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Research Article

Creation of Some Fuzzy Ultranorm Spaces and Examining of Their Properties

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Abstract

In this article, first of all, the definition of fuzzy sets was made, and the different aspects of fuzzy sets from classical sets are underlined. The operations in fuzzy sets are shown. Fuzzy numbers formed by fuzzy sets and fuzzy sequences are emphasized. Fuzzy norm and fuzzy ultranorm definitions are made. The α-cut sequences of fuzzy numbers are shown in figures, and various sequences of fuzzy numbers are given. The definition of ultrametric space is created in fuzzy sets, it is proved that the fuzzy number set is ultrametric space and that it is a complete ultrametric space by researching their completeness. Finally, some fuzzy ultra-sequence sets are defined. Then the set of fuzzy ultra-convergent, fuzzy ultra-null, and fuzzy ultra-bounded sequences are denoted and their coverage states are examined. The differences between fuzzy sequences and fuzzy ultra-sequences are emphasized. In addition, the properties of some fuzzy ultra-sequences created are shown. Fuzzy ultra-bounded sequence spaces are proven to be complete and ultra-isomorphic.

Keywords: Fuzzy set, fuzzy ultrametric spaces, fuzzy ultranorm, fuzzy ultra-sequence, ultra-isomorphic.

Bazı Bulanık Ultranorm Uzayların Oluşturulması ve Özelliklerinin İncelenmesi

Öz

Bu makalede başlangıçta bulanık kümeler açıklanmış, bulanık kümeler ve klasik kümelerin birbirinden farklı yönlerinin altı çizilmiştir. Bulanık kümelerdeki işlemler gösterilmiştir. Bulanık kümeler ve bulanık dizilerden oluşan bulanık sayılar üzerinde durulmuştur. Bulanık norm ve bulanık ultranorm tanımları yapılmıştır. Bulanık sayıların α-kesim dizileri şekillerde gösterilmiş ve çeşitli bulanık sayı dizileri verilmiştir. Ultrametrik uzayın tanımı bulanık kümeler halinde yapılmış olup, bulanık sayı kümesinin ultrametrik uzay olduğu ve tamlıkları araştırılarak tam bir ultrametrik uzay olduğu kanıtlanmıştır. Son olarak bazı bulanık ultra dizi kümeleri tanımlanmıştır. Daha sonra bulanık ultra-yakınsak, bulanık ultra-sıfır ve bulanık ultra-sınırlı diziler kümesi belirtilmiş ve kapsam durumları incelenmiştir. Bulanık diziler ve bulanık ultra diziler arasındaki farklar vurgulanmıştır. Ayrıca oluşturulan bazı bulanık ultra dizilerin özellikleri gösterilmiştir. Bulanık ultra sınırlı dizi uzaylarının tam ve ultra izomorfik olduğu kanıtlanmıştır.

Anahtar Kelimeler: Bulanık küme, bulanık ultrametrik uzaylar, bulanık ultranorm, bulanık ultra diziler, ultra izomorfiklik.

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1. Introduction

The concept of fuzzy sets was considered as a generalization of crisp sets and has continued to evolve over time and it has begun to be used in many areas. Because it paved the way for graded evaluation by taking into account the membership degree assigned to each member. Fuzzy logic has been studied by Lukasiewicz and Tarski [1] since the 1920s. Then, the Fuzzy set theory was proposed by Zadeh [2] in 1965. Kaleva [3], Seikkala [4] and Matloka [5] examined fuzzy metric spaces and their properties and obtained significant results. Katsaras [6], C. Felbin [7], and Cheng-Mordeson [8], on the other hand, studied fuzzy metric spaces and fuzzy normed spaces and tried to improve them by Kramosil, Michalek [9]. Bag and Samanta [10] studied fuzzy-bounded linear operators on fuzzy normed spaces in 2005. Fuzzy normed spaces and their topological properties were studied by Kia and Sadeqi [11] in 2009. Xia and Guo [12] investigated the completeness of fuzzy metric space and fuzzy closed set cases. Some new fuzzy sequence spaces were studied by Vakeel A. Khan, Mobeen Ahmad, and Masood Alam [13]. Maria Manuel Clementino and Andrea Montoli were researched and interested in ultrametric groups [14]. Thus, many researchers have developed fuzzy functional analysis and its applications.

Cases of some consequences of metrics induced by a fuzzy ultrametric method were studied by Li [25]. Regarding probability measurement in the category of fuzzy ultrametric spaces, it was examined by Savchenko et al. [26]. Additionally, the completeness of fuzzy metric spaces was examined by Gregori et al. [27]. Fuzzy ultametric spaces and their applications were also encountered in the decision process, Khameneh et al. [28] worked on the subject. The properties of ultranormed spaces were introduced and studied by Şanlıbaba [29,30]. Intuitionistic fuzzy normed rings and the generation of their basic properties were investigated by Abed Alhaleem et al [31].

