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Measures of Distance and Entropy Based on the Fermatean Fuzzy-Type Soft Sets Approach

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Abstract

The definition of Fermatean fuzzy soft sets and some of its features are introduced in this study. A Fermatean fuzzy soft set is a parameterized family of Fermatean fuzzy sets and a generalization of intuitionistic and Pythagorean fuzzy soft sets. This paper presents a definition of the Fermatean fuzzy soft entropy. Also acquired are the formulae for standard distance measures such as Hamming and Euclidean distance. Other formulas have also been proposed for calculating the entropy and distance measurements of FFSSs. Even if the entropy and distance measures are defined for other set extensions, they cannot be applied directly to Fermatean fuzzy soft sets. It can be used to determine the uncertainty associated with a Fermatean fuzzy soft set, discover similarities between any two Fermatean fuzzy soft sets using the proposed distance measures, and compare it to other existing structures in the literature. Fermatean fuzzy soft set applications in decision-making and pattern recognition difficulties are also examined. Finally, comparison studies with other known equations are performed.

1. Introduction

1.1. Motivation

In generalized set theory, measures of entropy and distance are crucial notions. Distance is the difference between two patterns. The pattern could be a scalar number, vector, matrix, or other numeric data type. Distance metrics are effective for identifying parallels and differences in patterns. The dissimilarity, or distance, between two patterns, is zero if they are identical. As the difference between patterns develops, so does the dissimilarity or distance. Distance measuring between objects is essential in many fields, including information retrieval, data mining, and machine learning. The items under discussion are frequently made up of numerous components or groups of another object. A distance can be easily defined if the set or tuple is sorted and may be shown as a vector. However, it is expected to mistakenly believe that the $i-th$ index of a vector x equals the $i-th$ index of a vector y. One must rely on a less accurate distance measure if such a correlation does not exist. Entropy measures the degree of ambiguity. A system's entropy is directly proportional to its irregularity. If the entropy of each system is known, one can determine which is more stable. Entropy is the amount of practical work that can be produced from the heat energy emitted into the environment. You cannot work if the heat energy is equal inside and outside the engine. For example, suppose there is an energy difference between the engine and the external environment. The situation changes if it is cold outside and the engine's pistons are hot. The energy will flow from the hot to the cold, and at the same time, it will start the engine. Ludwig Boltzmann later generalized this definition: Accordingly, the particle configurations (low temperature) corresponding to the equal spread of energy in space are different from the combinations of energy concentration at a single point (high temperature). After all, space is more significant than a single point. If the energy is evenly distributed, the entropy always increases because we cannot do work. This is why heat flows from hot to cold. It is not impossible for the air in a room to spontaneously collect in one corner. However, this is such a low probability that you will not see it in your lifetime. Thus, Boltzmann showed that entropy is statistical.

Shannon [1] established the concept of information entropy. In information theory, entropy measures the uncertainty associated with a random variable. Entropy can be conceived of as a system's hidden information. Entropy, in more technical terms, is a measure of the

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amount of information that may be acquired by measuring the system. In this context, very detailed information can be obtained if the air molecule particles collected in the corner of the room are measured. As a result, trillions of particles are measured at once. On the other hand, the air molecules surrounding the room provide more information. As for why the entropy is high, A particle gathered in a single corner of the room gives clear information about other particles. After all, they are all in the same place. However, if one of the air molecules emitted into the room is measured, not much information can be obtained about the location of the other molecules. According to Shannon entropy, the information does not disappear; in this case, it is hidden from view. The number of positional combinations of the air molecules occupying the room is higher than the number of molecules collected in one corner. In short, the more the different combinations of particles that make up a system look like the same thing at first glance (in this case, the air surrounding the room), the more information is hidden. Various entropy and distance measures are available in the literature, helpful in solving real-life problems. FS-type entropy and distance measures were later defined and studied in SS, IFSS, and PFSS. Fuzzy entropy is a term used to indicate the degree of uncertainty, and finding the entropy of a set is one of the essential applications of fuzzy set theory. It has yet to be suggested that these studies' definitions of entropy and distance measures be extended to include FFSSs. Filling this gap is the primary motivation of this study.

1.2. Literature

Multi-Criteria Decision Making (MCDM) is a collection of analytical approaches that evaluate the advantages and disadvantages of alternatives based on many criteria. MCDM methods are used to support the DM process and to select one or more alternatives from a set of alternatives with different characteristics according to conflicting criteria or to rank these alternatives. In other words, in MCDM methods, decision-makers rank the alternatives with different characteristics by evaluating them according to many criteria. MCDM is a set of methods frequently used in all areas of life and at all levels. There are many studies in the literature about MCDM in various fields [2]- [11].

Fuzzy sets(FS) and their expansions are a more effective tool for describing vague and imprecise information and explaining it in a way that is close to human thinking. Although the FSs that emerged with the membership function have made an innovative contribution to the solution of uncertainties, it is impossible to explain the problems and uncertainties in real life only through membership. Real life consists of degrees of non-membership and even hesitations as much as degrees of membership. This situation naturally leaves the solution to uncertainties incomplete. FS expansions proposed by many researchers, especially Atanassov [12], have been powerful tools to solve the problem. However, FSs must be more comprehensive in explaining uncertainties in real-life problems. Despite all of the possible responses, these theories have several drawbacks. These limitations include the inability to properly consider the parametrization tool and how to set the membership function for each unique item. Because of these restrictions, it is challenging for DMRs to make wise judgments throughout the analysis.

Since the formation of the membership function (MF) differs for each individual, the formation of more than one MF and its belonging to the set varies according to everyone. Thereupon, Molodtsov [13] initiated the SS theory. While the soft set (SS) theory deals with the set-valued function, FSs remove the uncertainty with the real-valued function. The problem of establishing an MF does not exist in the SS. So, the SS is much more useful. An SS is a parameterized family of sets extended into different hybrid structures, such as Fuzzy soft sets (FSS), intuitionistic fuzzy soft sets (IFSS), and Pythagorean fuzzy soft sets (PFSS). Since the Fermatean fuzzy set (FFS) can deal with vagueness or uncertainty, the parameterized family of FFSs, the Fermatean fuzzy soft sets (FFSS), also performs well. An FFS is obtained in the case of FFSSs, corresponding to each parameter. FFSs can manage several real-life situations where the intuitionistic fuzzy sets (IFS) and Pythagorean fuzzy sets (PFS) fail to explain. Suppose there is a case in which someone expresses his satisfaction to particular criteria as 0.6, and the degree of dissatisfaction is 0.7. Then, their sum exceeds one, but the square sum does not. So, FFSs can handle this. Thus, a FFSS is an effective parameterizing tool and an excellent medium to represent vagueness in many real-life situations. An SS is a parameterized family of sets that can be expanded into various hybrid structures such as FSSs, IFSSs, and PFSSs. The parameterized family of FFSs, the FFSS, also performs well because the FFS is highly competent in dealing with vagueness or uncertainty. In the case of FFSSs, an FFS is obtained for each parameter. FFSs can handle a variety of real-world circumstances that IFSs and PFSs cannot explain. Assume someone expresses his pleasure with specific criteria as 0.7 and his degree of discontent is 0.8. The amount then surpasses one, but not the square sum. As a result, FFSs can manage it. As a result, an FFSS is an effective parameterizing tool and an ideal medium for representing ambiguity in many real-world circumstances.

The concept of an FS proposed by Zadeh [14] was used to demonstrate the ambiguity and vagueness of a membership degree(MD). The IFS developed by Atanassov [12] can more fully explain evaluation information by linking an element's non-membership degree(ND) to an item. In light of the IFS mentioned above weakness, Yager [15] pioneered the PFS concept to increase the range of MD and ND so that $MD^2+ND^2 \leq 1$. As a result, PFS provides additional evaluation opportunities for professionals to voice their opinions on numerous objectives. As the decision-making environment becomes more complex, it becomes increasingly challenging for professionals to provide more credible evaluation information. The notions of IFS and PFS have been supported to impact the vagueness and ambiguity created by the complicated subjectivity of human cognition. The FFS was the first to broaden the area of information statements by including the cubic sum of MD and ND. As a result, FFS is a more efficient and practical strategy than IFS and PFS for dealing with indeterminacy of choice difficulties. Due to its advantages in displaying ambiguous information and providing additional possibilities for professionals, academics have pushed to create many DM systems to handle genuine DM and evaluation problems.

Senepati and Yager [16] are the creators of the FFS. FFSs explain uncertainties better than IFSs and PFSs. Senapati and Yager [17] expanded on this article by examining a range of novel operations and arithmetic mean techniques over FFSs. They used the FF-weighted product model to handle MCDM problems. FFS-related novel aggregation operators have been defined, and [18] has investigated their properties. Many studies on FFS have appeared in the literature([17]- [31]).

The SS defines a distinct scenario to address ambiguity and vagueness [13]. A set of features produces a family of subsets regarded as approximation definitions of a notion (one for each property-defined viewpoint). Many academics with diverse interests were rapidly drawn to soft sets([26], [32]- [40]). With the advancement of communication and technology, many complex topics require more than one analytical instrument. In this respect, Maji et al. [35] demonstrated that FS and SS theories can coexist. Many articles ([41]- [46]) studied these models further. Researchers expanded on this sort of hybridization ([40], [47]- [49]). Of course, the motivations for studying these generalizations of IFS sets are reasonable.

In FS theory, which has gotten much attention in recent years, the distance measure is useful for representing the difference between two FSs. Many authors have proposed various distance measurements for IFSs and PFSs in IFS and PFS theories. Szmidt and Kacprzyk [50] defined and explained various distance measures for IFSs using the geometric method. Several forms of distance and similarity measurements for FS, IFS, and PFSs have been introduced since the evolution of FS theory. Several scholars have recently focused on distance or similarity measurements, which are significant mathematical instruments in DM and pattern recognition problems([51]- [58]). There are also new FFSS studies in the literature([19]- [21], [27], [59], [60]).

In the FSs theory, Zadeh [61] was the first to mention entropy as a measure of fuzziness or ambiguous information. De Luca and Termini [62] defined the entropy of FSs using Shannon's function and gave the hypotheses that the fuzzy entropy must follow. According to information entropy, the volume and quality of accessible information are the most significant factors of the accuracy and reliability of the decision to be made in a DM situation [63]. Kaufmann [64] showed how to determine an FS's entropy by measuring the distance between the FS and the nearest crisp set. In contrast, Yager [65] measured the distance between the FS and its complement.

Higashi and Klir [66] expanded Yager's [65] approach to a fairly general class of fuzzy complements. Using entropy in DM processes increases the method's usefulness in uncertain contexts because entropy is crucial for gauging uncertain information. Mohagheghi et al. [67] used the idea of entropy to weigh the importance of each criterion. So, the criteria' relevance was addressed directly by the DMs and indirectly by computing a weight based on the ideas obtained. Peng et al. [68] provide axiomatic definitions of PF-information metrics such as entropy, distance measure, inclusion measure, and similarity measure.

