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AN APPLICATION OF CARATHÉODORY FUNCTIONS

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Abstract. Let $\mathcal{P}(\alpha)$ be the class of functions $p(z)$ which are Carathéodory functions of order $\alpha\,(0 \leqq \alpha < 1)$ in the open unit disk $\mathbb{U}$. In view of the extremal function $L_0(\alpha; z)$ for the class $\mathcal{P}(\alpha)$, a new class $\mathcal{Q}(\beta)$ of functions $q(z)$ is introduced. The object of the present paper is to discuss some interesting coefficient inequalities for $q(z)$ in the class $\mathcal{Q}(\beta)$.

1. Introduction

Let $\mathcal{P}$ be the class of functions $p(z)$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Also, let $\mathcal{P}(\alpha)$ denote the subclass of $\mathcal{P}$ consisting of functions $p(z)$ which satisfy

$$\operatorname{Re}(p(z)) > \alpha \qquad (z \in \mathbb{U}) \tag{1.2}$$

for some real $\alpha\,(0 \leqq \alpha < 1)$. We say that $p(z)$ is a Carathéodory function in $\mathbb{U}$ if $p(z) \in \mathcal{P}(0)$ (see [1], [2], [3]). Therefore, we call that $p(z)$ is a Carathéodory function of order α in $\mathbb{U}$ if $p(z) \in \mathcal{P}(\alpha)$ (see [4], [5], [6]). It is well known that a function $L_0(\alpha; z)$ given by

$$L_0(\alpha; z) = \frac{1 + (1 - 2\alpha)z}{1 - z} = 1 + 2(1 - \alpha) \sum_{n=1}^{\infty} z^n \tag{1.3}$$

is the extremal function for the class $\mathcal{P}(\alpha)$. For this extremal function $L_0(\alpha; z)$ given by (1.3), we consider a function $q_0(\beta; z)$ defined by

$$q_0(\beta; z) = \frac{1 + (1 - 2\alpha)z}{1 - \sqrt{z}} = 1 + \sqrt{z} + 2(1 - \alpha) \sum_{n=1}^{\infty} z^{\frac{n+1}{2}} \qquad (z \in \mathbb{U}), \tag{1.4}$$

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where we consider the principal value for $\sqrt{z}$ and

$$\beta = \begin{cases} 3\alpha - 1 & (0 \leqq \alpha < \frac{1}{2}) \\ \\ 1 - \alpha & (\frac{1}{2} \leqq \alpha < 1). \end{cases} \tag{1.5}$$

Then, it follows that

$$\operatorname{Re}\left(q_0(\beta; z)\right) = \operatorname{Re}\left(\frac{1 + (1 - 2\alpha)z}{1 - \sqrt{z}}\right) \tag{1.6}$$

$$= \alpha - (1 - 2\alpha) cos\,\frac{\theta}{2} > \beta$$

for $z = e^{i\theta}$.
From the above, we define the class $\mathcal{Q}$ of functions $q(z)$ of the form

$$q(z) = 1 + \sum_{n=1}^{\infty} a_{\frac{n}{2}} z^{\frac{n}{2}} \tag{1.7}$$

which are analytic in $\mathbb{U}$, where we consider the principal value for $\sqrt{z}$. Further, if $q(z)$ given by (1.7) satisfies

$$\operatorname{Re}\left(q(z)\right) > \beta \quad (z \in \mathbb{U}) \tag{1.8}$$

for some real β, then we say that $q(z) \in \mathcal{Q}(\beta)$, where β is given by (1.5).
Then we know that the function $q(z)$ given by (1.4) is the extremal function for the class $\mathcal{Q}(\beta)$.

2. Coefficient inequalities

We try to discuss some coefficient inequalities for $q(z)$ belonging to the class $\mathcal{Q}(\beta)$.

