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GENERALIZED STOLZ MAPPINGS

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Abstract. In this paper, we introduce the class of generalized Stolz mappings. Also we prove that the class of ℓ^p-type mappings is included in the class of generalized Stolz mappings and give a new quasi-norm equivalent to $\|T\|_{\phi_{(p)}}$. Finally, we present some properties of the class of generalized Stolz mappings.

1. Introduction

In functional analysis, the theory of operator ideals has a special importance since it has many applications in spectral theory, geometry of Banach spaces, theory of distribution, etc. One of the most important methods to construct operator ideals is via s-numbers (for more details see [2],[4],[5],[18],[21]).

In this study, we denote the set of all natural numbers by $\mathbb{N}$.

Let E and F be real or complex Banach spaces and $L(E,F)$ denote the space of all bounded linear operators from E to F and L denotes the space of all bounded linear operators between any two arbitrary Banach spaces.

A map $s=(s_n):L\to\mathbb{R}^+$ assigning to every operator $T\in L$ a non-negative scalar sequence $(s_n(T))_{n\in\mathbb{N}}$ is called an $s-$number sequence if the following conditions are satisfied:

$(S1)$ $\|T\|=s_1(T)\geq s_2(T)\geq\ldots\geq 0$ for every $T\in L(E,F)$.

$(S2)$ $s_{m+n-1}(S+T)\leq s_m(S)+s_n(T)$ for every $S,T\in L(E,F)$ and $m,n\in\mathbb{N}$.

$(S3)$ $s_n(RST)\leq\|R\|\,s_n(S)\,\|T\|$ for some $R\in L(F,F_0)$, $S\in L(E,F)$ and $T\in L(E_0,E)$, where E_0,F_0 are arbitrary Banach spaces.

$(S4)$ If $rank(T)\leq n$, then $s_n(T)=0$.

$(S5)$ $s_n(I:l_2^n\to l_2^n)=1$, where I denotes the identity operator on the $n-$dimensional Hilbert space l_2^n

$s_n(T)$ denotes the n^{th} $s-$number of the operator T [12].

For $T\in L(E,F)$, $a_n(T)$, the n^{th} approximation number, is defined in [1] as

$$a_n(T)=\inf\{\ \|T-A\|:A\in L(E,F),\ rank(A)<n\ \}$$

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which is an example of $s-$number.

Approximation numbers satisfy the following relations:

$(a1)$ $a_{2n-1}(T_1+T_2) \leq a_n(T_1)+a_n(T_2)$ for $n=1,2,\ldots$

$(a2)$ $a_n(\alpha T)=|\alpha|\,a_n(T)$, where α is a scalar.

Let ℓ_∞ be the space of all bounded real sequences and $K \subset \ell_\infty$ be the set of all sequences $x=(x_k)$ such that $card\{i\in\mathbb{N}:\ x_i\neq 0\}<n$ and $x_1\geq x_2\geq\ldots\geq 0$.

A function $\phi : K\to\mathbb{R}$ is called symmetric norming function, if it satisfies the following conditions for $x=(x_k)\in K$ and $y=(y_k)\in K$:

$(\phi1)$ $\phi(x)>0$ for all $x\neq(0,0,\ldots,0,\ldots)$.

$(\phi2)$ $\phi(\alpha x)=\alpha\phi(x)$ for all $\alpha\geq 0$.

$(\phi3)$ $\phi(x+y)\leq\phi(x)+\phi(y)$.

$(\phi4)$ $\phi(1,0,0,\ldots)=1$.

$(\phi5)$ If the inequality $\sum\limits_{i=1}^{k}x_i\leq\sum\limits_{i=1}^{k}y_i$ holds for $k=1,2,\ldots,$ then $\phi(x)\leq\phi(y)$ holds.

In [16],[19], it is given that the function $\phi_{(p)}$ defined as

$$\phi_{(p)} : (x_i)\in K\to\left(\phi\left(\{x_i^p\}\right)\right)^{\frac{1}{p}},1\leq p<\infty$$

is also a symmetric norming function for all symmetric norming functions ϕ.

For more results related to symmetric norming functions, we refer to [7],[13],[14],[16],[20].