1.3. Necessity

"Keep the certain, avoid the uncertain" instructions are familiar to everyone. A person tends to choose what he knows, even if it is terrible, because that way, he feels more secure. This makes people distant and anxious about the "new". The goal is to live safely in a clear and specific world. The serenity of foretelling what might happen to one makes the story, which promises pain, very appealing. Unexpected situations, social protests, the virus, the evolution of education, and natural disasters convince us that uncertainty is life itself. In other words, uncertainty is an inevitable human reality. However, man is not faced with uncertainties only outside himself. One's expressions, expectations, what one does, and what one wants to do always contain uncertainties. Therefore, people have to make decisions based on these uncertainties at all times and everywhere, from the work they do during the day to the work they plan to do in the future.

The cognitive continuum, which leads to choosing a faith or plan of action from a range of potential possibilities, is known as decision-making. It might be logical or illogical. Making a choice is a method of deliberation based on the decision-maker's (DMR) values, preferences, and beliefs. Every decision-making(DM) continuum ends with a final alternative that may or may not be followed by action. Data are used in DM to minimize or completely remove ambiguity. Decisions are regularly taken while it is unknown whether they will be beneficial or harmful. Due to time constraints, a lack of knowledge, carelessness, and their ability to comprehend information, decision-makers often need more clarity and accurate information to address problems.

IFS and PFS-based techniques, by definition, cannot capture data in FFS format. It is also evident that IFS and PFS-based decision-making systems cannot withstand scenarios in which experts supply preference values in FFSs. Approaches based on FFS, such as the recommended technique, effectively obtain and analyze information to rate available options based on predefined criterion values. The FFS environment manages more information and covers a broader range of themes for dealing with uncertain information since it includes both IFSs and PFSs into a single platform. As a result, more information is needed in this collection.

An SS is a bag that contains an approximate representation of the objects. It is made up of two parts: a predicate and an approximate value set. It states the object-related information more accurately and precisely. Traditional mathematics machinery fails because the beginning description is approximate; however, the SS can manage several challenges in this respect. As a result, it is an effective instrument for dealing with ambiguous and perplexing knowledge during the DM process. Fuzzy-type soft sets (FSS, IFSS, PFSS, and others) outperform all other mathematical tools and produce significantly better outcomes, particularly in decision-making procedures. Existing SSs from the IFS and PFS are considered a subset of the proposed SSs. Furthermore, the proposed SSs can handle more data than the current ones. As a result, the presented method contains significantly more information than existing methods for dealing with data uncertainties in the IFS and PFS contexts.

Since the Fermatean fuzzy soft set is an excellent tool to handle more than IFSS and PFSS, obtaining the entropy is also relevant. It is discovered that entropy can be used in DM problems, and hence, DM problems handle certain broader domains. Distance measures can be used to solve problems such as DM, pattern identification, and machine learning. Hamming distance and Euclidean distance are the most commonly used distance measurements. The measure of distance is inversely proportional to the measure of similarity. As a result, determining the similarity between sets is beneficial. Senapati and Yager [16] provide an FFS distance measure and illustrate with a numerical example that the proposed distance measures are realistic and appropriate. This definition of FFSs is expanded to define the FFSS distance measure-both the proposed entropy and distance measurements aid in adequately understanding real-life events. Based on the inspiration of the soft set structure and the benefits of FFSs in dealing with uncertain and imprecise information, this work investigates the theory of FFSSs by establishing some new information measures called entropy and distance measures.

1.4. Contribution

The core contributions of the present work can be expressed as:

(i) The article sets up a Fermatean fuzzy-type soft set. Further, the main structures of FFSSs are investigated.
(ii) New measures of distance and entropy based on FFSSs are provided to measure uncertain information.
(iii) The theoretical background of new measures of distance and entropy has been given in detail.
(iv) DM algorithms related to new distance and entropy measures have been given.
(v) The suggested techniques were corroborated by numerical examples related to medical DM ve PR.
(vi) The new method is compared with PFS elements to see the advantages.

The following are the benefits of this work:

(i) It can be used to determine the level of uncertainty associated with an FFSS.
(ii) It can be used to identify the similarity between any two FFSSs using the provided distance measures.
(iii) It is comparable to other existing structures in the literature.

Entropy and distance metrics in another type of generalized structure will be compared in future work. In addition, topological, algebraic, and order theoretical structures for FFSSs can be introduced and examined.

Structure: We propose the notion of a FFSS. The essential characteristics of FFSS, such as FFS-subsets, "AND" and "OR" operators, union, intersection, and complement, are investigated in Section 3. Section 4 focused on the measures of entropy and distance concerning FFSS. Novel methods for DM and PR problems are devised, and concrete examples are provided in Section 5.

2. Preliminaries

Let $Z=\{z_1,z_2,\ldots,z_n\}$ and $E=\{e_1,e_2,\ldots,e_m\}$ be the universal and the parameter sets, respectively.

Definition 2.1.
- *An fuzzy set FS is defined as $F=\{(z,m_F(z)):z\in Z\}$, where $m_F(z):Z\to[0,1]$ is called MD [14].*
- *An IFS F is defined as $F=\{(z,m_F(z),n_F(z)):z\in Z\}$ such that $m_F(z)+n_F(z)\leq 1$, where $m_F(z),n_F(z):Z\to[0,1]$ is called MD and ND, respectively [12].*
- *An PFS F is defined as $F=\{(z,m_F(z),n_F(z)):z\in Z\}$ such that $m_F^2(z)+n_F^2(z)\leq 1$, where $m_F(z),n_F(z):Z\to[0,1]$ [15].*

Definition 2.2 ([16])**.** *The set $F=\{<k,m_F(k),n_F(k)>: k\in Z\}$ is called FFS, where $0\leq m_F^3(k)+n_F^3(k)\leq 1$ and $m_F,n_F:Z\to[0,1]$.*

The hesitancy degree (HD) of F is $h_F(k)=\sqrt[3]{1-(m_F^3(k)+n_F^3(k))}$.
For FFSs $F=(m_F,n_F)$, $F_1=(m_{F_1},n_{F_1})$ and $F_2=(m_{F_2},n_{F_2})$, [16]:

(i) $F_1\cap F_2=[\min(m_{F_1},m_{F_2}),\max(n_{F_1},n_{F_2})]$;
(ii) $F_1\cup F_2=[\max(m_{F_1},m_{F_2}),\min(n_{F_1},n_{F_2})]$;
(iii) $F^t=(n_F,m_F)$;
(iv) $F_1\boxplus F_2=\left(\sqrt[3]{m_{F_1}^3+m_{F_2}^3-m_{F_1}^3m_{F_2}^3},n_{F_1}n_{F_2}\right)$;
(v) $F_1\boxtimes F_2=\left(m_{F_1}m_{F_2},\sqrt[3]{n_{F_1}^3+n_{F_2}^3-n_{F_1}^3n_{F_2}^3}\right)$;
(vi) $\alpha F=\left(\sqrt[3]{1-(1-m_F^3)^\alpha},n_F^\alpha\right)$;
(vii) $F^\alpha=\left(m_{F_1}^3,\sqrt[3]{1-(1-n_F^3)^\alpha}\right)$.

The properties of complement of FFS [16]:

(i) $(F_1\cap F_2)^c=F_1^c\cup F_2^c$;
(ii) $(F_1\cup F_2)^c=F_1^c\cap F_2^c$;
(iii) $(F_1\boxplus F_2)^c=F_1^c\boxtimes F_2^c$;
(iv) $(F_1\boxtimes F_2)^c=F_1^c\boxplus F_2^c$;
(v) $\alpha(F)^c=(F^\alpha)^c$;
(vi) $(F^c)^\alpha=(\alpha F)^c$.

Proposition 2.3 ([16])**.** *Let three FSSs F,G,H. Then,*

(i) If $F\subseteq G$ and $G\subseteq H$, then $F\subseteq H$;
(ii) $(F^c)^c=F$;
(iii) The properties commutative, associative, and distributive are applied for the union and intersection;
(iv) Union and intersection provide DeMorgan's laws.

Definition 2.4. *The soft sets (SS) are a parameterized family of subsets of Z. That is, for the function $F:E\to SS(Z)$, (F,E) is denoted a SS, where $SS(Z)$ is a set of all subsets of Z.*

According to Definition 2.4, if $SS(Z)$ is selected as Z's F-, IF-, and PF-subsets, then (F,N) will be defined as fuzzy soft set (FSS) [35], intuitionistic fuzzy soft set (IFSS) [36] and Pythagorean fuzzy soft set (PFSS) [40], respectively. The definitions of FSS, IFSS, and PFSS are given as follows:

Definition 2.5. • *The pair (F,E) is called FFS, if the function $F : E \to FS(Z)$ is a mapping from E into set of all fuzzy sets in Z, where if $FS(Z)$ is a set of all subsets of Z [35].*
- *The pair (F,E) is called IFFS, if the function $F : E \to IS(Z)$ is a mapping from E into set of all intuitionistic fuzzy power sets in Z, where if $IS(Z)$ is a set of all subsets of Z [36].*
- *The pair (F,E) is called PFFS, if the function $F : E \to PS(Z)$ is a mapping from E into set of all Pythagorean fuzzy sets in Z, where if $PS(Z)$ is a set of all subsets of Z [40].*

Definition 2.6. *Let d be a mapping $d : IFSS(Z) \times IFSS(Z) \to \mathbb{R}^+ \bigcup \{0\}$, where $\mathbb{R}^+ \bigcup \{0\}$ denotes the set of non-negative real numbers. For two $IFSS(Z)$ A, B, if $d(A,B)$ satisfies the following properties:*

- *$d(A,B) \geq 0$;*
- *$d(A,B) = d(B,A)$;*
- *$d(A,B) = 0$ if and only if $A = B$;*
- *For any $C \in IFSS(Z)$, $d(A,B) + d(B,C) \geq d(A,C)$.*

Then $d(A,B)$ is a distance measure between IFSSs A and B [69].

If the sets A and B in Definition 2.6 are taken as PFSSs and $d : PFSS(Z) \times PFSS(Z) \to \mathbb{R}^+ \bigcup \{0\}$, then the d transformation is called the "distance measure between PFSSs A and B" [32].

Definition 2.7. *A real function $T : IFFS(Z) \to \mathbb{R}^+$ is called a intuitionistic fuzzy soft entropy(IFSE) on $IFFS(Z)$ [69], if T has following properties;*

- *$T(p) = 0$ if and only if $p \in S(Z)$.*
- *Let $p = (F,E) = [a_{ij}]_{m \times n}$, $T(p) = mn$ if and only if $m_{F(e)}(z) = 0 = n_{F(e)}(z)$, $\forall e \in E, \forall z \in Z$.*
- *$T(p) = T(p^c)$ $p \in IFSS(Z)$.*
- *If $p \preceq \bar{p}$, then $T(p) \geq T(\bar{p})$ where $(F,T) = p$ and $(G,T) = \bar{p}$.*

If $IFSS(Z)$ in Definition 2.7 is taken as $PFSS(Z)$, then the T transformation is called a Pythagorean fuzzy soft entropy(PFSE) on $PFFS(Z)$ [32].