With a different approach, we have managed to go even further in this article. Fuzzy norm, fuzzy ultranorm, and fuzzy ultrametric spaces are introduced and their properties are studied. Fuzzy ultranorm spaces created by using ultranorm 3rd property (N3)' $\|u+v\|\leq\max\{\|u\|,\|v\|\}$ are emphasized as a more special case of fuzzy normed spaces, and various interesting results are obtained. Especially in the situations between fuzzy sets and fuzzy ultrametric sets, their coverage conditions are examined. For this reason, it is quite exciting to discover some kind of new type of fuzzy space that can be beneficial for solving various problems in fuzzy work. In addition, some theorems about fuzzy ultrametric space and fuzzy ultra-bounded sequence spaces are proved, and examples are shown.

The structure of this paper is prepared as follows: Some preliminary results and basic definitions are mentioned in section 2. In section 3 fuzzy ultranorm, fuzzy ultrametric, and some fuzzy ultra-sequence spaces are proven and main results are discussed. In section 4 Some of the conclusions emphasized in the article are summarized.

2. Preliminaries

Let X be a universal set, $\mu_A(x)$ is the degree of membership of x in A. Then a fuzzy set A of X is defined by the following transform:

$$\mu_A: X \to [0,1]$$

With another expressed as:

$$A=\frac{\mu_A(x_1)}{x_1}+\frac{\mu_A(x_2)}{x_2}+\frac{\mu_A(x_3)}{x_3}+\cdots+\frac{\mu_A(x_n)}{x_n}=\sum\frac{\mu_A(x_n)}{x_n} \quad [15].$$

As we know, classical sets are in particular fuzzy sets. Many algebraic operations used in classical sets are also valid for fuzzy sets and De Morgan's laws are not always valid in fuzzy sets. Therefore, it is obvious that fuzzy set operations are slightly different from classical set operations.

Let the membership functions of fuzzy sets A and B be $\mu_A(x)$ and $\mu_B(x)$, respectively. The union, intersection, complement, and inclusion of two fuzzy sets are shown below [16].

$\mu_{A\cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}$ $\quad \forall x \in X$ (Union)

$\mu_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}$ $\quad \forall x \in X$ (Intersection)

$\mu_{\bar{A}}(x) = 1 - \mu_A(x)$ $\quad \forall x \in X$ (Complement)

$\mu_A(x) \leq \mu_B(x)$, $(A \subseteq B)$ $\quad \forall x \in X$ (Inclusion)

Definition 2.1. Consider a fuzzy subset of the real line u: $\mathbb{R} \to [0,1]$. Then u is called a fuzzy number if it provides the following conditions:

1. u is normal, i.e., $\exists x_0 \in \mathbb{R}$ with $u(x_0) = 1$;
2. u is fuzzy convex, i.e., $u(tx + (1-t)y) \geq \min\{u(x), u(y)\}$, $\forall t \in [0,1]$ ve $\forall x, y \in \mathbb{R}$.
3. u is upper semi-continuous on $\mathbb{R}$. $\forall \varepsilon > 0, \exists \delta > 0$ such that $u(x) - u(x_0) < \varepsilon$ $|x - x_0| < \delta$
4. u is compactly supported, i.e., $\overline{u^0} = \overline{\{x \in \mathbb{R}: u(x) > \alpha\}}$ is compact. [17],[18].

Let's suppose that E_i is the be set of all closed and bounded intervals on $\mathbb{R}$, $E_i = \{u = [u^-, u^+]: u^- \leq x \leq u^+, u^- and\ u^+ \in \mathbb{R}\}$.

From this definition, $\forall \alpha \in [0,1]$, α −level set defined by $u^\alpha = \{x \in \mathbb{R}: u(x) \geq \alpha\}$ is in E_i, that is $u^\alpha = [u^{\alpha-}, u^{\alpha+}]$ and the set of all fuzzy numbers are denoted by E'.

E' is the set of fuzzy numbers, f: $\mathbb{N} \to E'$ $\quad k \to f(k) = u_k$, for $\forall k \in \mathbb{N}$, f is called sequences of fuzzy numbers with $u_k \in E'$,

$f: \mathbb{N} \to E' \quad k \to f(k) = u_k,$

If the sequences of fuzzy numbers are denoted by $u = (u_k)$ it is clear that u^{α} is called $\alpha -$ cut sequences given in form $(u^{\alpha}) = (u_k^{\alpha}) = (u^{\alpha}{}_1, u^{\alpha}{}_2, \dots, u_k^{\alpha}, \dots)$ [19],[20].

