ x'.y'\leq 0\right\}$$

such that the both of them pass through O and the coordinate axis parallel to line l is tangent to them at O, (Figure 2). It is seen that the axes of symmetry of the parabolas containing the parabola arcs in the image are parallel to the sides of the inversion circle. Since the reflection in the coordinate axis perpendicular to the line l maps each of the parabola arcs to the other, the image is also symmetric about the coordinate axis. Similar results are obtained immediately if the l line is parallel to the other coordinate axis.

In the case of $a_1=\pm a$ and $a_2=\pm b$, line l is a separator line. Considering the separator line with equation $ax+by+a_3=0$, the points on the image hold

$$ar^2x'+br^2y'+a_3(a\left|x'\right|+b\left|y'\right|)^2=0. \tag{3}$$

The equality (3) means that the image is the closed curve consisting the union of the line segment which is

$$\left\{(x',y'):ax+by+\frac{r^2}{a_3}=0,\ x'.y'\geq 0\right\}$$

and the parabola arc which is

$$\left\{(x',y') : ar^2x' + br^2y' + a_3(ax' - by')^2 = 0, x'.y' \leq 0\right\}, \text{ (Figure 3).}$$

The parabola arc in image is on the parabola that its vertex is

$$T = \left(\frac{(b^2-a^2)(3a^2+b^2)}{8aa_3(a^2+b^2)^2}r^2, -\frac{(b^2-a^2)(3b^2+a^2)}{8ba_3(a^2+b^2)^2}r^2\right),$$

its directrix and its axis of symmetry are the line

$$bx' + ay' - \frac{\sqrt{a^2+b^2}r^2}{8aba_3} = 0,$$

and the line

$$ax' - by' - \frac{(b^2-a^2)r^2}{2a_3(a^2+b^2)} = 0,$$

respectively. It is observed that the image pass through O and the separator line $ax+by=0$ is tangent to $I_{\mathscr{C}}(l)$ at O. Furthermore, the image is symmetric about the separator line $ax' - by' = 0$. ■

The following results are immediately obtained from Theorem 9.

Corollary 10. *The inversion in the generalized taxicab circle maps a gradual or steep line not passing through the inversion center to the closed curve having the following properties:*

i. It passes through the inversion center,

ii. It consists of two parabola arcs on parabolas whose axes of symmetry are the separator lines with the opposite signed slopes,

iii. Its tangent at the inversion center is the line parallel to the given line, (Figure 1).

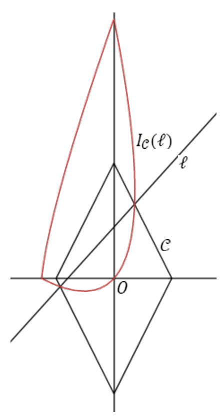

Fig. 1. The image of a gradual line

Corollary 11. *The inversion in the generalized taxicab circle maps a horizontal or vertical line not passing through the inversion center to the closed curve having the following properties:*

i. It passes through the inversion center,

ii. It consists of two parabola arcs on parabolas whose axes of symmetry are the separator lines with the opposite signed slopes,

iii. Its tangent at the inversion center is the coordinate axis parallel to the given line,

iv. It is symmetric about the coordinate axis perpendicular to the given line, (Figure 2).

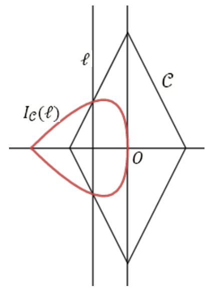

Fig. 2. The image of a vertical line

Corollary 12. *The inversion in the generalized taxicab circle maps the separator line not passing through the inversion center to the closed curve having the following properties:*

i. It passes through the inversion center,

ii. It consists of the line segment parallel to the given separator line and the parabola arc on parabola whose axis of symmetry is the separator line with the opposite signed slopes of given line,

iii. It is symmetric about the separator line passing through the inversion center,

iv. Its tangent at the inversion center is the separator line parallel to the given separator line, (Figure 3).