3. Fermatean Fuzzy Soft Sets

Definition 3.1. *For $M \subseteq E$, the FFSSs is defined as the pair (F,M) where $F : E \to FFS(Z)$ and $FFS(Z)$ is the set of all Fermatean fuzzy subsets of Z.*

For any parameter $e \in E$, $F(e)$ can be wirtten as a FFS such that

$$F(e) = \{(z, m_{F(e)}(z), n_{F(e)}(z)) : z \in Z\}$$

where $m_{F(e)}(z)$ and $n_{F(e)}(z)$ are the MD and ND with condition $m^3_{F(e)}(z) + n^3_{F(e)}(z) \leq 1$. Further, $h_{F(e)}(z) = \sqrt[3]{1 - (m_{F(e)}(z))^3 - (n_{F(e)}(z))^3}$.

Example 3.2. *The diseases set $Z = \{z_1, z_2, z_3\}$ and the symptoms set $M = \{e_1 = symptom1, e_2 = symptom2, e_3 = symptom3\}$. Hence*

$$F(e_1) = \{< z_1, 0.6, 0.9) >, < z_2, 0.8, 0.7) >, < z_3, 0.8, 0.9) >\}$$
$$F(e_2) = \{< z_1, 0.7, 0.9) >, < z_2, 0.9, 0.5) >, < z_3, 0.8, 0.8) >\}$$
$$F(e_3) = \{< z_1, 0.8, 0.7) >, < z_2, 0.8, 0.9) >, < z_3, 0.9, 0.6) >\}$$

and table representation as follows (Table 1):

	z_1	z_2	z_3
e_1	(0.6, 0.9)	(0.8, 0.7)	(0.8, 0.9)
e_2	(0.7, 0.9)	(0.9, 0.5)	(0.8, 0.8)
e_3	(0.8, 0.7)	(0.8, 0.9)	(0.9, 0.6)

Table 1: (F,M)

Definition 3.3. *Let $M,N \subset E$, and $(F,M),(G,N)$ be two FFSS (F,M) is called a FF soft subset of (G,N) (denoted by $(G,N)\hat{\subset}(F,M)$) if*

(i) $M \subseteq N$,
(ii) *For all $z \in Z$, $e \in M$, $m_M(z) \geq m_N(z)$ and $n_M(z) \leq n_N(z)$.*

Example 3.4. *Let $M = \{e_1 = symptom1\} \subset E$. Hence, we can written FFSS (G,N) as:*

$$G(e_1) = \{< z_1, 0.6, 0.8 >, < z_2, 0.6, 0.8) >, < z_3, 0.7, 0.9) >\}$$

This shows us that $(G,N)\hat{\subset}(F,M)$.

Definition 3.5. *Choose the two FFSS $(F,M),(F,N)$.*

(i) $(F,N)\hat{=}(F,M)$, *if $(G,N)\hat{\subset}(F,M)$ and $(F,M)\hat{\subset}(G,N)$.*
(ii) *The complement of (f,M) is identified $(F,M)^c$, where $F^c : M \to FFSS(Z)$ and $F^c(e) = (F(e))^c$ for every $e \in M$.*

Further, $((F,M)^c)^c = (F,M)$.

Example 3.6. *From Example 3.4. If*

$$G(e_1) = \{<z_1, 0.6, 0.8>, <z_2, 0.5, 0.7>, <z_3, 0.7, 0.6>\},$$

then

$$G^c(e_1) = \{<z_1, 0.8, 0.6>, <z_2, 0.7, 0.5>, <z_3, 0.6, 0.7>\}.$$

Definition 3.7. *Choose the two FFSS $(F,M),(G,N)$.*

- ***AND Operator:*** *(F,M) AND (G,N) is FFSS denoted by $(F,M)\wedge(G,N)$ is defined by $(F,M)\wedge(G,N) = (H, A\times B)$ where $H(\alpha,\beta) = F(\alpha)\cap G(\beta)$, $\forall \alpha,\beta \in A\times B$. That is, $H(\alpha,\beta)(z) = (z, \min\{m_{F(\alpha)(z)}, m_{G(\beta)(z)}\}, \max\{n_{F(\alpha)(z)}, n_{G(\beta)(z)}\})$, $\forall \alpha,\beta \in A\times B$ and $\forall z \in Z$.*
- ***OR Operator:*** *(F,M) OR (G,N) is FFSS denoted by $(F,M)\vee(G,N)$ is defined by $(F,M)\vee(G,N) = (H, A\times B)$ where $H(\alpha,\beta) = F(\alpha)\cup G(\beta)$, $\forall \alpha,\beta \in A\times B$. That is, $H(\alpha,\beta)(z) = (z, \max\{m_{F(\alpha)(z)}, m_{G(\beta)(z)}\}, \min\{n_{F(\alpha)(z)}, n_{G(\beta)(z)}\})$, $\forall \alpha,\beta \in A\times B$ and $\forall z \in Z$.*

Example 3.8. *Choose $N = \{e_1, e_2\}$. Then, FFSS (G,N) as:*

$$G(e_1) = \{<z_1, 0.6, 0.8>, <z_2, 0.4, 0.8>, <z_3, 0.8, 0.5>\},$$
$$G(e_2) = \{<z_1, 0.6, 0.7>, <z_2, 0.6, 0.4>, <z_3, 0.9, 0.4>\}.$$

It is seen that $(F,M)\hat{\subset}(G,N)$. AND and OR operations are shown by Tables 2 and 3.

	(e_1,e_1)	(e_1,e_2)	(e_2,e_1)	(e_2,e_2)	(e_3,e_1)	(e_3,e_2)
z_1	(0.6, 0.9)	(0.6, 0.9)	(0.6, 0.8)	(0.6, 0.7)	(0.6, 0.9)	(0.4, 0.9)
z_2	(0.4, 0.9)	(0.6, 0.9)	(0.4, 0.8)	(0.6, 0.5)	(0.4, 0.8)	(0.6, 0.8)
z_3	(0.8, 0.7)	(0.8, 0.7)	(0.8, 0.9)	(0.8, 0.9)	(0.8, 0.6)	(0.9, 0.6)

Table 2: $(F,M)\wedge(G,N)$

	(e_1,e_1)	(e_1,e_2)	(e_2,e_1)	(e_2,e_2)	(e_3,e_1)	(e_3,e_2)
z_1	(0.6, 0.8)	(0.6, 0.7)	(0.8, 0.7)	(0.8, 0.7)	(0.8, 0.8)	(0.8, 0.7)
z_2	(0.7, 0.8)	(0.7, 0.4)	(0.9, 0.5)	(0.9, 0.4)	(0.8, 0.8)	(0.8, 0.4)
z_3	(0.8, 0.5)	(0.9, 0.4)	(0.8, 0.5)	(0.9, 0.4)	(0.9, 0.5)	(0.9, 0.4)

Table 3: $(F,M)\vee(G,N)$

Theorem 3.9. *For two FFSSs (F,M) and (G,N),*

- *(i.) $((F,M)\wedge(G,N))^c = (F,M)^c \vee (G,N))^c$*
- *(ii.) $((F,M)\vee(G,N))^c = (F,M))^c \wedge (G,N))^c$.*

Proof. (i) First, take $(F,M)\wedge(G,N) = (H, M\times N)$, where $H(\alpha,\beta) = F(\alpha)\cap G(\beta)$, $\forall \alpha,\beta \in M\times N$. That is,

$$H(\alpha,\beta)) = (z, \min\{m_{F(\alpha)}(z), m_{G(\beta)}(z)\}, \max\{n_{F(\alpha)}(z), n_{G(\beta)}(z)\}), \quad \text{for all} \quad (\alpha,\beta)\in M\times N \quad \text{and} \quad z\in Z.$$

Second, $((F,M)\wedge(G,N))^c = (H, M\times N)^c = (H^c, M\times N)$. That is, $\forall \alpha,\beta \in M\times N$ and for all $z\in Z$,

$$H^c(\alpha,\beta)) = (z, \max\{n_{G(\alpha)}, n_{G(\beta)}\}, \min\{m_{F(\alpha)}, m_{F(\beta)}\}). \tag{3.1}$$

Let $(F,M)^c \vee (G,N)^c = (F^c, M)\vee(G^c, N) = (I, (M\times N))$, where $I(\alpha,\beta)) = F^c(\alpha)\cup G^c(\beta)$, $\forall(\alpha,\beta)\in M\times N$. So, for $z\in Z$, we get

$$\begin{aligned} I(\alpha,\beta) &= (z, \max\{m_{F^c(\alpha)}(z), m_{G^c(\beta)}(z)\}, \min\{n_{F^c(\alpha)}(z), n_{G^c(\beta)}(z)\}) \\ &= (z, \max\{n_{F(\alpha)}(z), n_{G(\beta)}(z)\}, \min\{m_{F(\alpha)}(z), m_{G(\beta)}(z)\}) \end{aligned} \tag{3.2}$$

We obtain $((F,M)\wedge(G,N))^c = (F,M)^c \vee (G,N)^c$, from (3.1) and (3.2).

(ii) can be proved similarly to (i).

□

Definition 3.10. *For two FFSSs (F,M) and (G,N), the union (H,P) of (F,M) and (G,N) $(F,M)\cup(G,N) = (H,P)$, is described as*

$$H(e) = \begin{cases} F(e) & , \quad e\in M/N \\ G(e) & , \quad e\in N/M \\ F(e)\cup G(e) & , \quad e\in M\cap N. \end{cases}$$

if $P = M\cup N$ and for all $e\in P$. So, for all $e\in M\cap N$, we have $F(e)\cup G(e) = (z, \max(m_{F(e)}(z), m_{G(e)}(z)), \min(n_{F(e)}(z), n_{G(e)}(z)) >: z\in Z$.

Theorem 3.11. *The union (H,P) is a FFSS.*

Proof. Using Definition 3.10, $\forall e \in P$ if $e \in M/N$ or $e \in N/M$, then $H(e) = F(e)$ or $H(e) = G(e)$. Therefore, $H(e)$ is FFSS.

If $e \in M \cap N$, for a fixed $z \in Z$, consider $m_{F(e)}(z) \leq m_{G(e)}(z)$, then,

$$\begin{aligned} m^3_{H(e)}(z) + n^3_{H(e)}(z) &= (m^3_{F(e)}(z) \vee m^3_{G(e)}(z)) + (n^3_{F(e)}(z) \wedge n^3_{G(e)}(z)) \\ &= m^3_{F(e)}(z) + (n^3_{F(e)}(z) \wedge n^3_{G(e)}(z)) \\ &\leq m^3_{G(e)}(z) + n^3_{G(e)}(z) \leq 1. \end{aligned}$$

Then, union (H,P) is a FFSS. □

Definition 3.12. *For two FFSSs (F,M) and (G,N), the union (H,P) of (F,M) and (G,N) $(F,M) \cup (G,N) = (H,P)$, is described as*

$$H(e) = \begin{cases} F(e) & , \quad e \in M/N \\ G(e) & , \quad e \in N/M \\ F(e) \cap G(e) & , \quad e \in M \cap N. \end{cases}$$

if $P = M \cup N$ and for all $e \in P$. So, for all $e \in M \cap N$, we have $F(e) \cap G(e) = (z, \min(m_{F(e)}(z), m_{G(e)}(z)), \max(n_{F(e)}(z), n_{G(e)}(z))) : z \in Z$.