Additionally, the sets of sequences of fuzzy numbers are indicated with $w(E') = \{u = (u_k) \in E' : k \in \mathbb{N}\} = \{f: \mathbb{N} \to E',\ f(k) = u_k\}$. The algebraic structure of $w(E')$ in Figure 2.1 is as follows:

$+: w(E') \times w(E') \to w(E'), u = (u_1, u_2, \dots)$ and $(v_1, v_2, \dots) \in w(E')$

$$(u, v) \to u + v = u_k + v_k$$

$$= (u_1 + v_1, u_2 + v_2, \dots, u_k + v_k, \dots)$$

$\cdot : \lambda \in \mathbb{R}\ , \lambda(u_k) = (\lambda u_k) = (\lambda u_1, \lambda u_2, \dots, \lambda u_k, \dots)$

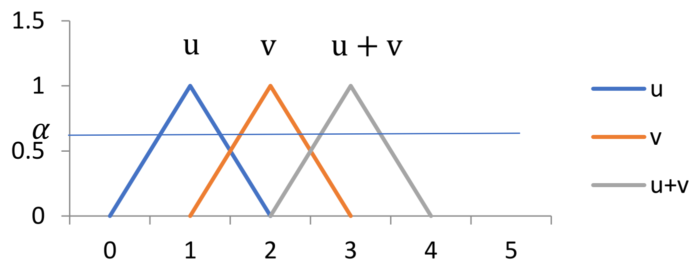

Figure 2.1. $\alpha -$ cut sequence and operations on fuzzy sequences

In scalar multiplication $(u_k) \in w(E')$ and $u_k^{\alpha} = [u_k^{\alpha-}, u_k^{\alpha+}]$ if $\alpha -$ cuts sequences are taken as:

$$u_k^{\alpha} = [u_k^{\alpha-}, u_k^{\alpha+}];$$

$$\lambda u_k = \lambda u_k^{\alpha-} = \lambda[u_k^{\alpha-}, u_k^{\alpha+}] = \begin{cases} [\lambda u_k^{\alpha-}, \lambda u_k^{\alpha+}], & \lambda \geq 0 \\ [\lambda u_k^{\alpha+}, \lambda u_k^{\alpha-}], & \lambda < 0 \end{cases}$$

Since the set of closed intervals does not have an inverse with respect to addition, the set of fuzzy numbers in the classical sense has no inverse with respect to addition. Therefore, the same is true for $w(E')$ elements. Therefore, $w(E')$ cannot be transformed into a vector space using scalar multiplication and addition in the Zadeh sense.

3. Main Theorem and Proof

In this section, firstly fuzzy ultrametric spaces and their definitions are given, then the fact that $F(E')$ which is the main subject of the research is an ultrametric space is proved by Theorem

3.1, and its completeness is shown in Theorem 3.2. In the last part, it is proved by Theorem 3.6 that the space of ultra-bounded sequences $l'_\infty(E')$ is ultra-isometric.

3.1. Fuzzy Ultrametric Spaces and Fuzzy Ultra-Sequences

The metric space and completeness of sequences of fuzzy sets have been shown in various articles. In this section, it will be proved that fuzzy numbers are ultrametric space and ultranorm. Then, the definitions of different fuzzy sequence spaces will be made and their properties will be examined.

Definition 3.1. Let G be the set of fuzzy numbers but nonnegative and $\mathrm{u}, \mathrm{v} \in E'$. The function $d_f: E'\mathrm{x}E' \rightarrow$ G is called fuzzy metric if it is satisfies the (F1), (F2), (F3) and (F4) properties:

(F1) $d_f(u, v) \geq 0$

(F2) $d_f(u, v) = 0$ if and only if $u = v$

(F3) $d_f(u, v) = d_f(v, u)$

(F4) For all $u, v, w \in E'$, $d_f(u, v) \leq d_f(u, w) + d_f(w, v)$ [21].

If is taken $(\mathrm{FU4})'$ $d_{fu}(u, v) \leq \max\{d_{fu}(u, w), d_{fu}(w, v)\}$ instead of (F4) then the function d_{fu} is called fuzzy ultrametric. For any $u, v \in E'$, if d_{fu} is a fuzzy ultrametric on E', the pair of (E', d_{fu}) is called fuzzy ultrametric space.

Definition 3.2. Let $\mathrm{u} = (\mathrm{u_k})$ sequence of fuzzy number and $\mathrm{u}, \mathrm{v} \in E'$. $\lambda(E')$ the subset of sequences spaces of fuzzy number, H is the set of all nonnegative fuzzy numbers, and $\|\cdot\|: \lambda(E') \rightarrow H$. If the function satisfies the following $(N1)$, $(N2)$, and $(N3)$ properties, it is called a fuzzy norm and fuzzy module [22].