By using the properties of a symmetric norming function ϕ and the sequence $(a_n(T))$, the class $L_\phi(E,F)$ is defined in [13] and [15] as follows

$$L_\phi(E,F)=\{T\in L(E,F):\phi(\{a_n(T)\})<\infty\}.$$

By using the properties of the function ϕ, $(a1)$ and $(a2)$, one can see that $\|T\|_\phi=\phi(\{a_n(T)\})$ and $\|T\|_{\phi_{(p)}}=\phi_{(p)}(\{a_n(T)\})$ are quasinorms.

In [1] the class of ℓ^p-type mappings $L_p(E,F)$ has introduced by Pietsch as follows

$$L_p(E,F)=\left\{T\in L(E,F):\sum_{n=1}^{\infty}a_n^p(T)<\infty,\quad\text{for } 0<p<\infty\right\}.$$

Iseki [8] defined the class of Stolz mappings as follows

(1.1)
$$L_{STOL,p}(E,F)=\left\{T\in L(E,F):\sum_{n=1}^{\infty}\left[\frac{1}{\alpha_1+\alpha_2+\ldots+\alpha_n}\sum_{i=1}^{n}\alpha_i a_i(T)\right]^p<\infty\right\},$$

$0<p<\infty$ where $\alpha_1\geq\alpha_2\geq\ldots>0$.

If we take $\alpha_1=\alpha_2=\ldots=1$ in (1.1), we obtain the class of Cesaro $p-$type mappings (see [6]).

If we have $\lim\limits_{n\to\infty}\alpha_n\neq 0$, it is proved in [9] that the result $L_{STOL,p}(E,F)=L_p(E,F)$ holds.

In [10], the authors gave some properties of $L_{STOL,p}(E,F)$.

In this paper, we introduce the class of generalized Stolz mappings. Also, we prove that the class of ℓ^p-type mappings are included in the class of generalized Stolz mappings and we define a new quasi-norm equivalent to $\|T\|_{\phi_{(p)}}$. Further, we present some properties of the class of generalized Stolz mappings.

2. Main Results

Throughout this study, (u_n) and (w_n) are sequences of non-negative real numbers such that $u_1 \geq u_2 \geq ... \geq u_n \geq \ldots$ and $w_1 \leq w_2 \leq ... \leq w_n \leq ...$ and $w_n \leq n \leq \frac{w_n}{u_n}$. In this study, we define the class of generalized Stolz mappings $L_{GSTOL,p}(E,F)$ as

$$L_{GSTOL,p}(E,F) = \left\{ T \in L(E,F) : \sum_{n=1}^{\infty} \left[\frac{1}{w_n} \sum_{i=1}^{n} u_i a_i(T) \right]^p < \infty \right\}, \quad 0 < p < \infty.$$

In the following theorem, we prove that ℓ^p- type mappings are included in the class of generalized Stolz mappings.

Theorem 2.1. *If $\lim_{n\to\infty} u_n = u \neq 0$, the class of ℓ^p- type mappings are included in the class of generalized Stolz mappings for $1 \leq p < \infty$.*

Proof. Let $T \in L(E,F)$ and (u_n), (w_n) be sequences of non-negative real numbers. Assume that $\lim_{n\to\infty} u_n \neq 0$. Then, we can write

$$\sum_{n=1}^{\infty} \left(\frac{1}{w_n} \sum_{i=1}^{n} u_i a_i(T) \right)^p \leq \sum_{n=1}^{\infty} \left(\frac{u_1}{n u_n} \sum_{i=1}^{n} a_i(T) \right)^p = \left(\frac{u_1}{u} \right)^p \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^{n} a_i(T) \right)^p.$$

Since $\sum_{n=1}^{\infty} a_n^p(T) < \infty$, we obtain from Hardy's inequality that

$$\left(\frac{u_1}{u} \right)^p \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^{n} a_i(T) \right)^p \leq \left(\frac{u_1}{u} \right)^p \left(\frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} a_n^p(T) < \infty.$$

It follows that

$$\sum_{n=1}^{\infty} \left(\frac{1}{w_n} \sum_{i=1}^{n} u_i a_i(T) \right)^p < \infty.$$

Hence the class of ℓ^p-type mappings are included in the class of generalized Stolz type mappings for $1 \leq p < \infty$. □