Theorem 3.13. *The intersection (H,P) is a FFSS.*

Proof. Using Definition 3.12, $\forall e \in P$ if $e \in M/N$ or $e \in N/M$, then $H(e) = F(e)$ or $H(e) = G(e)$. Therefore, $H(e)$ is FFSS.

If $e \in M \cap N$, for a fixed $z \in Z$, consider $n_{F(e)}(z) \leq n_{G(e)}(z)$, then, we have,

$$\begin{aligned} m^3_{H(e)}(z) + n^3_{H(e)}(z) &= (m^3_{F(e)}(z) \wedge m^3_{G(e)}(z)) + (n^3_{F(e)}(z) \vee n^3_{G(e)}(z)) \\ &= (m^3_{F(e)}(z) \wedge m^3_{G(e)}(z)) + (n^3_{F(e)}(z) \wedge n^3_{G(e)}(z)) \\ &\leq m^3_{F(e)}(z) + n^3_{G(e)}(z) \leq 1. \end{aligned}$$

Hence, intersection (H,P) is a FFSS. □

Theorem 3.14. *Let (F,M), (G,N) and (H,P) be three FFSSs.*

(i) $(F,M) \cup (F,M) = (F,M)$
(ii) $(F,M) \cap (F,M) = (F,M)$
(iii) $(F,M) \cup (G\,N) = (G,N) \cup (F,M)$
(iv) $(F,M) \cap (G,N) = (G,N) \cap (F,M)$
(v) $((F,M) \cup (G,N)) \cup (H,P) = (F,M) \cup ((G,N) \cup (H,P))$
(vi) $((F,M) \cap (G,N)) \cap (H,P) = (F,M) \cap ((G,N) \cap (H,P))$.

Proof. The proof is obtained by Proposition (2.3), Definitions (3.10) and (3.12). □

Theorem 3.15. *Let (F,M) and (G,N) be two FFSSs.*

(i) $((F,M) \cap (G,N))^c = (F,M)^c \cup (G,N)^c$
(ii) $((F,M) \cup (G,N))^c = (F,M))^c \cap (G,N))^c$

Proof. If we take $P = M \cup N$ and $e \in P$, then $(F,M) \cap (G,N) = (H,P)$,

$$H(e) = \begin{cases} F(e) & , \quad e \in M/N \\ G(e) & , \quad e \in N/M \\ F(e) \cap G(e) & , \quad e \in M \cap N. \end{cases}$$

So, for all $e \in M \cap N$, we have $(F(e) \cap G(e) = (z, \min(m_{F(e)}(z), m_{G(e)}(z)), \max(n_{F(e)}(z), n_{G(e)}(z)) : z \in Z)$. So that, $((F,M) \cap (G,N))^t = (H,P))^c$ and $H^c(e) = (H(e))^c$. Then,

$$(H(e))^c = \begin{cases} (F(e))^c & , \quad e \in M/N \\ (G(e))^c & , \quad e \in N/M \\ (F(e) \cap G(e))^c & , \quad e \in M \cap N. \end{cases}$$

That is, $\forall e \in M \cap N$, we get

$$\begin{aligned} (F(e) \cap G(e))^c &= (z, \min(m_{F(e)}(z), m_{G(e)}(z)), \max(n_{F(e)}(z), n_{G(e)}(z)) : z \in Z >^c \\ &= (z, \max(n_{F^c(e)}(z), n_{G^c(e)}(z)), \min(m_{F^c(e)}(z), m_{G^c(e)}(z)) : z \in Z). \end{aligned}$$

Now, $(F,M))^c = (F^c, M)$ and $(G,N))^c = (G^c, N)$. So that $(F,M)^c \cup (G,N)^c = (F^c, M) \cup (G^c, N) = (H^c, P)$, where $P = M \cup N$ and

$$H^c(e) = \begin{cases} F^c(e) & , \quad e \in M/N \\ G^c(e) & , \quad e \in N/M \\ F(e)^c \cap G^c(e) & , \quad e \in M \cap N. \end{cases}$$

So, for all $e \in M \cap N$, $F^c(e) \cup G^c(e) = (k, \max(n_{F^c(e)}(k), n_{G^c(e)}(k)), \min(m_{F^c(e)}(k), m_{G^c(e)}(k)) : k \in Z)$. Hence, $((F,M) \cap (G,N))^c = (F,M)^c \cup (G,N)^c$. □

Example 3.16. *Take the diseases set* $Z = \{z_1, z_2, z_3, z_4\} = \{disease1, disease2, disease3, disease4\}$. *Select* $E = \{e_1, e_2, e_3, e_4, e_5\} = \{symptom1, symptom2, symptom3, symptom4, symptom5\}$ *as parameter set. Consider that* (F,M), $(F,M)^c$, (G,N), *and* (H,P) *are four FFSSs over* Z *given by* $M = \{e_1, e_2\}$, $N = \{e_1, e_2, e_4\}$ *and* $P = \{e_1, e_3, e_4\}$ *defined as follows (Tables (4)-(7)) :*

	e_1	e_2
z_1	(0.64, 0.88)	(0.81, 0.72)
z_2	(0.73, 0.79)	(0.94, 0.53)
z_3	(0.85, 0.59)	(0.92, 0.49)
z_4	(0.83, 0.67)	(0.67, 0.85)

Table 4: (F,M)

	e_1	e_2
z_1	(0.88, 0.64)	(0.72, 0.81)
z_2	(0.79, 0.73)	(0.53, 0.94)
z_3	(0.59, 0.85)	(0.49, 0.92)
z_4	(0.67, 0.83)	(0.85, 0.67)

Table 5: (F^c, M)

Table 6: (G,N)

	e_1	e_2	e_4
z_1	(0.82, 0.73)	(0.92, 0.57)	(0.85, 0.67)
z_2	(0.66, 0.78)	(0.75, 0.62)	(0.54, 0.91)
z_3	(0.84, 0.49)	(0.72, 0.39)	(0.71, 0.81)
z_4	(0.43, 0.87)	(0.67, 0.59)	(0.76, 0.37)

	e_1	e_3	e_4
z_1	(0.44, 0.95)	(0.57, 0.69)	(0.86, 0.59)
z_2	(0.56, 0.81)	(0.68, 0.69)	(0.79, 0.38)
z_3	(0.68, 0.56)	(0.92, 0.35)	(0.72, 0.65)
z_4	(0.63, 0.76)	(0.84, 0.37)	(0.95, 0.29)

Table 7: (H,P)

For the four FFSSs, the operations are in Tables 8 -11:

	e_1	e_2	e_3	e_4
z_1	(0.64, 0.88)	(0.81, 0.72)	(0.57, 0.69)	(0.86, 0.59)
z_2	(0.73, 0.81)	(0.94, 0.53)	(0.68, 0.69)	(0.79, 0.38)
z_3	(0.85, 0.56)	(0.92, 0.49)	(0.92, 0.35)	(0.72, 0.65)
z_4	(0.83, 0.67)	(0.67, 0.85)	(0.84, 0.37)	(0.95, 0.29)

Table 8: $(F,M) \cup (H,P)$

	e_1	e_2	e_3	e_4
z_1	(0.44, 0.95)	(0.81, 0.72)	(0.57, 0.69)	(0.86, 0.59)
z_2	(0.56, 0.81)	(0.94, 0.53)	(0.68, 0.69)	(0.79, 0.38)
z_3	(0.68, 0.59)	(0.92, 0.49)	(0.92, 0.35)	(0.72, 0.65)
z_4	(0.63, 0.76)	(0.67, 0.85)	(0.84, 0.37)	(0.95, 0.29)

Table 9: $(F,M) \cap (H,P)$

	(e_1,e_1)	(e_1,e_2)	(e_1,e_4)	(e_2,e_1)	(e_2,e_2)	(e_2,e_4)
z_1	(0.64, 0.88)	(0.64, 0.88)	(0.64, 0.88)	(0.81, 0.73)	(0.81, 0.72)	(0.81, 0.72)
z_2	(0.66, 0.79)	(0.73, 0.79)	(0.54, 0.91)	(0.66, 0.78)	(0.75, 0.62)	(0.54, 0.91)
z_3	(0.84, 0.59)	(0.72, 0.59)	(0.71, 0.81)	(0.84, 0.49)	(0.72, 0.49)	(0.71, 0.81)
z_4	(0.43, 0.87)	(0.67, 0.67)	(0.76, 0.67)	(0.43, 0.87)	(0.67, 0.85)	(0.67, 0.85)

Table 10: $(F,M)\wedge(G,N)$

	(e_1,e_1)	(e_1,e_2)	(e_1,e_4)	(e_2,e_1)	(e_2,e_2)	(e_2,e_4)
z_1	(0.82, 0.73)	(0.92, 0.57)	(0.85, 0.67)	(0.82, 0.72)	(0.92, 0.57)	(0.85, 0.67)
z_2	(0.73, 0.78)	(0.75, 0.62)	(0.73, 0.79)	(0.94, 0.53)	(0.94, 0.53)	(0.94, 0.53)
z_3	(0.85, 0.49)	(0.85, 0.39)	(0.85, 0.59)	(0.92, 0.49)	(0.92, 0.39)	(0.92, 0.49)
z_4	(0.83, 0.67)	(0.83, 0.59)	(0.83, 0.37)	(0.67, 0.85)	(0.67, 0.59)	(0.76, 0.37)

Table 11: $(F,M)\vee(G,N)$

4. Fermatean Fuzzy Measures

A crucial technique for quantifying uncertain information is entropy. One can quickly determine whether information is more stable if the entropy is lower because lower entropies also mean lower levels of uncertainty. Due to its greater generalization, the FFSS can represent information where other structures cannot. Therefore, introducing the measure of entropy is crucial in the current situation. The equations for entropy and distance measure for FFSSs are obtained and demonstrated with samples in this section by introducing various concepts and results.

4.1. Entropy meausre

Definition 4.1. *Take the two FFSSs (F,M) and (G,N). For all $z\in Z$ and $e\in E$, $m_{F(e)}(z)\leq m_{G(e)}(z)$ and $n_{F(e)}(z)\leq n_{G(e)}(z)$, $(F,M)\preceq(G,N)$ means that (F,M) is less than or equal to (G,N).*

The following definition is about a mapping that maps every FFSS to an FSS. It is also shown that the collection of images of FFSSs with $x\in[0,1]$ and with the relation $\subseteq$ is a totally ordered family of FSSs.

Definition 4.2. *For $x\in[0,1]$, the function $f_x:FFSS(Z)\to FSS(Z)$ is described as $f_x((F,E))=(F_xE)$, for each FFSS (F,E) with MV $m_{F(e)}$ and NV $n_{F(e)}$ and $F_x(e)=f_x(F_e)$ and,*

$$f_x(F_e)=(z,m^3_{F(\rho)}(z)+x.h^3_{F(e)}(z),1-m^3_{F(e)}(z)-x.h^3_{F(e)}(z):z\in Z). \tag{4.1}$$

As a result, every FFSS is given an FSS by the map f_x. A modification of [70] is the f_x. In contrast to the f_x described in [70], which is to assign an FFSS to an FS, the operator f_x is assigned to an FFSS to an FSS.