$(N1)$ $\|\mathrm{u}\| = \theta \Leftrightarrow \mathrm{u} = \theta$

$(N2)$ $\|\alpha\mathrm{u}\| = |\alpha|\|\mathrm{u}\|$

$(N3)$ $\|\mathrm{u} + \mathrm{v}\| \leq \|\mathrm{u}\| + \|\mathrm{v}\|$

$(N3)'$ $\|\mathrm{u} + \mathrm{v}\| \leq \max\{\|\mathrm{u}\|, \|\mathrm{v}\|\}$

Also, if it satisfies $(N1)$, $(N2)$ and $(N3)'$ properties, it is called fuzzy ultranorm.

Theorem 3.1. The set of all fuzzy numbers is represented by $\mathrm{F}(E')$ and $\mathrm{F}(E') =$

$\{\mathrm{u}|\mathrm{u}: \mathbb{R} \rightarrow [0,1]\}$ it is remembered that $\mathrm{F}(E')$ is normal, convex, and upper semi-continuous $\overline{\mathrm{u}^0}$ that it is also compact. This is stated from the definition of fuzzy numbers given in definition 2.1. Then, let $d(\mathrm{u}, \mathrm{v}) = \sup_{\alpha \in [0,1]}\left\{\left|\mathrm{u}^{(\alpha)-} - \mathrm{v}^{(\alpha)-}\right|, \left|\mathrm{u}^{(\alpha)+} - \mathrm{v}^{(\alpha)+}\right|\right\}$, for $\mathrm{u}, \mathrm{v} \in \mathrm{F}(E')$. The d function satisfies the ultranorm conditions, and $(\mathrm{F}(E'), d)$ is a fuzzy ultrametric space.

Proof. Let $\mathrm{u}, \mathrm{v} \in \mathrm{F}(E')$;

$d(\mathrm{u}, \mathrm{v}) = 0 \Leftrightarrow \mathrm{u} = \mathrm{v}$

$d(\mathrm{u}, \mathrm{v}) = \sup_{\alpha\in[0,1]}\{|\mathrm{u}^{(\alpha)-} - \mathrm{v}^{(\alpha)-}|, |\mathrm{u}^{(\alpha)+} - \mathrm{v}^{(\alpha)+}|\} = 0 \ \Leftrightarrow$

$|\mathrm{u}^{(\alpha)-} - \mathrm{v}^{(\alpha)-}| = 0 \ \wedge \ |\mathrm{u}^{(\alpha)+} - \mathrm{v}^{(\alpha)+}| = 0 \Leftrightarrow \mathrm{u} = \mathrm{v}$

$d(\mathrm{u}, \mathrm{v}) = \sup_{\alpha\in[0,1]}\{|\mathrm{u}^{(\alpha)-} - \mathrm{v}^{(\alpha)-}|, |\mathrm{u}^{(\alpha)+} - \mathrm{v}^{(\alpha)+}|\}$

$= \sup_{\alpha\in[0,1]}\{|\mathrm{v}^{(\alpha)-} - \mathrm{u}^{(\alpha)-}|, |\mathrm{v}^{(\alpha)+} - \mathrm{u}^{(\alpha)+}|\} = d(\mathrm{v}, \mathrm{u})$

$d(\mathrm{u}, \mathrm{v}) = \sup_{\alpha\in[0,1]}\{\max\{|\mathrm{u}^{(\alpha)-} - \mathrm{v}^{(\alpha)-}|, |\mathrm{u}^{(\alpha)+} - \mathrm{v}^{(\alpha)+}|\}\}$

$= \sup_{\alpha\in[0,1]}\{\max\{|\mathrm{u}^{(\alpha)-} - \mathrm{w}^{(\alpha)-} + \mathrm{w}^{+(\alpha)} - \mathrm{v}^{(\alpha)-}|, |\mathrm{u}^{(\alpha)+} - \mathrm{w}^{(\alpha)+} + \mathrm{w}^{(\alpha)+} - \mathrm{v}^{(\alpha)+}|\}\}$

$\leq \sup_{\alpha\in[0,1]}\left\{\max\{|\mathrm{u}^{(\alpha)-} - \mathrm{w}^{(\alpha)-}|, |\mathrm{u}^{(\alpha)+} - \mathrm{w}^{(\alpha)+}|\}\right\}, \quad \sup_{\alpha\in[0,1]}\left\{\max\{|\mathrm{w}^{(\alpha)-} - \mathrm{v}^{(\alpha)-}|, |\mathrm{w}^{+(\alpha)} - \mathrm{v}^{(\alpha)+}|\}\right\}$

$= \max\left\{\sup_{\alpha\in[0,1]}\{|\mathrm{u}^{(\alpha)-} - \mathrm{w}^{(\alpha)-}|, |\mathrm{u}^{(\alpha)+} - \mathrm{w}^{(\alpha)+}|\}\right\}, \max\left\{\sup_{\alpha\in[0,1]}\{|\mathrm{w}^{(\alpha)-} - \mathrm{v}^{(\alpha)-}|, |\mathrm{w}^{+(\alpha)} - \mathrm{v}^{(\alpha)+}|\}\right\}$

$d(\mathrm{u}, \mathrm{v}) \leq \max\{d(\mathrm{u}, \mathrm{w}), d(\mathrm{w}, \mathrm{v})\}.$

Since it satisfies the ultrametric axioms $(\mathrm{F}(E'), d)$, it is an ultrametric space.