Theorem 2.2. *Let $\lim_{n\to\infty} u_n \neq 0$, then the quasi-norm $\|T\|_{\phi_{(p)}}$ is equivalent to*

$$\|T\|_{\phi_{(p)}}^{\gamma} = \phi_{(p)} \left(\left\{ \frac{1}{w_n} \sum_{i=1}^{n} u_i a_i(T) \right\} \right) \text{ for } 1 \leq p < \infty.$$

Proof. Since the sequences (u_n) and $(a_n(T))$ are decreasing, we can write

$$\frac{1}{n} n u_n a_n(T) \leq \frac{1}{w_n} \sum_{i=1}^{n} u_i a_i(T) \leq \frac{1}{n u_n} u_1 \sum_{i=1}^{n} a_i(T).$$

Summing from $n=1$ to k, we get

$$\sum_{n=1}^{k}(u_n a_n(T))^p \leq \sum_{n=1}^{k}\left(\frac{1}{w_n}\sum_{i=1}^{n}u_i a_i(T)\right)^p \leq \sum_{n=1}^{k}\left(\frac{u_1}{n u_n}\sum_{i=1}^{n}a_i(T)\right)^p .$$

If $\lim_{n\to\infty} u_n = u \neq 0$, then we obtain

$$u^p\sum_{n=1}^{k}a_n^p(T) \leq \sum_{n=1}^{k}\left(\frac{1}{w_n}\sum_{i=1}^{n}u_i a_i(T)\right)^p \leq \left(\frac{u_1}{u}\right)^p\sum_{n=1}^{k}\left(\frac{1}{n}\sum_{i=1}^{n}a_i(T)\right)^p$$

for every $k\in\mathbb{N}$. By using Hardy's inequality, we get

$$u^p\sum_{n=1}^{k}a_n^p(T) \leq \sum_{n=1}^{k}\left(\frac{1}{w_n}\sum_{i=1}^{n}u_i a_i(T)\right)^p \leq \left(\frac{u_1}{u}\right)^p\left(\frac{p}{p-1}\right)^p\sum_{n=1}^{k}a_n^p(T)$$

for every $k\in\mathbb{N}$. From the properties of the function ϕ, we obtain that

$$u\left\|T\right\|_{\phi_{(p)}} \leq \left\|T\right\|_{\phi_{(p)}}^{\gamma} \leq \left(\frac{u_1}{u}\right)\left(\frac{p}{p-1}\right)\left\|T\right\|_{\phi_{(p)}} .$$

□

Corollary 2.1. *For the particular case, if we choose $u_i=\alpha_i$ and $w_n=\alpha_1+\alpha_2+...+\alpha_n$ in Theorem 2, then we obtain Theorem 1.4 in* [12], *where $\alpha_1\leq 1$. If we take $u_i=1$ and $w_n=n$ in Theorem 2, then we obtain Proposition 1.2 in* [12].

Theorem 2.3. *If $S\in L_{GSTOL,s,q}(E,F)$ and $T\in L_{GSTOL,t,r}(E,F)$, then*

$ST\in L_{GSTOL,p}(E,F)$, where $1=\frac{1}{s}+\frac{1}{t}$, $\frac{1}{p}=\frac{1}{q}+\frac{1}{r}$, $1\leq p<\infty$ and

$$L_{GSTOL,s,q}(E,F)=\left\{T\in L(E,F):\left(\sum_{n=1}^{\infty}\left(\frac{1}{w_n}\sum_{i=1}^{n}u_i a_i^s(T)\right)^{\frac{q}{s}}\right)^{\frac{1}{q}}\right\}<\infty .$$

Proof. We know from [17] that

$$\sum_{i=1}^{n}a_i(ST)\leq 2\sum_{i=1}^{n}\left[a_i(S)a_i(T)\right]\ n=1,2,.... \tag{2.1}$$