Example 4.3. *Take* $(F,E)=[a_{ij}]=\begin{pmatrix}(0.8,0.7) & (0.7,0.4)\\ (0.5,0.8 & (0.9,0.6)\end{pmatrix}$. *Choose $x=0.8$. Hence,*

$$F_x(e_1)=f_x[(k_1,0.8,0.7),(k_2,0.7,0.4)]=\{(k_1,0.628,0.372),(k_2,0.8174,0.1826)\}$$
$$F_x(e_2)=f_x[(k_1,0.5,0.8),(k_2,0.9,0.6)]=\{(k_1,0.4154,0.5846),(k_2,0.773,0.227)\}$$

Therefore, FSS is symbolized by the matrix $\begin{pmatrix}(0.628,0.372) & (0.8174,0.1826)\\ (0.4154,0.5846 & (0.773,0.227)\end{pmatrix}$

Theorem 4.4. *Let $\rho,\bar{\rho}\in FFSS(Z)$ and $x,y\in[0,1]$. Then,*

(i) If $x\leq y\Rightarrow f_x(\rho)\subset f_y(\rho)$.
(ii) If $\rho\subset\bar{\rho}\Rightarrow f_x(\rho)\subset f_x(\bar{\rho})$.
(iii) $f_x(f_y(\rho))=f_y(\rho)$
(iv) $(f_x(\rho^c))^c=f_{1-x}(\rho)$.

Proof. For $\forall e\in E$, take

$$\rho=(F,E),\quad F(e)=\{(z,m_{F(e)}(z),n_{F(e)}(z)):z\in Z\},$$
$$\bar{\rho}=(G,E),\quad G(e)=\{(z,m_{G(e)}(z),n_{G(e)}(z)):z\in Z\},$$

and $f_x(\rho)=(F_x,E)$, where

$$f_x(e)=(z,m^3_{F(e)}(z)+x.h^3_{F(e)}(z),1-m^3_{F(e)}(z)-x.h^3_{F(e)}(z):z\in Z).$$

(i) For $x \leq y$, for all $z \in Z$ and $e \in E$,

$$m^3_{F(e)}(z) + x.h^3_{F(e)}(z) \leq m^3_{F(e)}(z) + x.h^3_{F(e)}(z).$$

Thus,

$$m_{F_x(e)}(z) \leq m_{F_y(e)}(z)$$

for all $z \in Z$ and $e \in E$. Hence, $f_x(\rho) \subset f_y(\rho)$.

(ii) Take $\rho \subset \bar{\rho}$. Therefore,

$$m_{F(e)}(z) \leq m_{G(e)(z)} \quad and \quad n_{F(e)}(z) \geq n_{G(e)}(z), \quad for\ all z \in Z, e \in E.$$

Then,

$$\begin{aligned} m_{F_x(e)}(z) &= m^3_{F(e)}(z) + x.h^3_{F(e)}(z) \\ &= m^3_{F(e)}(z) + x.(1 - m^3_{F(e)}(z) - n^3_{F(e)}(z)) \\ &= m^3_{F(e)}(z)(1-x) + x - x.n^3_{F(e)}(z) \\ &\leq m^3_{G(e)}(z)(1-x) + x - x.n^3_{G(e)}(z) \\ &= m^3_{G(e)}(z) + x.h^3_{G(e)}(z) = m_{G_x(e)}(z). \end{aligned}$$

Hence, $m_{F_x(e)}(z) \leq m_{G_x(e)}(z)$ and $f_x(\rho) \subset f_y(\rho)$.

(iii) Let $f_x(f_y(F,E)) = f_x(F_y,E) = ((F_y)_x,E)$ where $((F_y)_x(e) = f_x(F_y(e)) = f_x(f_y(F(e)))$ for all $e \in E$. It will be shown as $f_x(f_y(F(e))) = f_y(F(e))$. Since

$$((f_y)(F(e)) = \{(z, m^3_{F(e)}(z) + y.h^3_{F(e)}(z), 1 - m^3_{F(e)}(z) - y.h^3_{F(e)}(z) >: z \in Z\},$$

any $e \in E$, we get

$$\begin{aligned} f_x[f_y(F(e))] =& f_x(\{(z, m^3_{F(e)}(z) + y.h^3_{F(e)}(z), 1 - m^3_{F(e)}(z) - y.h^3_{F(e)}(z) >: z \in Z\}) \\ =& \{(z, (m^3_{F(e)}(z) + y.h^3_{F(e)}(z)) + xa.[1 - (m^3_{F(e)}(z) + ya.h^3_{F(e)}(z)) \\ & - (1 - m^3_{F(e)}(z) - y.\pi^3_{F(e)}(z))], 1 - [(m^3_{F(e)}(z) + y.\pi^3_{F(e)}(z)) \\ & + \alpha.(1 - (m^3_{F(e)}(z) + y.h^3_{F(e)}(z)) - (1 - m^3_{F(e)}(z) - y.h^3_{F(e)}(z)))]) : z \in Z\} \\ =& \{(z, m^3_{F(e)}(z) + y.h^3_{F(e)}(z), 1 - m^3_{F(e)}(z) - y.h^3_{F(e)}(z)) : z \in Z\} = f_y(F(e)) \end{aligned}$$

(iv) For all $e \in \neg E$, Take

$$\rho^c = (F,E)^c = (F^c, \neg E) = \{(z, n_{F(e)}(z), m_{F(e)}(z)) : z \in Z\}, \quad f_x(\rho^c) = f_x(F^c, \neg E) = ((F^c)_x, \neg E)$$

where

$$(F^c(x))(e) = \{(z, n^3_{F(\neg e)}(z) + x.h^3_{F(\neg e)}(z), 1 - n^3_{F(\neg e)}(z) - x.h^3_{F(\neg e)}(z) : z \in Z\}$$

$$(F^c_x)^c(e) = \{(z, 1 - n^3_{F(e)}(z) - x.h^3_{F(e)}(z)), n^3_{F(e)}(z) + x.h^3_{F(e)}(z) : z \in Z\}$$

for all $e \in E$.

$$\begin{aligned} f_{1-x}(\rho) &= f_{(1-x)}((F,E)) = f_{1-x}(\{(z, m_{F(e)}(z), n_{F(e)}(z)) : z \in Z\}) \\ &= (m^3_{F(e)}(z) + (1-x)h^3_{F(e)}(z), 1 - m^3_{F(e)}(z) - (1-x)h^3_{F(e)}(z), \\ & 1 - n^3_{F(e)}(z) - x.(1 - m^3_{F(e)}(z) - n^3_{F(e)}(z)), n^3_{F(e)}(z) + x.h^3_{F(e)}(z)) \\ &= F^3_{x^c}(e). \end{aligned}$$

Thus, $(f_x(\rho^c))^c = f_{1-x}(\rho)$.

□

Definition 4.5. *If the properties are satisfies, a real mapping $T : FFSS(Z) \to \mathbb{R}^+$ is called a FFSE on FFSS(Z):*

(i) $T(\rho) = 0 \Leftrightarrow \rho \in FSS(Z)$
(ii) Let $\rho = (F,T) = [a_{ij}]_{m\times n}$, for all $z \in Z$, $T(\rho) = mn \Leftrightarrow m_{F(e)}(z) = n_{F(e)}(z) = 0$, for all $e \in T$.
(iii) $T(\rho) = T(\rho^c)$, $\rho \in FFSS(Z)$.
(iv) For $(F,T) = \rho$ and $(G,T) = \bar{\rho}$, if $\rho \preceq \bar{\rho} \Rightarrow T(\rho) \geq T(\bar{\rho})$.

From the definition, entropy is minimum(zero) when the FFSS degenerates into SS. The following theorem discusses the case when the entropy is maximum.

Theorem 4.6. *FFSE ρ is maximum $\Leftrightarrow \rho=(F,T)=[a_{ij}]_{m\times n}=[0]_{m\times n}$. So, $m_{F(e_j)}(z_i)=n_{F(e_j)}(z_i)=0$, for all $e_j\in T$, $z_i\in Z$ where $i\in\{0,1,\cdots,m\}$ and $j\in\{0,1,\cdots,n\}$ and $\rho\in FFSS(Z)$.*

Proof. Take $\rho=(F,T)=[0]m\times n$. Choose $\bar{\rho}=(G,T)$ be any FFSS. Since $m_{F(e_j)}(z_i)\geq 0$ and $n_{F(e_j)}(z_i)\leq 0$ for all $e_j\in T, z_i\in Z$, where $i\in\{0,1,\cdots,m\}$ and $j\in\{0,1,\cdots,n\}$, by Definition 4.1, $\rho\preceq\bar{\rho}$. Therefore, from Definition 4.5 $T(\rho)\geq T(\bar{\rho})$ for all $\bar{\rho}$. Then, $T(\rho)$ is maximum.

Moreover, let $T(\rho)$ be the maximum. If we take $\rho=(F,T)\neq[0]_{m\times n}$, then there exist $e_j\in T$ and $z_i\in Z$ such that $m_{F(e_j)}(z_i)\neq 0$ or $n_{F(e_j)}(z_i)\neq 0$. Build the FFSS as $\bar{\rho}=(G,T)$ with $m_{G(e_j)}(z_i)=m_{F(e_j)}(z_i)/2$ and $n_{G(e_j)}(z_i)=n_{F(e_j)}(z_i)/2$ for all $e_j\in T$ and $z_i\in F$. Hence, using the Definition 4.1, $\rho\leq\bar{\rho}$. Thus $T(\bar{\rho})\geq T(\rho)$ is obtained. This is a contradiction. Therefore, $\rho=[0]_{m\times n}$. □

The objective is to provide a statement enabling entropy generation for FFSSs. The method is the same as the one used to calculate concrete entropies for FFSSs: Let's build $\Lambda_{\mathscr{K}}:\mathscr{K}\to[0,1]$ using the set $\mathscr{K}=\{(u,v)\in[0,1]\times[0,1]:u^3+v^3\leq 1\}$ given below, which meets the requirements listed below.

(i) $\Lambda_{\mathscr{K}}(u,v)=1\Leftrightarrow(u,v)=(0,1)\quad or\quad(u,v)=(1,0)$
(ii) $\Lambda_{\mathscr{K}}(u,v)=0\Leftrightarrow u=v=0$
(iii) $\Lambda_{\mathscr{K}}(u,v)=\Lambda_{\mathscr{K}}(v,u)$
(iv) If $u\leq u'$ and $v\leq v'$ then $\Lambda_{\mathscr{K}}(u,v)\leq\Lambda_{\mathscr{K}}(u',v')$.