Similarly with $\|u\| = \sup_k |u_k|$, properties $(N1), (N2)$ are clear. Property $(N3)'$ is following denoted.

$\|u + v\| = \sup_k |u_k + v_k| \leq \sup_k \{\max\{|u_k|, |v_k|\}$

$= \sup_k \{\max\{|u_k + \theta|, |v_k + \theta|\}$

$\leq \sup_k \{\max\{\max\{|u_k|, \theta\}, \max\{|v_k|, \theta\}\}\}$

$= \max\{\sup_k \{\max\{|u_k|, \theta\}, \sup_k \{\max\{|v_k|, \theta\}\}\}$

$= \max\{\sup_k d(u_k, \theta), \sup_k d(v_k, \theta)\}$

$= \max\{\|u\|, \|v\|\}$

Consequently, d is fuzzy ultranorm. □

Fuzzy ulltranorm spaces are also special cases of fuzzy norm spaces. Now, considering the definitions and theorems explained so far, the relationship between fuzzy metric space, fuzzy ultrametric space, fuzzy ultranorm space, and fuzzy normed spaces is easily shown in diagram 3.1. below.

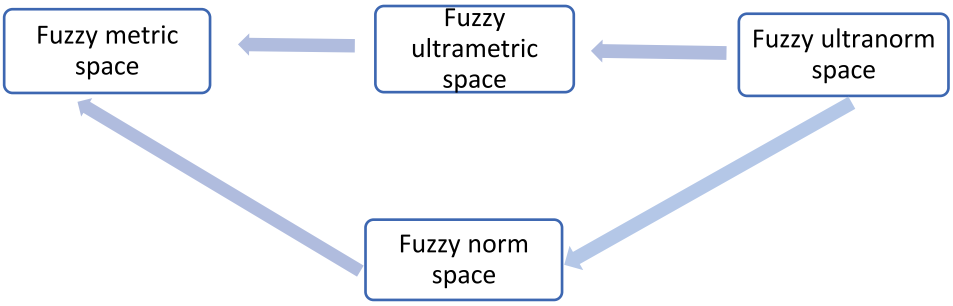

Diagram 3.1. The connection between fuzzy ultrametric space and fuzzy ultranorm space.

Let u be a sequence of fuzzy numbers. For $\forall\varepsilon > 0$, $\forall \mathrm{k} \geq \mathrm{k}_0$ and $\exists \mathrm{k} \in \mathbb{N}$, if the inequality $\mathrm{d}(\mathrm{u}_\mathrm{k}, \mathrm{u}_0) = \sup_{\alpha\in[0,1]}\overline{\mathrm{d}}(\mathrm{u}_\mathrm{k}{}^\alpha, \mathrm{u}_0{}^\alpha) < \varepsilon$, then (u_k) is called to convergent to $(\mathrm{u}_0) \in \mathrm{F}(E')$.

$\bar{d}(\mathrm{u}_\mathrm{k}{}^\alpha, \mathrm{u}_0{}^\alpha) = \max\left\{\left|\mathrm{u}_\mathrm{k}{}^{(\alpha)-} - \mathrm{u}_0{}^{(\alpha)-}\right|, \left|\mathrm{u}_\mathrm{k}{}^{(\alpha)+} - \mathrm{u}_0{}^{(\alpha)+}\right|\right\}$. Briefly, it is denoted by $\lim \mathrm{u}_\mathrm{k} = \mathrm{u}_0$. If in the closed interval $d([\mathrm{a},\mathrm{b}],[\mathrm{c},\mathrm{d}]) = \max\{|\mathrm{a}-\mathrm{c}|, |\mathrm{b}-\mathrm{d}|\}$.

Lemma 3.1. Let $E' = \{[\mathrm{a},\mathrm{b}] | \mathrm{a} \leq \mathrm{b} \text{ and } \mathrm{a},\mathrm{b} \in \mathbb{R}\}$ be the set of all closed intervals. Since $\tilde{d}: E' \times E' \to \mathbb{R}$, $(E', \tilde{d})$ is ultrametric space.

Defined by $([\mathrm{a},\mathrm{b}],[\mathrm{c},\mathrm{d}]) \to \tilde{d}([\mathrm{a},\mathrm{b}],[\mathrm{c},\mathrm{d}]) = \max\{|\mathrm{a}-\mathrm{c}|, |\mathrm{b}-\mathrm{d}|\}$, $\tilde{d}$ provides ultrametric properties.

