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Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 73, Number 2, Pages 391–398 (2024)
DOI:10.31801/cfsuasmas.1380675
ISSN 1303-5991 E-ISSN 2618-6470

*Research Article; Received: October 25, 2023; Accepted: December 19, 2023*

# GENERAL LOGARITHMIC CONTROL MODULO AND TAUBERIAN REMAINDER THEOREMS

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Abstract. Let $\lambda = (\lambda_n)$ be a nondecreasing sequence of positive numbers such that $\lambda_n \to \infty$. A sequence $(\xi_n)$ is called $\lambda$-bounded if

$$\lambda_n(\xi_n - \alpha) = O(1)$$

with the limit $\lim_{n\to\infty} \xi_n = \alpha$. In this work, we obtain several Tauberian remainder theorems on $\lambda$-bounded sequences for the logarithmic summability method with help of general logarithmic control modulo of the oscillatory behavior. Tauber conditions in our main results are on the generator sequence and the general logarithmic control modulo.

## 1. Introduction

Let $\xi = (\xi_n)$ be a sequence of real numbers. Throughout this work, the notation of $(\xi_n) = O(1)$ means that the sequence of $(\xi_n)$ is bounded for large enough $n$. The $(C,1)$ mean of $(\xi_n)$ is defined by $\sigma_n^{(1)}(\xi) = \frac{1}{n+1}\sum_{k=0}^{n} \xi_k$ and the logarithmic mean of $(\xi_n)$ is defined by $\ell_n^{(1)}(\xi) = \frac{1}{\gamma_n}\sum_{k=0}^{n} \frac{\xi_k}{k+1}$, where $\gamma_n = \sum_{k=0}^{n} \frac{1}{k+1} \sim \log n$, where for two sequences $(u_n)$ and $(v_n)$ of positive numbers, we write $u_n \sim v_n$ if $\lim_{n\to\infty} \frac{u_n}{v_n} = 1$. A sequence $(\xi_n)$ is said to be $(C,1)$ summable to a finite number $\alpha$ if the limit

$$\lim_{n\to\infty} \sigma_n^{(1)}(\xi) = \alpha \tag{1}$$

2020 *Mathematics Subject Classification.* 40E05, 40G05.
*Keywords.* Tauberian remainder theorem, $\lambda$-bounded sequence, logarithmic summability method, logarithmic general control modulo.
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exists and we say that a sequence $(\xi_n)$ is logarithmic summable to a finite number $\alpha$, if

$$\lim_{n\to\infty} \ell_n^{(1)}(\xi) = \alpha \tag{2}$$

[1]. It is well known that if a sequence $(\xi_n)$ is convergent, then (1) and (2) are exist. In other words, these two methods are regular methods. Also the existence of (1) implies the existence of (2). However the converse implications are not always true. For example the sequence $(\xi_n) = (-1)^n(2n+1)$ is neither ordinary convergent nor $(C,1)$ convergent. But it is logarithmic convergent to 0.

For a sequence $(\xi_n)$, we have the following identity:

$$\xi_n - \ell_n^{(1)}(\xi) = v_n^{(0)}(\Delta\xi), \tag{3}$$

where $v_n^{(0)}(\Delta\xi) = \dfrac{1}{\gamma_n}\displaystyle\sum_{k=1}^{n}\gamma_{k-1}(\Delta\xi_k)$. The identity (3) is called the logarithmic Kronecker identity and the sequence $(v_n^{(0)}(\Delta\xi))$ is called the generator sequence of $(\xi_n)$. For each integer $k \geq 1$, $\ell_n^{(k)}(\xi)$ is defined by

$$\ell_n^{(k)}(\xi) = \frac{1}{\gamma_n}\sum_{t=0}^{n}\frac{\ell_t^{k-1}(\xi)}{t+1}, \tag{4}$$

where $\ell_n^{(0)}(\xi) = \xi_n$ and $\ell_n^{(1)}(\xi) = \ell_n(\xi)$.

If we get the logarithmic mean of the sequence of $(v_n^{(0)}(\Delta\xi))$, then we obtain

$$\ell_n^{(1)}(v^{(0)}(\Delta\xi)) = \frac{1}{\gamma_n}\sum_{k=0}^{n}\frac{v_k^{(0)}(\Delta\xi)}{k+1} = v_n^{(1)}(\Delta\xi).$$

By getting the logarithmic mean of $(v_n^{(1)}(\Delta\xi))$, then we obtain

$$\ell_n(v^{(1)}(\Delta\xi)) = \frac{1}{\gamma_n}\sum_{k=0}^{n}\frac{v_k^{(1)}(\Delta\xi)}{k+1} = v_n^{(2)}(\Delta\xi).$$