Theorem 4.7. *Let $T:FFSS(Z)\to\mathbb{R}^+$ and $\rho=(F,T)=[a_{ij}]_{m\times n}\in FFSS(Z)$. If $T(\rho)=\sum_{j=1}^{n}\sum_{i=1}^{m}[1-(\Lambda_{\mathscr{K}}(m_{F(e_j)}(z_i),n_{F(e_j)}(z_i)))]$ where $\Lambda_{\mathscr{K}}$ satisfies the conditions (i)-(iv) of FFSE.*

Proof. $T(\rho)=0\Leftrightarrow T(\rho)=\sum_{j=1}^{n}\sum_{i=1}^{m}[1-(\Lambda_{\mathscr{K}}(m_{F(e_j)}(z_i),n_{F(e_j)}(z_i)))]=0\Leftrightarrow\Lambda_{\mathscr{K}}(m_{F(e_j)}(z_i),n_{F(e_j)}(z_i)))=1,\forall e_j\in T\quad and z_i\in Z\Leftrightarrow m_{F(e_j)}(z_i)=1, n_{F(e_j)}(z_i)=0\quad or\quad m_{F(e_j)}(z_i)=0, n_{F(e_j)}(z_i)=1\Leftrightarrow\rho\quad is\ a\ SS$. Thus, T satisfies property (i) of Definition 4.5. $T(\rho)=mn\Leftrightarrow T(\rho)=\sum_{j=1}^{n}\sum_{i=1}^{m}[1-(\Lambda_{\mathscr{K}}(m_{F(e_j)}(z_i),n_{F(e_j)}(z_i)))]=mn\Leftrightarrow(\Lambda_{\mathscr{K}}(m_{F(e_j)}(z_i),n_{F(e_j)}(z_i)))=0,\forall e_j\in T\quad and z_i\in Z\Leftrightarrow m_{F(e_j)}(z_i)=0=n_{F(e_j)}(z_i)\ \forall e_j\in N$ and $z_i\in Z$. Therefore, T satisfies property (ii) of Definition 4.5. For $F^c(e)=\{(z_i,n_{F(e_j)}(z_i),m_{F(e_j)}(z_i)):z_i\in Z\}$, $\forall\neg e_j\in\neg T$, since $\rho=(F,T)^c=(F^c,\neg T)$, therefore,

$$\begin{aligned}T(\rho)&=\sum_{j=1}^{n}\sum_{i=1}^{m}[1-(\Lambda_{\mathscr{K}}(m_{F(e_j)}(z_i),n_{F(e_j)}(z_i)))]\\&=\sum_{j=1}^{n}\sum_{i=1}^{m}[1-(\Lambda_{K}(n_{F(e_j)}(z_i),m_{F(e_j)}(z_i)))]=T(\rho^c)\end{aligned}$$

This property (iii) is provided for T. Let $\bar{\rho}=(G,T)=[b_{ij}]_{m\times n}$. If $\rho\leq\bar{\rho}$ then, $m_{F(e_j)}(z_i)\leq m_{G(e_j)}(z_i)$ and $n_{F(e_j)}(z_i)\leq n_{G(e_j)}(z_i)$ which implies $\Lambda_{\mathscr{K}}(m_{F(e_j)}(z_i),n_{F(e_j)}(z_i))\leq\Lambda_{\mathscr{K}}(m_{G(e_j)}(k_i),n_{G(e_j)}(k_i))$. $T(\rho)\geq T(\bar{\rho})$ and so, property (iv) is provided for T. Therefore, T is a FFSE. □

Example 4.8. $T(\rho)=\sum_{j=1}^{n}\sum_{i=1}^{m}[1-(m^4_{F(e)},n^4_{F(e)})]$*. We must show that $T(\rho)$ is FFSE. To demonstrate this, it is necessary to prove that $m^4_{F(e)}+n^4_{F(e)}$ meets the $\Lambda_{\mathscr{K}}$ requirements. $\Lambda_{\mathscr{K}}:\mathscr{K}=\{(m_{F(e)},n_{F(e)})\in[0,1]\times[0,1]:u^3+v^3\leq 1\}\to[0,1]$, where $\Lambda_{\mathscr{K}}(u,v)=m^4_{F(e)}+m^4_{F(e)}$. Further, $m^4_{F(e)}+n^4_{F(e)}=1\Leftrightarrow m_{F(e)}=1,\quad n_{F(e)}=0\quad or\quad m_{F(e)}=0, n_{F(e)}=1$ in the domain $\mathscr{K}$.*

Definition 4.9. *Let $\Gamma,\Gamma':[0,1]\to[0,1]$, if $u^3+v^3\leq 1$, then $\Gamma(u^3)+\Gamma'(v^3)\leq 1$ with $u,v\in[0,1]$. Define the function $T_{\Gamma,\Gamma'}$ of the FFSS $\rho=(F,T)=[a_{ij}]_{m\times n}$ to $\mathbb{R}^+$ as,*

$$T_{\Gamma,\Gamma'}=mn-\sum_{j=1}^{n}\sum_{i=1}^{m}\Gamma[m_{F(e_j)}(z_i)]+\Gamma'[n_{F(e_j)}(z_i)] \tag{4.2}$$

Obviously $0\leq T_{\Gamma,\Gamma'}(\rho)\leq mn$ and $\forall\rho=[a_{ij}]_{m\times n}$ belonging to FFSS(Z).

Theorem 4.10. *Let $\Gamma:[0,1]\to[0,1]$ provide the following items:*

(i) Γ is increasing
(ii) $\Gamma(u)=0\Leftrightarrow u=0$
(iii) $\Gamma(u)+\Gamma(v)=1\Leftrightarrow(u,v)=(0,1)\quad or\quad(u,v)=(1,0)$.

Therefore, $\Gamma(u)+\Gamma(v)$ provides the properties (i)-(iv) of the $\Lambda_{\mathscr{K}}$.

Proof. The property (iii) of this theorem is identical to the condition (i) of $\Lambda_{\mathscr{K}}$ if $\Lambda_{\mathscr{K}}(u,v)=\Gamma(u)+\Gamma(v)$ is taken into account. $\Lambda_{\mathscr{K}}(u,v)=\Gamma(u)+\Gamma(v)=0$ if and only if $\Gamma(u)=0=\Gamma(v)$ from property (ii), $u=v=0$. As a result, the second condition of $\Lambda_{\mathscr{K}}$ is obtained. Additionally, as $\Gamma(u)+\Gamma(v)=\Gamma(v)+\Gamma(u)$, $\Lambda_{\mathscr{K}}(u,v)=\Lambda_{\mathscr{K}}(v,u)$. Γ is increasing, hence condition (iv) of $\Lambda_{\mathscr{K}}$ is obtained. Therefore, the $\Lambda_{\mathscr{K}}$ function's conditions (i) to (iv) are satisfied by $\Gamma(u)+\Gamma(v)$. □

Theorem 4.11. *Let $T:FFSS(Z)\to\mathbb{R}^+$, $\Gamma:[0,1]\to[0,1]$, and $\rho=(F,T)=[a_{ij}]_{m\times n}\in FFSS(Z)$. T is FFSE and a $T_{\Gamma,\Gamma}-function\Leftrightarrow T(\rho)=\sum_{j=1}^{n}\sum_{i=1}^{m}\left(1-\Gamma(m_{F(e_j)(z_i)})+\Gamma(n_{F(e_j)(z_i)})\right)$.*

Proof. Take $\Lambda : [0,1]\times[0,1] \to [0,1]$ with $\Lambda(u,v) = \Gamma(u)+\Gamma(v)$, and $\{(u,v) \in [0,1]\times[0,1] : u^3+v^3 \le 1\}$. Restrict the $\Lambda_{\mathscr{K}}$ function from $\mathscr{K}$ to $[0,1]$. If $T(\rho) = \sum_{j=1}^{n}\sum_{i=1}^{m}\left(1-\Gamma(m_{F(e_j)(z_i)})+\Gamma(n_{F(e_j)(z_i)})\right)$, then $T(\rho)$ is a FFSE. It is enough to prove that T is an $T_{\Gamma,\Gamma}$-function.

Let $\alpha,\beta \in [0,1]$ and $\alpha^3+\beta^3 \le 1$ to prove $\Gamma(\alpha^3)+\Gamma(\beta^3) \le 1$, construct the FFSS:

$$[a_{ij}]_{m\times n} = \begin{bmatrix} (\alpha^3,\beta^3) & (1,0) & \cdots & (1,0) \\ (\alpha^3,\beta^3) & (1,0) & \cdots & (1,0) \\ \vdots & \vdots & \vdots & \vdots \\ (\alpha^3,\beta^3) & (1,0) & \cdots & (1,0) \end{bmatrix}$$

Thus,

$$\begin{aligned} T(p) &= \sum_{j=1}^{n}\sum_{i=1}^{m}\left(1-\Gamma(m_{F(e_j)(z_i)})+\Gamma(n_{F(e_j)(z_i)})\right) \\ &= mn - m\left(\Gamma(\alpha^3)+\Gamma(\beta^3)\right) - m(n-1)(\Gamma(\alpha(1))+\Gamma(\beta(0))). \end{aligned}$$

Using Theorem 4.10, $\Gamma(\alpha(1))+\Gamma(\beta(0)) = 1$. Hence, $T(\rho) = mn - m\left(\Gamma(\alpha^3)+\Gamma(\beta^3)\right) - m(n-1) = m\left(\Gamma(\alpha^3)+\Gamma(\beta^3)\right)$. $T(\rho) \ge 0$ because T is entropy. $m\left(1-(\Gamma(\alpha^3)+\Gamma(\beta^3))\right) \ge 0$ which implies that $\Gamma(\alpha^3)+\Gamma(\beta^3) \le 1$. Therefore, T is entropy and a $T_{\Gamma,\Gamma}$ function.

On the other hand, if T is an entropy and $T_{\Gamma,\Gamma}$ are functions, then T has the form $T(\rho) = \sum_{j=1}^{n}\sum_{i=1}^{m}\left(1-\Gamma(m_{F(e_j)(z_i)})+\Gamma(n_{F(e_j)(z_i)})\right)$.

(i) Let $\alpha \le \beta$, $\alpha,\beta \in [0,1]$ construct the following FFSSs:

$$\rho = (F,T) = [a_{ij}]_{m\times n} = \begin{bmatrix} (\alpha,0) & (0,0) & \cdots & (0,0) \\ (\alpha,0) & (0,0) & \cdots & (0,0) \\ \vdots & \vdots & \vdots & \vdots \\ (\alpha,0) & (0,0) & \cdots & (0,0) \end{bmatrix}$$

and

$$\tilde{\rho} = (G,T) = [b_{ij}]_{m\times n} = \begin{bmatrix} (\beta,0) & (0,0) & \cdots & (0,0) \\ (\beta,0) & (0,0) & \cdots & (0,0) \\ \vdots & \vdots & \vdots & \vdots \\ (\beta,0) & (0,0) & \cdots & (0,0) \end{bmatrix}.$$

Thus,

$$\begin{aligned} T(\rho) &= \sum_{j=1}^{n}\sum_{i=1}^{m}\left(1-\Gamma(m_{F(e_j)(z_i)})+\Gamma(n_{F(e_j)(z_i)})\right) \\ &= mn - m(\Gamma(\alpha)+\Gamma(0)) - m(n-1)2\Gamma(0), \end{aligned}$$

and

$$\begin{aligned} T(\tilde{\rho}) &= \sum_{j=1}^{n}\sum_{i=1}^{m}\left(1-\Gamma(m_{F(e_j)(z_i)})+\Gamma(n_{F(e_j)(z_i)})\right) \\ &= mn - m(\Gamma(\Gamma(\beta)+\Gamma(0)) - m(n-1)2\Gamma(0). \end{aligned}$$

$T(\rho) \ge T(\tilde{\rho})$, because $\alpha \le \beta$, $p \preceq \tilde{\rho}$. Therefore,

$$mn - m(\Gamma(\alpha)+\Gamma(0)) - m(n-1)2\Gamma(0) \ge mn - m(\Gamma(\Gamma(\beta)+\Gamma(0)) - m(n-1)2\Gamma(0)$$

implies $\Gamma(\alpha) \le \Gamma(\beta)$. Therefore, Γ is increasing.