$\tilde{d}([\mathrm{a},\mathrm{b}],[\mathrm{c},\mathrm{d}]) = 0 \quad \Leftrightarrow \quad \max\{|\mathrm{a}-\mathrm{c}|, |\mathrm{b}-\mathrm{d}|\} = 0 \quad \Leftrightarrow$ and $\mathrm{a} = \mathrm{c}$, $\mathrm{b} = \mathrm{d}$, $[\mathrm{a},\mathrm{b}] = [\mathrm{c},\mathrm{d}]$.

$\tilde{d}([\mathrm{a},\mathrm{b}],[\mathrm{c},\mathrm{d}]) = \tilde{d}([\mathrm{c},\mathrm{d}],[\mathrm{a},\mathrm{b}])$ it is clear that.

$$\begin{aligned}
\tilde{d}([\mathrm{a},\mathrm{b}],[\mathrm{c},\mathrm{d}]) &= \max\{|\mathrm{a}-\mathrm{c}|, |\mathrm{b}-\mathrm{d}|\} \\
&= \max\{|\mathrm{a}-\mathrm{c}+\mathrm{e}-\mathrm{e}|, |\mathrm{b}-\mathrm{d}+\mathrm{f}-\mathrm{f}|\} \\
&\leq \max\left\{\{|\mathrm{a}-\mathrm{e}| + |\mathrm{e}-\mathrm{c}|\}, \{|\mathrm{b}-\mathrm{f}| + |\mathrm{f}-\mathrm{d}|\}\right\} \\
&= \max\left\{\{|\mathrm{a}-\mathrm{e}|, |\mathrm{b}-\mathrm{f}|\} + \{|\mathrm{e}-\mathrm{c}|, |\mathrm{f}-\mathrm{d}|\}\right\} \\
&\leq \max\left\{\{|\mathrm{a}-\mathrm{e}|, |\mathrm{b}-\mathrm{f}|\}, \{|\mathrm{e}-\mathrm{c}|, |\mathrm{f}-\mathrm{d}|\}\right\} \\
&= \max\{\tilde{d}([\mathrm{a},\mathrm{b}],[\mathrm{e},\mathrm{f}]), \tilde{d}([\mathrm{e},\mathrm{f}],[\mathrm{c},\mathrm{d}])\}
\end{aligned}$$

Since it provides ultrametric properties $(E', \tilde{d})$, it is an ultrametric space. $\square$

Let's suppose that $u_k = ([x_k, y_k])$ is a Cauchy sequence. So for all $\varepsilon > 0$, it is $\exists n_0 \in \mathbb{N}$ such that for $\forall k, m \geq n_0$ it is $d([x_k, y_k], [a_m, b_m]) < \varepsilon$.

Since $\max\{|x_k - x_m|, |y_k - y_m|\} < \varepsilon$, $|x_k - x_m| < \varepsilon$ and $|y_k - y_m| < \varepsilon$. The sequences (x_k) and (y_k) are also a Cauchy sequence in $\mathbb{R}$. Because $\mathbb{R}$ is complete, $\lim_k x_k = x_0$ and $\lim_k y_k = y_0$ exist. Since $x_k \leq y_k$ is $x_0 \leq y_0$. So $\lim_k u_k = \lim_k [x_k, y_k] = [x_0, y_0]$. Namely, (u_k) Cauchy sequence is convergent and since it is convergent $(E', \tilde{d})$ it is a complete ultrametric space.

Theorem 3.2. $F(E')$ is the complete ultrametric space.

Proof. It can be proved that $F(E')$ is also complete using Lemma 2.1. It can be taken as $u \in F(E') \Leftrightarrow u^\alpha = [u^{\alpha-}, u^{\alpha+}]$. Because for $(u_k) \in F(E')$ and $\forall \alpha \in [0,1]$, α −cut $({u_k}^\alpha)$ gives a sequence of E'. As a result, it is clear from the complete of E' that $F(E')$ is complete. □

Definition 3.3. The fuzzy norm of the fuzzy number u is denoted which notation to the fuzzy distance from u to 0, as follows:

$$\|u\|_{E'} = \sup_k \sup_{\lambda \in [0,1]} \lambda[|u_k{}^- - \bar{0}|, |u_k{}^+ - \bar{0}|] \tag{3.1}$$

Let $u = (u_k)$ be a sequence of fuzzy numbers, $(u_k) \in E'$ and $\|\cdot\|$ be a fuzzy norm.

It means that the sequences (u_k) converge to $(u_0) \in E'$ with the fuzzy norm $\|\cdot\|$, there is an integer n_0 such that $\|u_k - u_0\| < [\varepsilon, \varepsilon] = \varepsilon$ for $k \geq n_0$, if for any given $\varepsilon > 0$. The sequence (u_k) is said to be fuzzy norm $\|\cdot\|$ if $\sup_k \|u_k\| < \infty$, for all $k \in \mathbb{N}$. Respectively $c(E')$, $c_0(E')$, and $l_\infty(E')$ are written for the the fuzzy sets of all fuzzy convergent, fuzzy null, fuzzy bounded sequences [23].