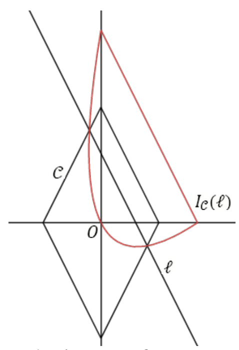

Fig. 3. The image of a separator line

Considering that $I_{\mathscr{C}}$ and l_0 are the inversion in the generalized taxicab circle $\mathscr{C}$ with the center $O=(0,0)$ and the radius r and the line passing through O with equation $a_1x+a_2y=0$, where $a_1^2+a_2^2 \neq 0$, respectively. The pencil of all lines parallel to the line l_0 is actually the set $\{l : l \parallel l_0\}$, where l has equation $a_1x+a_2y+a_3=0$, $a_3 \in \mathbb{R}$. The image of this pencil under $I_{\mathscr{C}}$ is the set of inverses of all lines parallel to the line l_0. In Theorem 9, the image of the line under the inversion $I_{\mathscr{C}}$ is examined according to whether it is a separator line, a horizontal or vertical line and a gradual or steep line. Thus, it is clear that $I_{\mathscr{C}}$ maps the lines in pencil (except l_0) to the closed curves through the inversion center O such that the line l_0 is tangent to them at the inversion center. The set of these closed curves is the image of the pencil under $I_{\mathscr{C}}$. The following corollaries provide properties relating to the images of pencil of the parallel lines under inversion in the generalized taxicab circle, which depends on the position of the lines.

Corollary 13. *The image of the pencil of horizontal (or vertical) parallel lines (except for the line at the inversion center) under the inversion in the generalized taxicab circle is the pencil of the closed curves such that they are formed by the union of two parabola arcs on parabolas whose axes of symmetry are the separator lines. And, the image of the pencil is symmetric about the coordinate axis perpendicular to the pencil. Also, the line passing through the inversion center in the pencil is tangent to the image of the pencil, (Figure 4).*

Corollary 14. *The image of the pencil of gradual (or steep) parallel lines (except for the line at the inversion center) under the inversion in the generalized taxicab circle is the pencil of the closed curves such that they are formed by the union of two parabola arcs on parabolas whose axes of symmetry are the separator lines. Also, the line passing through the inversion center in the pencil is tangent to the image of the pencil, (Figure 5).*

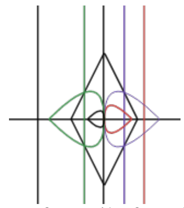

Fig. 4. The image of pencil of vertical parallel lines

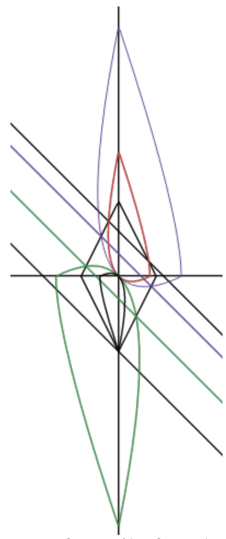

Fig. 5. The image of pencil of gradual parallel lines

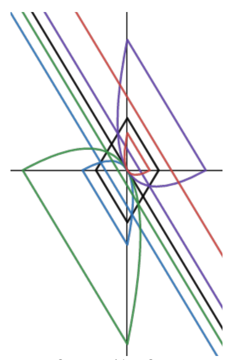

Fig. 6. The image of pencil of separator parallel lines

Corollary 15. *The image of the pencil of separator parallel lines (except for the line at the inversion center) under the inversion in the generalized taxicab circle is the pencil of the closed curves such that they are formed by the union of two parabola arcs on parabolas whose axes of symmetry are the separator lines. Also, the line passing through the inversion center in the pencil is tangent to the image of the pencil, (Figure 6).*

4. Conclusions

This study delved into the examination of lines under the inversion in the generalized taxicab circle. Comparing with the well-known results the inversion in Euclidean circle, it was observed that the lines passing through the inversion center in the generalized taxicab circle were also invariant. However, the lines not passing through the inversion center transformed into closed curves passing through the inversion center. Analyzing the image of lines under these inversions led to several significant findings. These observations provide to our understanding of how lines and pencils of parallel lines are inverted by the inversion in the generalized taxicab circle. Consequently, it is thought that the results obtained in this study contribute to the literature including the subject of the inversion in non-Euclidean geometry.

References

[1] Akça, Z., & Kaya R. (1997). *On the taxicab trigonometry*. Journal of Inst. Math. Comput. Sci. Math. Ser., 10(3), 151–159.

[2] Akça, Z., & Kaya R. (2004). *On the distance formulae in three dimensional taxicab space*. Hadronic Journal, 27(5), 521-532.