(ii) To prove $\Gamma(\alpha) = 0 \Leftrightarrow \alpha = 0$;
If $\alpha = 0$:

$$\rho = (F,T) = [a_{ij}]_{m\times n} = \begin{bmatrix} (0,0) & (0,0) & \cdots & (0,0) \\ (0,0) & (0,0) & \cdots & (0,0) \\ \vdots & \vdots & \vdots & \vdots \\ (0,0) & (0,0) & \cdots & (0,0) \end{bmatrix}.$$

Thus,

$$T(\rho) = mn - mm(\Gamma(0)+\Gamma(0))$$

hence $\Gamma(0)=0$ and so, $\Gamma(\alpha)=0$.
If $\Gamma(\alpha)=0$:

$$\rho=(F,T)=[a_{ij}]_{m\times n}=\begin{bmatrix}(\alpha,0) & (0,0) & \cdots & (0,0)\\ (\alpha,0) & (0,0) & \cdots & (0,0)\\ \vdots & \vdots & \vdots & \vdots\\ (\alpha,0) & (0,0) & \cdots & (0,0)\end{bmatrix}.$$

Thus,

$$\begin{aligned}T(\rho)&=\sum_{j=1}^{n}\sum_{i=1}^{m}\left(1-\Gamma(m_{F(e_j)(z_i)})+\Gamma(n_{F(e_j)(z_i)})\right)\\ &=mn-m(\Gamma(\Gamma(\alpha)+\Gamma(0))-m(n-1)2\Gamma(0).\end{aligned}$$

Since $\Gamma(0)=0$ in the preceding section and $\Gamma(\alpha)=0$, the conclusion is that $T(\rho)=mn$. Thus, α must be equal to 0.

(iii) To prove $\Gamma(\alpha)+\Gamma(\beta)=1\Leftrightarrow(\alpha,\beta)=(0,1)\quad or\quad(1,0)$;

If $(\alpha,\beta)=(0,1)$ or $(1,0)$:

$$\rho=(F,T)=[a_{ij}]_{m\times n}=\begin{bmatrix}(\alpha,\beta) & (\alpha,\beta) & \cdots & (\alpha,\beta)\\ (\alpha,\beta) & (\alpha,\beta) & \cdots & (\alpha,\beta)\\ \vdots & \vdots & \vdots & \vdots\\ (\alpha,\beta) & (\alpha,\beta) & \cdots & (\alpha,\beta)\end{bmatrix}.$$

Then $\rho\in SS(Z)$. As a result, $T(\rho)=0$. Hence,

$$T(\rho)=\sum_{j=1}^{n}\sum_{i=1}^{m}\left(1-\Gamma(m_{F(e_j)(z_i)})+\Gamma(n_{F(e_j)(z_i)})\right)=0.$$

Then, $\Gamma(\alpha)+\Gamma(\beta)=1$.

If $\Gamma(\alpha)+\Gamma(\beta)=1$:

$$\rho=([a_{ij}]_{m\times n}=\begin{bmatrix}(\alpha,\beta) & (1,0) & \cdots & (1,0)\\ (\alpha,\beta) & (1,0) & \cdots & (1,0)\\ \vdots & \vdots & \vdots & \vdots\\ (\alpha,\beta) & (1,0) & \cdots & (1,0)\end{bmatrix}.$$

Thus

$$\begin{aligned}T(\rho)&=\sum_{j=1}^{n}\sum_{i=1}^{m}\left(1-\Gamma(m_{F(e_j)(z_i)})+\Gamma(n_{F(e_j)(z_i)})\right)\\ &=mn-m(\Gamma(\alpha)+\Gamma(\beta))-m(n-1)(\Gamma(1)+\Gamma(0)).\end{aligned}$$

Given that $\Gamma(\alpha)+\varphi(\beta)=1$ and $\Gamma(0)+\Gamma(1)=1$ respectively, Then, $T(\rho)=0$. $\rho\in SS(Z)$, or $(\alpha,\beta)=(1,0)$ or $(0,1)$.

□

Let's note that: Let $\rho=(F,T)=[a_{ij}]_{m\times n}\in FFSS(Z)$, then entropy of ρ is,

$$T(\rho)=\sum_{j=1}^{n}\sum_{i=1}^{m}\left(1-\Gamma(m^{t}_{F(e_j)(z_i)})+\Gamma(n^{t}_{F(e_j)(z_i)})\right),\quad t=3,4,5,\ldots$$

4.2. Distance measure

Definition 4.12. *Let $\rho=(F,M)$ and $\tilde{\rho}=(G,N)$ be two FFSSs. Let U be a mapping given by $U:FFSS(Z)\times FFSS(Z)\to\mathbb{R}^{+}\cup\{0\}$ and $U(\rho,\tilde{\rho})$ satisfies the following axioms:*

(i) $0\leq U(\rho,\tilde{\rho})\leq 2^{1/2}$,
(ii) $U(\rho,\tilde{\rho})=U(\tilde{\rho},U(\rho,\tilde{\rho}))$,
(iii) $U(\rho,\tilde{\rho})=0\Leftrightarrow\rho=\tilde{\rho}$,
(iv) *For any* $\sigma=(H,P)\in FFSS(Z)$, $U(\rho,\tilde{\rho})+U(\tilde{\rho},\sigma)\geq U(\rho,\sigma)$.

Then $U(\rho,\tilde{\rho})$ is a distance measure between FFSSs ρ and $\tilde{\rho}$.

Definition 4.13. *Let $\rho_1 = (F,E), \rho_2 = (G,E)$ be two FFSSs over Z. Then normalized Euclidean distance between ρ_1, ρ_2 is defined as follows:*

$$U_E(\rho_1,\rho_2) = \left[\frac{1}{4mn}\sum_{j=1}^{m}\sum_{i=1}^{n}\left((m^3_{F(e_j)}(z_i) - m^3_{G(e_j)}(z_i))^2 + (n^3_{F(e_j)}(z_i) - n^3_{G(e_j)}(z_i))^2 + (h^3_{F(e_j)}(z_i) - h^3_{G(e_j)}(z_i))^2\right)\right]^{1/2}$$

Theorem 4.14. *Properties of Definition 4.12 are provided for normalized Euclidean distances of FFSSs.*

Theorem 4.15. *For three FFSSs $\rho_1 = (F,E), \rho_2 = (G,E), \rho_3 = (H,E)$ over Z, if $\rho_1 \leq \rho_2 \leq \rho_3$, then $U_E(\rho_1,\rho_2) \leq U_E\rho_1,\rho_3)$ and $U_E(\rho_2,\rho_3) \leq U_E(\rho_1,\rho_3)$.*

5. Applications

5.1. Entropy application

In this subsection, we will practice DM using entropy.

Algortihm:

Step 1: Input each of the FFSSs $\rho_1, \rho_2, \cdots \rho_k$

Step 2: Compute the entropy of each FFSS using the expression

$$T(\rho) = \sum_{j=1}^{n}\sum_{i=1}^{m}[1 - (m^3_{F(e_j)}(a_i) + n^3_{F(e_j)}(a_i))].$$

Step 3: Obtain ρ_r with the minimum of $T(\rho_i)$,

Step 4: The optimum result is to choose the ρ_r to get from Step 3.

Step 5: If more than one ideal solution is discovered, the user may select any of them.

Because the FFSS is an extension of existing sets such as the IFSS and PFSS, it is an excellent tool for representing information during decision-making. Consider a set of k options $V_1, V_2, ..., V_k$ examined by n experts $P_1, P_2, ..., P_n$. Each expert P_j evaluates the alternatives using the parameters $K = k_1, k_2, ..., k_m$ and assigns ratings to FFSNs. The challenge then seeks to select the best option among them. The provided methodology offers a method for solving the problem above using entropy measures.

Example 5.1. *Consider the selection of the car from a particular company. For it, a person wants to select a car from three different alternatives $V_1, V_2, \cdots, V_n$. To address it thoroughly and remove the hesitation between them, they hire three experts E_1, E_2, E_3 to evaluate each alternative under the three significant set of parameters K.*

Consider the purchase of an automobile from a specific firm. A customer wants to choose an automobile from three options: A, B, C. To address it adequately and remove any doubts, they appoint three experts, E_1, E_2, E_3, to analyze each possibility using the three critical sets of parameters $K = \{k_1, k_2, k_3\}$, where k_1 = expensive, k_2 = good engine capacity and k_3 = warranty.
Step 1: *Build $(F,A), (G,B), (H,C)$:*

$$F(k_1) = \{(E_1,(0.75,0.58)),(E_2,(0.98,0.15)),(E_3,(0.47,0.83))\}$$
$$F(k_2) = \{(E_1,(0.82,0.66)),(E_2,(0.59,0.51)),(E_3,(0.26,0.95))\}$$
$$F(k_3) = \{(E_1,(0.54,0.79)),(E_2,(0.73,0.55)),(E_3,(0.87,0.51))\}$$

$$G(k_1) = \{(E_1,(0.63,0.87)),(E_2,(0.80,0.72)),(E_3,(0.56,0.68))\}$$
$$G(k_2) = \{(E_1,(0.72,0.80)),(E_2,(0.51,0.92)),(E_3,(0.67,0.71))\}$$
$$G(k_3) = \{(E_1,(0.82,0.53)),(E_2,(0.88,0.45)),(E_3,(0.73,0.66))\}$$

$$H(k_1) = \{(E_1,(0.42,0.93)),(E_2,(0.56,0.70)),(E_3,(0.88,0.62))\}$$
$$H(k_2) = \{(E_1,(0.67,0.79)),(E_2,(0.77,0.64)),(E_3,(0.76,0.39))\}$$
$$H(k_3) = \{(E_1,(0.68,0.52)),(E_2,(0.91,0.36)),(E_3,(0.74,0.67))\}$$

Step 2: *Compute the FFSEs:*

$$T(F,A) = \sum_{j=1}^{n}\sum_{i=1}^{m}[1 - (m^4_{F(e_j)}(k_i) + n^4_{F(e_j)}(k_i))] = 4.60312528$$
$$T(G,B) = 3.96972281$$
$$T(H,C) = 4.17700393.$$

Step 3: *Find the FFSS with (G,B) as its entropy value, which is the smallest.*

Step 4: *The optimum decision is to choose* (G,B).

Step 5: *The B most likely has an infectious condition because there is only one best course of action.*

It is seen from these computed results that the best alternative for the given problem is B while the worst one is either C or A.

5.2. Distance measure application

PR is the process of identifying patterns in data and categorizing them so that there is a strong correlation between patterns belonging to the same category and a weak correlation between patterns belonging to different categories. FSs, SSs, FSSs, and other tools are helpful for modeling patterns. Due to the premise that similarity is a parallel concept to distance measurement, sets with reduced distances are presumed to be similar. FFSS can also illustrate patterns with a more precise representation of ambiguity.