$\|\cdot\|$ is satisfies the property $(N3)'$ with the norm defined by $sup_k \|u_k\|$. (3.2)

$u_{l'_\infty(E')}(k) = u_{c'(E')}(k) = u_{c'_0(E')}(k)$.

Then the set of fuzzy ultra-convergent, fuzzy ultra-null, and fuzzy ultra-bounded sequences are denoted by $c'(E')$, $c'_0(E')$ and $l'_\infty(E')$ respectively, and defined as follows:

$$c'(E') = \{u = (u_k) \in w(E'): \lim_k \sup_{\lambda \in [0,1]} \lambda[|u_k{}^- - \bar{0}|, |u_k{}^+ - \bar{0}|] = \varphi, \varphi \in E' \}$$

$$c'_0(E') = \left\{u = (u_k) \in w(E'): \lim_k \sup_{\lambda \in [0,1]} \lambda[|u_k{}^- - \bar{0}|, |u_k{}^+ - \bar{0}|] = \theta \right\}$$

$$l'_\infty(E') = \left\{u = (u_k) \in w(E'): \sup_k \sup_{\lambda \in [0,1]} \lambda[|u_k{}^- - \bar{0}|, |u_k{}^+ - \bar{0}|] < \infty \right\}$$

Theorem 3.3. $c'_0(E') \subset c'(E') \subset l'_\infty(E')$ coverage is available.

Theorem 3.4. $c'_0(E') \subset c_0(E') \subset c'(E') \subset c(E') \subset l'_\infty(E') \subset l_\infty(E')$. It is clear that it has coverage.

Theorem 3.5. $l'_\infty(E') = \{\mathrm{u} = (\mathrm{u_k}) \in E' : \sup_k \sup_{\lambda\in[0,1]} \lambda[|\mathrm{u}_k{}^- - \bar{0}|, |\mathrm{u}_k{}^+ - \bar{0}|] < \infty\}$ fuzzy ultra-bounded sequences is a complete ultranorm space, with the norm given in (3.2).

Proof. It is easy to prove that a fuzzy ultra-bounded sequence is a fuzzy ultranormed space. Hence it will be proved below that a fuzzy ultra-bounded sequence is complete.

Let's assume that $(\mathrm{u_k})$ is a Cauchy sequence in $l'_\infty(E')$. If (u_k) is a constant sequence then the case is clear. If (u_k) is not a constant sequence then there is an integer n_0 such that $m, n \geq n_0$ and;

$$\|u^m - u^n\| = sup_k |u_k{}^m - u_k{}^n| < \varepsilon \tag{3.3}$$

From here, $|u_k{}^m - u^n| < \varepsilon$ is obtained when $m, n \geq n_0$ for every arbitrary but constant k with $k = 1,2, \dots$ and $\varepsilon > 0$. So for every constant k, $(u_k^1, u_k^2, u_k^3, \dots)$ is also a Cauchy sequence in E'. Since E' is complete $u_k^m \to u \in E'$. With the help of these limits to be obtained for each natural number k, the sequence $u = (u_1, u_2, \dots)$ is formed in E'.

$$\begin{array}{l}
u_1 = (u_1^1, u_2^1, \dots, u_n^1, \dots) \\
u_2 = (u_1^2, u_2^2, \dots, u_n^2, \dots) \\
u_3 = (u_1^3, u_2^3, \dots, u_n^3, \dots) \qquad . \\
\quad\vdots \\
u_k = \left(u_1^k, u_2^k, \dots, u_n^k, \dots\right) \\
\quad\vdots \\
u_m = (u_1^m, u_2^m, \dots, u_n^m, \dots) \\
\ \downarrow \qquad \downarrow \quad \downarrow \cdots \quad \downarrow \cdots \\
\ u = (u_1, u_2, \ \dots, u_n, \dots)
\end{array}$$

Now it will be denoted that $u_n \to u$, $n \to \infty$, and $u \in l'_\infty(E')$. In (3.2) if $n \to \infty$ is done $\|u_k{}^n - u_k\|_{l'_\infty(E')} < \varepsilon$ is obtained. Since $u_n = (u_k^n) \in l'_\infty(E')$, there is t_n real sequence such that $\|u_k{}^n\| \leq t_n$, for $k = 1,2, \dots$

From the strong triangle inequality $\|u_k\| = \|u_k - u_k{}^n + u_k{}^n\| \leq \max\{\|u_k - u_k{}^n\|, \|u_k{}^n\|\} \leq \max\{\varepsilon, t_n\} < \infty$. Considering that for all k this inequality is valid, $(u_k) \in l'_\infty(E')$. □

Definition 3.4. $Z' = (z_{nk})$ and given the sequence $u = (u_k)$, consider the transformation $v_k = pu_k + (1-p)u_{k-1}$, which is called Zweier transformation by the following definition [24].