Continuing in this way, we obtain the following sequence:

$$\ell_n(v^{(m-1)}(\Delta\xi)) = \frac{1}{\gamma_n}\sum_{k=0}^{n}\frac{v_k^{(m-1)}(\Delta\xi)}{k+1} = v_n^{(m)}(\Delta\xi),$$

for $m \geq 1$. Hence, all these given sequences can be written as follows:

$$v_n^{(m)}(\Delta\xi) = \begin{cases} \frac{1}{\gamma_n}\displaystyle\sum_{k=0}^{n}\frac{v_k^{(m-1)}(\Delta\xi)}{k+1}, & m \geq 1 \\ v_n(\Delta\xi), & m = 0. \end{cases}$$

For a sequence $(\xi_n)$, classical logarithmic control modulo is defined by

$$\omega_n^{(0)}(\xi) = (n+1)\gamma_{n-1}\Delta\xi_n. \tag{5}$$

The general logarithmic control modulo of the oscillatory behavior of integer order $m \geq 1$ of a sequence $(\xi_n)$ is defined by

$$\omega_n^{(m)}(\xi) \quad = \quad \omega_n^{(m-1)}(\xi) - \ell_n^{(1)}(\omega^{(m-1)}(\xi)). \tag{6}$$

Assume that $\lambda = (\lambda_n)$ be a nondecreasing sequence of positive numbers such that $\lambda_n \to \infty$. A sequence $(\xi_n)$ is called bounded with the rapidity $(\lambda_n)$ if

$$\lambda_n(\xi_n - \alpha) = O(1) \tag{7}$$

with $\lim_{n\to\infty} \xi_n = \alpha$. Shortly, we say that the sequence $(\xi_n)$ is $\lambda$-bounded and the set of all $\lambda$-bounded sequences is denoted by $m^\lambda$.

Also a sequence $(\xi_n)$ is called $\lambda$-bounded by logarithmic method of summability if

$$\lambda_n(\ell_n^{(1)}(\xi) - \alpha) = O(1) \tag{8}$$

with $\lim_{n\to\infty} \ell_n^{(1)}(\xi) = \alpha$. The set of all logarithmic $\lambda$-bounded sequences is denoted by $(\ell, m^\lambda)$.

Tauberian theory for the logarithmic method have been studied by various authors. A number of authors such as Kwee [2] and Ishiguro [3–5] obtained some Tauberian theorems for the logarithmic method and generalized some classical Tauberian theorems to logaritmic method. Móricz [6] presented some classical type Tauberian theorems for logarithmic method of sequences and established some Tauberian theorems by introducing logarithmic summability method of integrals.

Later, Okur and Totur [7,8] introduced general logarithmic control modulo and classical logarithmic control modulo for logarithmic method of integrals. And they extanded Tuaberian theorems which are given for $(C,1)$ method. Sezer and Çanak [9,10] investigated new Tauberian conditions with help of general logarithmic control modulo for logarithmic method of sequences and proved some Theorems for logarithmic method of power series.

On the other hand many researchers studied Tauberian remainder theorems for some summability methods such as Kangro [11] and Tammeraid [12–14] after Kangro's work [15] in which the author introduced the concepts of Tauberian remiander theorems by using summability with given rapidity $\lambda$. Meronen and Tammeraid [16] presented some Tauberian remainder theorems for $(C,1)$ summability method from a new perspective. In this work, they used the concept of general control modulo which was defined in [17]. Later Sezer and Çanak [18,19] and Totur and Okur [20,21] proved some results for weighted mean, Hölder and $(C,\alpha)$ summability methods. They also benefited from the concept of general control modulo to obtain Tauberian remainder theorems in these studies.

We aim in this paper to prove some Tauberian remainder theorems for the logarithmic summability method. Firstly, we prove 3 lemmas in section 2 and in each lemma, the relationship between the different-order general logarithmic control modulo of a sequence and its different-order logarithmic means is given. After that, the main theorems are presented in the next section. In the main theorems, we obtain $\lambda$-boundedness of a sequence from its logarithmic $\lambda$-boundedness by using conditions on generator sequence and general logarithmic control modulo of the given sequence.

## 2. Auxilary Results

For the proofs of our main results, we require the following lemmas.

**Lemma 1.** *The following equality is valid.*

$$\omega_n^{(1)}(\xi) = \omega_n^{(0)}(\xi) - \xi_n + \ell_n^{(1)}(\xi). \tag{9}$$

*Proof.* Taking $m = 1$ in (6) and using (5), we get

$$\begin{aligned} \omega_n^{(1)}(\xi) &= \omega_n^{(0)}(\xi) - \ell_n^{(1)}(\omega^{(0)}(\xi)) \\ &= \omega_n^{(0)}(\xi) - \frac{1}{\gamma_n}\sum_{k=0}^{n}\frac{(k+1)\gamma_{k-1}\Delta\xi_k}{k+1} \\ &= \omega_n^{(0)}(\xi) - v_n^{(0)}(\Delta\xi). \end{aligned}$$