Algorithm:

The supplied pattern is initially displayed in the feature space SSA as FFSSs $O_1, O_2, \cdots, O_k$. A single O_i, $i = 1, 2, \cdots, k$ should be used to identify the pattern, also represented as FFSS B. The pattern O_i with the shortest distance to B is then found by calculating the distance between each O_i and B. This OI most closely resembles pattern B. The PR algorithm is provided below:
Step 1: Enter the patterns $O_1, O_2, \ldots O_k$

Step 2: Enter the expectedly recognizable pattern B.

Step 3: In Steps 1 and 2, determine the Euclidean distance between each set.

Step 4: The O_i with the smallest Euclidean distance will be chosen at the end.

The scenario and evaluation method of the example below are taken from reference [71].

Example 5.2. *We have proposed a MAGDM method based on the novel FFSS entropy measure. In this example, the method will be used in selecting a missile position. In making a battle plan, staff officers must select a place as a missile position. The following are the main attributes they took into account:* $S = \{k_1, k_2, k_3, k_4, k_5\}$*, as* k_1*-the operational intentions of superiors;* k_2*-the geological conditions of positions;* k_3*-the efficiency of firepower exertion;* k_4*-maneuverability;* k_4*-battlefield viability.*

After a thorough screening and comparison, three locations—$\{O_1, O_2, O_3\}$*—have been tentatively chosen as alternatives. Three experts are asked to rate the options using their FFNs based on gathered knowledge, facts, and experiences to help them make better decisions. Let* $E = \{e_1, e_2\}$ *be given such that the parameters for choosing the most suitable location according to these features are defined as* e_1*-appropriate and* e_2*-not appropriate. Let a P missile location be predetermined.*

$$O_1 = \begin{Bmatrix} e_1 = (k_1,(0.65,0.45)),(k_2,(0.52,0.54)),(k_3,(0.11,0.62)),(k_4,(0.35,0.72)),(k_5,(0.42,0.78)) \\ e_2 = (k_1,(0.92,0.11)),(k_2,(0.76,0.62)),(k_3,(0.94,0.10)),(k_4,(0.83,0.44)),(k_5,(0.69,0.58)) \end{Bmatrix}$$
$$O_2 = \begin{Bmatrix} e_1 = (k_1,(0.86,0.14)),(k_2,(0.90,0.26)),(k_3,(0.73,0.52)),(k_4,(0.44,0.38)),(k_5,(0.68,0.60)) \\ e_2 = (k_1,(0.27,0.88)),(k_2,(0.16,0.82)),(k_3,(0.48,0.62)),(k_4,(0.65,0.54)),(k_5,(0.57,0.48)) \end{Bmatrix}$$
$$O_3 = \begin{Bmatrix} e_1 = (k_1,(0.89,0.31)),(k_2,(0.87,0.35)),(k_3,(0.74,0.52)),(k_4,(0.78,0.25)),(k_5,(0.73,0.28)) \\ e_2 = (k_1,(0.14,0.43)),(k_2,(0.29,0.47)),(k_3,(0.12,0.57)),(k_4,(0.32,0.57)),(k_5,(0.40,0.70)) \end{Bmatrix}$$

Build the FFSNs of pattern P

$$P = \begin{Bmatrix} e_1 = (k_1,(0.9,0.2)),(k_2,(0.8,0.3)),(k_3,(0.8,0.4)),(k_4,(0.7,0.5)),(k_5,(0.9,0.1)) \\ e_2 = (k_1,(0.1,0.9)),(k_2,(0.2,0.8)),(k_3,(0.3,0.8)),(k_4,(0.4,0.7)),(k_5,(0.2,0.9)) \end{Bmatrix}.$$

The Euclidean distance values are:

- *For* e_1, $U_E(O_1,P) = 0.173$; $U_E(O_2,P) = 0.1$; $U_E(O_3,P) = 0.073$
- *For* e_2, $U_E(O_1,P) = 0.202$; $U_E(O_2,P) = 0.11$; $U_E(O_3,P) = 0.098$.

Between O_3 *and P, the Euclidean distance is the smallest. As a result, pattern* O_3 *resembles pattern P more. It can be concluded that the predetermined location P should be the location* O_3.

5.3. Comparison

For the PFSSs F_1, F_2, F_3 and the attribute set $A = \{k_1, k_2, k_3\}$, let the following values be given.

$$F_1 = \left\{ \begin{array}{l} e_1 = (k_1, 0.3, 0.2), (k_2, 0.6, 0.0), (k_3, 0.5, 0.4), \\ e_2 = (k_1, 0.6, 0.3), (k_2, 0.7, 0.2), (k_3, 0.4, 0.3), \\ e_3 = (k_1, 0.8, 0.1), (k_2, 0.8, 0.1), (k_3, 0.6, 0.1) \end{array} \right\}$$

$$F_2 = \left\{ \begin{array}{l} e_1 = (k_1, 0.6, 0.2), (k_2, 0.8, 0.1), (k_3, 0.8, 0.1), \\ e_2 = (k_1, 0.5, 0.5), (k_2, 0.7, 0.2), (k_3, 0.5, 0.4), \\ e_3 = (k_1, 0.7, 0.1), (k_2, 0.6, 0.3), (k_3, 0.6, 0.3) \end{array} \right\}$$

$$F_3 = \left\{ \begin{array}{l} e_1 = (k_1, 0.5, 0.4), (k_2, 0.4, 0.1), (k_3, 0.6, 0.2), \\ e_2 = (k_1, 0.6, 0.2), (k_2, 0.7, 0.1), (k_3, 0.8, 0.1), \\ e_3 = (k_1, 0.9, 0.0), (k_2, 0.5, 0.1), (k_3, 0.6, 0.3) \end{array} \right\}$$

Let's compare the $I(w)$ given in Theorem 4 in the work of Jiang et al. [69], which offers the entropy measure related to IFSS, with the FFS entropy measure given in this article.

The IFSSVs are: $T_{IFSS}(F_1) = 2.12, T_{IFSS}(F_2) = 2.02, T_{IFSS}(F_3) = 2.11$.

The Pythagorean fuzzy soft entropy(PFSE) described in [32] is utilized to compare the proposed entropy metric for FFSSs.
The PFSSVs are: $T_{PFSS}(F_1) = 6.63, T_{PFSS}(F_2) = 6.13, T_{PFSS}(F_3) = 6.34$.

The FFSEVs were measured as $T_{FFSS}(F_1) = 4.17, T_{FFSS}(F_2) = 3.88, T_{FFSS}(F_3) = 3.95$.

It can be seen that IFSE values are very close to each other. However, the result is that F_2 has the lowest entropy and F_1 has the highest entropy, which corresponds to T_{IFSS}, T_{PFSS} and T_{FFSS}. Therefore, the entropy equations that have been proposed are consistent (Table 12).

	T_{IFSS}	T_{PFSS}	T_{FFSS}
F_1	2.12	6.63	7.37
F_2	2.02	6.13	7.08
F_3	2.11	6.34	7.11

Table 12: Comparison of IFSE, PFSE and FFSE.

Now let's make a comparison of distance measures. Let's compare the normalized Euclidean distance based on IFFS given in Definition 8 in [69] and the normalized Euclidean distance based on PFFS given in Definition 3.6 in [32] with the normalized Euclidean distance based on FFSS proposed in this study:

Euclidean distance values for IFSS, PFSS, and FFSS are obtained as follows, respectively:

$$\left\{ \begin{array}{llll} \text{For} \quad e_1, & U_{IFSS}(F_1, F_2) = 0.281; & U_{IFSS}(F_2, F_3) = 0.367; & U_{IFSS}(F_1, F_3) = 0.302, \\ \text{For} \quad e_2, & U_{IFSS}(F_1, F_2) = 0.274; & U_{IFSS}(F_2, F_3) = 0.312; & U_{IFSS}(F_1, F_3) = 0.291, \\ \text{For} \quad e_3, & U_{IFSS}(F_1, F_2) = 0.241; & U_{IFSS}(F_2, F_3) = 0.338; & U_{IFSS}(F_1, F_3) = 0.278, \end{array} \right.$$

$$\left\{ \begin{array}{llll} \text{For} \quad e_1, & U_{PFSS}(F_1, F_2) = 0.277; & U_{PFSS}(F_2, F_3) = 0.325; & U_{PFSS}(F_1, F_3) = 0.318, \\ \text{For} \quad e_2, & U_{PFSS}(F_1, F_2) = 0.253; & U_{PFSS}(F_2, F_3) = 0.316; & U_{PFSS}(F_1, F_3) = 0.286, \\ \text{For} \quad e_3, & U_{IFSS}(F_1, F_2) = 0.266; & U_{IFSS}(F_2, F_3) = 0.323; & U_{IFSS}(F_1, F_3) = 0.303, \end{array} \right.$$

and

$$\left\{ \begin{array}{llll} \text{For} \quad e_1, & U_{FFSS}(F_1, F_2) = 0.225; & U_{FFSS}(F_2, F_3) = 0.344; & U_{FFSS}(F_1, F_3) = 0.327, \\ \text{For} \quad e_2, & U_{FFSS}(F_1, F_2) = 0.212; & U_{FFSS}(F_2, F_3) = 0.285; & U_{FFSS}(F_1, F_3) = 0.266, \\ \text{For} \quad e_3, & U_{IFSS}(F_1, F_2) = 0.198; & U_{IFSS}(F_2, F_3) = 0.309; & U_{IFSS}(F_1, F_3) = 0.275. \end{array} \right.$$

	U_{IFSS}			U_{PFSS}			U_{FFSS}		
	e_1	e_2	e_3	e_1	e_2	e_3	e_1	e_2	e_3
(F_1, F_2)	0.281	0.274	0.241	0.277	0.253	0.266	0.225	0.212	0.198
(F_2, F_3)	0.367	0.312	0.338	0.325	0.316	0.323	0.344	0.285	0.309
(F_1, F_3)	0.302	0.291	0.278	0.318	0.286	0.303	327	266	275

Table 13: Comparison of distance measures

Distance measures are given in Table 13. When these values for IFSS, PFSS, and FFSS are examined, it is seen that for each of them, the distance between F_1 and F_2 is the smallest, while the distance between F_2 and F_3 is the largest.

6. Conclusion

This study aims to define FFSSs and provide entropy and distance metrics. First, the FFSS idea is explained. Then, several FFSS activities and properties are covered. The idea of FSSs has been generalized in the form of FFSSs. Also introduced are the entropy and distance measures of FFSSs. It is simple to state that FFSS is more accurate and reasonable than current soft-set models. Then, DM issues and pattern identification on FFSS are suggested as applications. The recommended entropy was determined to be consistent when compared to FFSS entropy and Pythagorean fuzzy entropy.

This study still has several problems. The first distinction is between risk and uncertainty. The impacts of risk preference rather than uncertainty preference are the main focus of this study. One form of uncertainty avoidance is risk aversion. This is also important because it can be challenging to determine the exact probability of real-world problems. Aside from the benefits of the provided FFSS-based technique, its inability to generate a complete ranking of the available alternatives limits its usefulness in particular DM contexts. Furthermore, when the number of criteria and possibilities is enormous, constructing FFSSs can become difficult.

This study has some limitations while being objective and quantitative for DM problems. The arithmetic operations of FFSNs are more difficult to calculate than crisp or FNs; thus, computing solutions must be developed to lessen the effort of specialists.

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