$$Z'_{nk} = \begin{cases} p, & n = k \\ 1-p, & n-1 = k \\ 0, & otherwise \end{cases} \quad , p \in \mathbb{R} - \{-1\} \qquad n, k \in \mathbb{N}$$

Theorem 3.6. Defined by $\boldsymbol{l'_\infty(E')} = \{\mathbf{u} = (\mathbf{u_k}) \in \boldsymbol{E'}: \mathbf{sup}_{\boldsymbol{k}}\mathbf{sup}_{\boldsymbol{\lambda\in[0,1]}}\boldsymbol{\lambda}[|\mathbf{u}_{\boldsymbol{k}}{}^{-} - \overline{\mathbf{0}}|, |\mathbf{u}_{\boldsymbol{k}}{}^{+} - \overline{\mathbf{0}}|] < \infty\}$ the set of the fuzzy ultra-bounded sequence is ultra-isomorphic.

Proof.

$Z': \tilde{l}_\infty(E') \to l'_\infty(E')$ consider the transformation $u \to Z'u = v,\ v = v_k,$

$$v_k = pu_k + (1-p)u_{k-1} \tag{3.4}$$

i) Z' is linear.

$\forall u, v \in \tilde{l}_\infty(E')$, $\forall \alpha \in E'$

$$\begin{aligned} Z'(u+v) &= p(u_k+v_k) + (1-p)(u_{k-1}+v_{k-1}) \\ &= pu_k + pv_k + (1-p)u_{k-1} + (1-p)v_{k-1} \\ &= pu_k + (1-p)u_{k-1} + pv_k + (1-p)v_{k-1} \\ &= Z'(u) + Z'(v) \end{aligned}$$

$$\begin{aligned} Z'(\alpha u) &= p(\alpha u_k) + (1-p)(\alpha u_{k-1}) \\ &= \alpha(pu_k + (1-p)u_{k-1}) = \alpha Z'(u). \end{aligned}$$

ii) Z' bijection.

$Z': \tilde{l}_\infty(E') \to l'_\infty(E')$ and $v_k = pu_k + (1-p)u_{k-1}$

When $\mathrm{u} = Z'^{-1}v$, every $u = (u_k) = \sum_{j=0}^{k}(-1)^{k-j}\frac{(1-p)^{k-j}}{p^{k-j+1}}v_j$ (3.5)

The form given has a v element. As a result, it is shown to be a bijection.

iii) In the case of $\tilde{l}_\infty(E')$ and $l'_\infty(E')$ ultranormed spaces, the Z' isomorphic is a linear space isomorphic that preserves the norm, that is, satisfies the $\|Z'u\| = \|u\|$ condition for each u (element) Z'.

In the $u = (u_k) = \sum_{j=0}^{k}(-1)^{k-j}\frac{(1-p)^{k-j}}{p^{k-j+1}}v_j$ sequence given in (3.5), with (3.4) $v_k = pu_k + (1-p)u_{k-1}$

$$\begin{aligned} \|Z'u\|_{\tilde{l}_\infty(E')} &= \sup_k \tilde{d}(Z'u_k, \overline{0}) \\ &= \sup_k \tilde{d}(\mathrm{p}u_k + (1-p)u_{k-1}, \overline{0}) \end{aligned}$$

$$= \sup_k \tilde{d}\left(p\sum_{j=0}^{k}(-1)^{k-j}\frac{(1-p)^{k-j}}{p^{k-j+1}}v_j + (1-p)\sum_{j=0}^{k}(-1)^{k-j}\frac{(1-p)^{k-j}}{p^{k-j+1}}v_j, \bar{0}\right)$$

$$= \sup_k \tilde{d}(u_k, \bar{0}) = \|u\|_{l'_\infty(E')}$$

To summarize, Z' is norm preserving. The spaces $\tilde{l}_\infty(E')$ and $l'_\infty(E')$ are isometric. $\square$

As a result, $l'_\infty(E')$ fuzzy ultra-bounded sequence space is shown to be ultra-isomorphic and the proof is completed.

4. Conclusion

In this study, fuzzy ultrametric space and fuzzy ultranorm definitions were made. It is proved that fuzzy sets are ultrametric space and ultranorm. It was emphasized that fuzzy ultranorm spaces are also special cases of fuzzy norm spaces. Additionally, a relationship was established between fuzzy ultrametric spaces and fuzzy ultranorm spaces. It has been shown that F(E') is a complete ultrametric space. By defining fuzzy ultra-sequence spaces $c'(E'), c'_0(E')$, and $l'_\infty(E')$ their coverage states are researched and shown. Finally, the fuzzy ultra-bounded sequence spaces are proven to be complete and ultra-isomorphic.

Ethics in Publishing

There are no ethical issues regarding the publication of this study.

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