By using the inequality (2.1) and Hölder's inequality we obtain that;

$$\begin{aligned}
\|ST\|_{GSTOL,p} &= \left(\sum_{n=1}^{\infty}\left(\frac{1}{w_n}\sum_{i=1}^{n}u_i a_i\left(ST\right)\right)^p\right)^{\frac{1}{p}} \\
&\leq 2\left(\sum_{n=1}^{\infty}\left(\frac{1}{w_n}\sum_{i=1}^{n}u_i a_i\left(S\right)a_i(T)\right)^p\right)^{\frac{1}{p}} \\
&\leq 2\left(\sum_{n=1}^{\infty}\left(\frac{\left(\sum_{i=1}^{n}u_i a_i^s\left(S\right)\right)^{\frac{1}{s}}}{w_n^{\frac{1}{s}}}\frac{\left(\sum_{i=1}^{n}u_i a_i^t\left(T\right)\right)^{\frac{1}{t}}}{w_n^{\frac{1}{t}}}\right)^p\right)^{\frac{1}{p}} \\
&\leq 2\left(\sum_{n=1}^{\infty}\left(\left(\frac{\sum_{i=1}^{n}u_i a_i^s\left(S\right)}{w_n}\right)^{\frac{1}{s}}\left(\frac{\sum_{i=1}^{n}u_i a_i^t\left(T\right)}{w_n}\right)^{\frac{1}{t}}\right)^p\right)^{\frac{1}{p}} \\
&\leq 2\left(\sum_{n=1}^{\infty}\left(\frac{\sum_{i=1}^{n}u_i a_i^s\left(S\right)}{w_n}\right)^{\frac{q}{s}}\right)^{\frac{1}{q}}\left(\sum_{n=1}^{\infty}\left(\frac{\sum_{i=1}^{n}u_i a_i^t\left(T\right)}{w_n}\right)^{\frac{r}{t}}\right)^{\frac{1}{r}} \\
&< \infty,
\end{aligned}$$

where $1=\frac{1}{s}+\frac{1}{t}$, $\frac{1}{p}=\frac{1}{q}+\frac{1}{r}$. Hence $ST\in L_{GSTOL,p}(E,F)$. □

In the following theorem, as in [10] we prove the tensor product stability of this new class if the sequence (u_n) satisfies the Tita property $u_{n^2}\leq\frac{C}{n}u_n$, for every $n=1,2,\ldots$.

Theorem 2.4. *The class $L_{GSTOL,p}\left(E,F\right)$ is tensor product stable for all tensor norms, if the sequence (u_n) satisfies $u_{n^2}\leq\frac{C}{n}u_n$ for every $n=1,2,\ldots$ where C is a constant (depending only on the sequence $u=(u_1,u_{2},\ldots)$).*

Proof. We know that the inequality

$$\sum_{n=1}^{k}u_n a_n\left(S\otimes T\right)\leq C\left(u\right)\sum_{n=1}^{k}u_n\left(a_n\left(S\right)+a_n\left(T\right)\right) \tag{2.2}$$

holds for $(fixed)$ $S, T \in L(E, F)$. (If we take $p = 1$ in [11], we get the inequality (2.2).)

If we use (2.2) and Minkowski inequality, we obtain that

$$\begin{aligned}
\|S \otimes T\|_{GSTOL,p} &= \left(\left(\sum_{n=1}^{\infty}\left(\frac{1}{w_n}\sum_{i=1}^{n} u_i a_i (S \otimes T)\right)^p\right)^{\frac{1}{p}}\right) \\
&\leq C(u)\left(\left(\sum_{n=1}^{\infty}\left(\frac{1}{w_n}\sum_{i=1}^{n} u_i \left[a_i(S) + a_i(T)\right]\right)^p\right)^{\frac{1}{p}}\right) \\
&= C(u)\left(\sum_{n=1}^{\infty}\left(\frac{1}{w_n}\sum_{i=1}^{n} u_i a_i(S) + \frac{1}{w_n}\sum_{i=1}^{n} u_i a_i(T)\right)^p\right)^{\frac{1}{p}} \\
&\leq C(u,p) \\
&\times \left(\left(\sum_{n=1}^{\infty}\left(\frac{1}{w_n}\sum_{i=1}^{n} u_i a_i(S)\right)^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty}\left(\frac{1}{w_n}\sum_{i=1}^{n} u_i a_i(T)\right)^p\right)^{\frac{1}{p}}\right) \\
&\leq C(u,p)\left[\|S\|_{GSTOL,p} + \|T\|_{GSTOL,p}\right] < \infty.
\end{aligned}$$

This completes the proof. □

3. Conclusion

In this study we defined a new Stolz mapping class by generalizing the class of Stolz mapping in reference [8] and a new quasi-norm. We proved that the class of l^p- type mappings are included in the new mappings. Also, we gave some properties of this new class.

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