[3] Akça, Z., & Çalış, C. (2021). *On the voronoi diagram and taxicab plane*. Erzincan University Journal of Science and Technology, 14(1), 175-181.

[4] Bayar A., & Ekmekçi, S. (2014). *On circular inversions in taxicab plane*. Journal of Advanced Research in Pure Mathematics, 6(4), 33-39.

[5] Kaya, R., Akça, Z., Günaltılı, İ. & Özcan, M. (2000). *General equation for Taxicab conics and their classification*. Mitt. Math. Ges. Hamburg, 19(0), 136-148.

[6] Krause, E. F. (1975). *Taxicab geometry*. Addison -Wesley Publishing Company, Menlo Park, California, USA.

[7] Çolakoğlu, H. B. (2018). *The generalized taxicab group*. International Electronic Journal of Geometry, 11(2), 83-89.

[8] Ekmekçi, S., Bayar, A., & Altıntaş, A. (2015). *On the group of isometries of the generalized Taxicab metric*. International Journal of Comtemporary Mathematical Sciences, 10(4), 159-166.

[9] Ekmekçi, S., Akça, Z., & Altıntaş, A. (2015). *On trigonometric functions and norm in the generalized Taxicab plane*. Mathematical Sciences And Applications E-Notes, 3(2), 27-33.

[10] Altıntaş, A. (2009). *The application of some geometric problems on Euclidean plane using generalized taxicab metric*. Master's Thesis, Eskişehir Osmangazi University, Eskişehir.

[11] Wallen, L. J. (1995). *Kepler, the taxicab metric, and beyond: An isoperimetric primer*. The College Mathematics Journal, 26(3), 178-190.

[12] Blair, D. E. (2000). *Inversion theory and conformal mapping*. Student Mathematical Library, American Math. Society.

[13] Gdawiec, K. (2014). *Star-shaped set inversion fractals*. Fractals, 22(4), 1450009-1-1450009-7.

[14] Childress, N. (1965). *Inversion with respect to the central conics*. Mathematics Magazine, 38(3), 147-149.

[15] Nickel, J. A. (1995). *A budget of inversion*. Math. Comput. Modelling, 21(6), 87-93.

[16] Ramirez, J. L. (2014). *Inversions in an ellipse*. Forum Geometricorum, 14, 107-115.

[17] Ramirez, J. L., & Rubiano G. N. (2014). *A geometrical construction of inverse points with respect to an ellipse*. International Journal of Mathematical Education in Science and Technology, 45(8), 1254-1259.

[18] Ramirez, J. L., Rubiano, G. N., & Zlobec, B. J. (2015). *A generating fractal patterns by using p-circle inversion*. Fractals, 23(4), 1550047-1-1550047-13.

[19] Ekmekçi, S. (2023). *A note on the maximum circle inverses of lines in the maximum plane*. Ikonion Journal of Mathematics, 5(2), 1-9.

[20] Gelişgen Ö., & Ermiş T. (2023). *Inversions and fractal patterns in Alpha plane*. Int. Electron. J. Geom., 16(1), 398-411.

[21] Bilgin, Y. (2023). *Inversions with respect to circles in the generalized Taxicab plane*. Master's Thesis, Eskişehir Osmangazi University, Eskişehir.

[22] Can, Z. (2022). *On spherical inversions in three dimensional tetrakis hexahedron space*. Erciyes Üniversitesi Fen Bilimleri Enstitüsü Dergisi, 38(1), 100-108.

[23] Pekzorlu, A., & Bayar, A. (2020). *On the Chinese Checkers spherical invesions in three dimensional Chinese Checker space*. Communications Faculty of Sciences University of Ankara Series A1: Mathematics and Statistics, 69(2), 1498-1507.

[24] Pekzorlu, A., & Bayar, A. (2020). *Taxicab spherical invesions in Taxicab space*. Journal of Mahani Mathematical Research Center, 9(1), 45-54.

[25] Pekzorlu, A., & Bayar, A. (2022). *On the Chinese Checkers circular inversions in the Chinese Checkers plane*. Hagia Sophia Journal of Geometry, 4(2), 28-34.

[26] Yüca, G., & Can, Z. (2020). *On The circular inversion in maximum plane*. Ikonion Journal of Mathematics, 2(2), 26-34.

[27] Cırık, Y., & Ekmekçi, S. (2022). *On the maksimum spherical inversions*. Erzincan University, Journal of Science and Technology, 15(1), 360-371.