Using (3) in the last equality, we obtain

$$\omega_n^{(1)}(\xi) = \omega_n^{(0)}(\xi) - \xi_n + \ell_n^{(1)}(\xi).$$

□

**Lemma 2.** *The following equality is valid.*

$$\omega_n^{(2)}(\xi) = \omega_n^{(0)}(\xi) - 2\xi_n + 3\ell_n^{(1)}(\xi) - \ell_n^{(2)}(\xi). \tag{10}$$

*Proof.* If we take $m = 2$ in (6), we obtain

$$\omega_n^{(2)}(\xi) = \omega_n^{(1)}(\xi) - \ell_n^{(1)}(\omega^{(1)}(\xi)).$$

Using (9), we get

$$\begin{aligned} \omega_n^{(2)}(\xi) &= \omega_n^{(0)}(\xi) - \xi_n + \ell_n^{(1)}(\xi) - \ell_n^{(1)}(\omega^{(0)}(\xi) - \xi + \ell^{(1)}(\xi)) \\ &= \omega_n^{(0)}(\xi) - \xi_n + \ell_n^{(1)}(\xi) - \frac{1}{\gamma_n}\sum_{k=0}^{n}\frac{1}{k+1}(\omega_k^{(0)}(\xi) - \xi_k + \ell_k^{(1)}(\xi)) \end{aligned}$$

From (4), we get

$$\omega_n^{(2)}(\xi) = \omega_n^{(0)}(\xi) - \xi_n + \ell_n^{(1)}(\xi) - v_n^{(0)}(\Delta\xi) + \ell_n^{(1)}(\xi) - \ell_n^{(2)}(\xi).$$

By (3), we conclude that

$$\omega_n^{(2)}(\xi) = \omega_n^{(0)}(\xi) - 2\xi_n + 3\ell_n^{(1)}(\xi) - \ell_n^{(2)}(\xi).$$

□

**Lemma 3.** *The following equality is valid.*

$$\omega_n^{(3)}(\xi) = \omega_n^{(0)}(\xi) - 3\xi_n + 6\ell_n^{(1)}(\xi) - 4\ell_n^{(2)}(\xi) + \ell_n^{(3)}(\xi). \tag{11}$$

*Proof.* By taking $m = 3$ in (6), we have

$$\omega_n^{(3)}(\xi) = \omega_n^{(2)}(\xi) - \ell_n^{(1)}(\omega^{(2)}(\xi)).$$

From (10), we obtain

$$\begin{aligned}\omega_n^{(3)}(\xi) &= \omega_n^{(0)}(\xi) - 2\xi_n + 3\ell_n^{(1)}(\xi) - \ell_n^{(2)}(\xi)\\ &\quad -\ell_n^{(1)}(\omega^{(0)}(\xi) - 2\xi + 3\ell^{(1)}(\xi) - \ell^{(2)}(\xi))\\ &= \omega_n^{(0)}(\xi) - 2\xi_n + 3\ell_n^{(1)}(\xi) - \ell_n^{(2)}(\xi)\\ &\quad -\frac{1}{\gamma_n}\sum_{k=0}^{n}\frac{1}{k+1}(\omega_k^{(0)}(\xi) - 2\xi_k + 3\ell_k^{(1)}(\xi) - \ell_k^{(2)}(\xi)).\end{aligned}$$

Now, using (4) in the last equality, we get

$$\begin{aligned}\omega_n^{(3)}(\xi) &= \omega_n^{(0)}(\xi) - 2\xi_n + 3\ell_n^{(1)}(\xi) - \ell_n^{(2)}(\xi)\\ &\quad -v_n^{(0)}(\Delta\xi) + 2\ell_n^{(1)}(\xi) - 3\ell_n^{(2)}(\xi) + \ell_n^{(3)}(\xi).\end{aligned}$$

Finally, from definition of the logarithmic Kronecker identity, we have

$$\omega_n^{(3)}(\xi) = \omega_n^{(0)}(\xi) - 3\xi_n + 6\ell_n^{(1)}(\xi) - 4\ell_n^{(2)}(\xi) + \ell_n^{(3)}(\xi).$$

□

## 3. Main Results

**Theorem 1.** *Let $\xi$ is $\lambda$-bounded by the $(\ell, 1)$ method. If*

$$\lambda_n v_n^{(0)}(\Delta\xi) = O(1), \tag{12}$$

*then $\xi$ is $\lambda$-bounded.*

*Proof.* Because of $\xi$ is $\lambda$-bounded by the $(\ell, 1)$ method, we have

$$\lambda_n\left(\ell_n^{(1)}(\xi) - \alpha\right) = O(1). \tag{13}$$

By the equality

$$\lambda_n(\xi_n-\alpha)=\lambda_n\left(\ell_n^{(1)}(\xi)-\alpha+\xi_n-\ell_n^{(1)}(\xi)\right),$$

we obtain

$$\lambda_n(\xi_n-\alpha)=\lambda_n\left(\ell_n^{(1)}(\xi)-\alpha\right)+\lambda_n v_n^{(0)}(\Delta\xi)$$

using (3). By combining (12) and (13) with the last equailty, we get

$$\lambda_n(\xi_n-\alpha)=O(1).$$

So, $\xi$ is $\lambda$-bounded and proof is completed. □

**Theorem 2.** *Let $\xi$ is $\lambda$-bounded by the $(\ell,1)$ method. If*

$$\lambda_n\omega_n^{(0)}(\xi)=O(1) \tag{14}$$

*and*

$$\lambda_n\omega_n^{(1)}(\xi)=O(1), \tag{15}$$

*then $\xi$ is $\lambda$-bounded.*

*Proof.* Benefit from Lemma 1, we get the following equality:

$$\lambda_n(\xi_n-\alpha)=\lambda_n\left(\ell_n^{(1)}(\xi)-\alpha-\omega_n^{(1)}(\xi)+\xi_n-\ell_n^{(1)}(\xi)+\omega_n^{(1)}(\xi)\right).$$

So, we conclude that

$$\lambda_n(\xi_n-\alpha)=-\lambda_n\omega_n^{(1)}(\xi)+\lambda_n\left(\ell_n^{(1)}(\xi)-\alpha\right)+\lambda_n\omega_n^{(0)}(\xi).$$

From $\lambda$-boundedness by the $(\ell,1)$ method, we have (13). Taking (14) and (15) into account we obtain

$$\lambda_n(\xi_n-\alpha)=O(1).$$

This result completed the proof. □

**Theorem 3.** *Let $\xi$ is $\lambda$-bounded by the $(\ell,1)$ method and the condition (14) is satisfied. If*

$$\lambda_n\omega_n^{(2)}(\xi)=O(1) \tag{16}$$

*and*

$$\lambda_n\left(\ell_n^{(2)}(\xi)-\alpha\right)=O(1), \tag{17}$$

*then $\xi$ is $\lambda$-bounded.*

*Proof.* Using Lemma 2, we obtain the equality of

$$\lambda_n(\xi_n-\alpha)=\lambda_n\left(-\omega_n^{(2)}(\xi)+\omega_n^{(0)}(\xi)-\alpha-\xi_n-\ell_n^{(2)}(\xi)+3\ell_n^{(1)}(\xi)\right).$$

Therefore we get the following result:

$$2\lambda_n(\xi_n-\alpha)=-\lambda_n\omega_n^{(2)}(\xi)+\lambda_n\omega_n^{(0)}(\xi)-\lambda_n\left(\ell_n^{(2)}(\xi)-\alpha\right)+3\lambda_n\left(\ell_n^{(1)}(\xi)-\alpha\right).$$

Using (13), (14), (16) and (17) we get the result of

$$\lambda_n\left(\xi_n-\alpha\right)=O(1).$$

It means that $\xi$ is $\lambda$-bounded. □

**Theorem 4.** *Let $\xi$ is $\lambda$-bounded by the $(\ell,1)$ method and the conditions (14) and (17) are satisfied. If*

$$\lambda_n\omega_n^{(3)}(\xi)=O(1) \tag{18}$$

*and*

$$\lambda_n\left(\ell_n^{(3)}(\xi)-\alpha\right)=O(1), \tag{19}$$

*then $\xi$ is $\lambda$-bounded.*

*Proof.* With the Lemma 3 we obtain the following equality:

$$\lambda_n\left(\xi_n-\alpha\right)=\lambda_n\left(-\omega_n^{(3)}(\xi)+\omega_n^{(0)}(\xi)+\ell_n^{(3)}(\xi)-4\ell_n^{(2)}(\xi)+6\ell_n^{(1)}(\xi)-2\xi_n-\alpha\right).$$

Then it follows that

$$\begin{aligned}3\lambda_n\left(\xi_n-\alpha\right) &= -\lambda_n\omega_n^{(3)}(\xi)+\lambda_n\omega_n^{(0)}(\xi)+\lambda_n\left(\ell_n^{(3)}(\xi)-\alpha\right)\\ &\quad -4\lambda_n\left(\ell_n^{(2)}(\xi)-\alpha\right)+6\lambda_n\left(\ell_n^{(1)}(\xi)-\alpha\right).\end{aligned}$$

If we combine (13), (14), (17), (18) and (19), we have the equality

$$\lambda_n\left(\xi_n-\alpha\right)=O(1).$$

Therefore we obtain that $\xi$ is $\lambda$-bounded. □

**Declaration of Competing Interests** The author declares that there is no competing interest regarding the publication of this paper.

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