Theorem 2.1 *If a function $q(z)$ given by (1.7) satisfies*

$$\sum_{n=1}^{\infty} \left| a_{\frac{n}{2}} \right| \leqq \gamma \tag{2.1}$$

for some real γ defined by

$$\gamma = 1 - \beta = \begin{cases} 2 - 3\alpha & (0 \leqq \alpha < \frac{1}{2}) \\ \\ \alpha & (\frac{1}{2} \leqq \alpha < 1), \end{cases} \tag{2.2}$$

then $q(z) \in \mathcal{Q}(\beta)$. The equality in (2.1) holds true for $q(z)$ given by

$$q(z) = 1 + \sum_{n=1}^{\infty} \frac{\gamma\varepsilon}{n(n+1)} z^{\frac{n}{2}} \quad (|\varepsilon| = 1). \tag{2.3}$$

Proof We note that $q(z)$ belongs to the class $\mathcal{Q}(\beta)$ if $q(z)$ satisfies

$$|q(z) - 1| < 1 - \beta \quad (z \in \mathbb{U}). \tag{2.4}$$

This is equivalent to

$$\left|\sum_{n=1}^{\infty} a_{\frac{n}{2}} z^{\frac{n}{2}}\right| < 1-\beta \quad (z \in \mathbb{U}). \tag{2.5}$$

Therefore, if $q(z)$ satisfies

$$\sum_{n=1}^{\infty} |a_{\frac{n}{2}}| \leqq 1-\beta = \gamma, \tag{2.6}$$

then $q(z) \in \mathcal{Q}(\beta)$. For the equality in (2.1), we consider a function $q(z)$ given by

$$q(z) = 1 + \sum_{n=1}^{\infty} \frac{\gamma\varepsilon}{n(n+1)} z^{\frac{n}{2}} \quad (|\varepsilon| = 1). \tag{2.7}$$

In this case

$$a_{\frac{n}{2}} = \frac{\gamma\varepsilon}{n(n+1)} \tag{2.8}$$

and we have that

$$\sum_{n=1}^{\infty} |a_{\frac{n}{2}}| = \gamma \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \gamma \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \gamma. \tag{2.9}$$

Thus $q(z)$ given by (2.7) satisfies the equality in (2.1).

Taking $\alpha = 0$ in Theorem 2.1, we have

Corollary 2.1 *If $q(z)$ given by (1.7) satisfies*

$$\sum_{n=1}^{\infty} |a_{\frac{n}{2}}| \leqq 2, \tag{2.10}$$

then $q(z) \in \mathcal{Q}(-1)$. The equality in (2.10) holds true for

$$q(z) = 1 + \sum_{n=1}^{\infty} \frac{2\varepsilon}{n(n+1)} z^{\frac{n}{2}} \quad (|\varepsilon| = 1). \tag{2.11}$$

Further, making $\alpha = \frac{1}{2}$ in Theorem 2.1, we have

Corollary 2.2 *If $q(z)$ given by (1.7) satisfies*

$$\sum_{n=1}^{\infty} |a_{\frac{n}{2}}| \leqq \frac{1}{2}, \tag{2.12}$$

then $q(z) \in \mathcal{Q}(\frac{1}{2})$. The equality in (2.12) holds true for

$$q(z) = 1 + \sum_{n=1}^{\infty} \frac{\varepsilon}{2n(n+1)} z^{\frac{n}{2}} \quad (|\varepsilon| = 1). \tag{2.13}$$

Next, we consider a function $q(z)$ given by (1.7) with

$$a_{\frac{n}{2}} = |a_{\frac{n}{2}}| e^{i(\pi - \frac{n}{2}\theta)} \quad (n = 1, 2, 3, \cdots) \tag{2.14}$$

for some $\theta\,(0 \leqq \theta < 2\pi)$. For such functions $q(z)$, we derive

Theorem 2.2 *Let $q(z)$ be given by* (1.7) *with* (2.14)*. Then $q(z)$ belongs to the class $\mathcal{Q}(\beta)$ if and only if*

$$\sum_{n=1}^{\infty} |a_{\frac{n}{2}}| \leqq 1-\beta, \tag{2.15}$$

where β is defined by (1.5)*. The equality in* (2.15) *is satisfied for*

$$q(z) = 1 + \sum_{n=1}^{\infty} \frac{(1-\beta)e^{i(\pi-\frac{n}{2}\theta)}}{n(n+1)} z^{\frac{n}{2}}. \tag{2.16}$$

Proof In view of Theorem 2.1, we see that $q(z)$ belongs to the class $\mathcal{Q}(\beta)$ if $q(z)$ satisfies the coefficient inequality (2.15).
On the other side, we suppose that $q(z) \in \mathcal{Q}(\beta)$. Then, letting $z = re^{i\theta}\,(0 < r < 1)$, we have that

$$\begin{aligned} \operatorname{Re}(q(z)) &= 1 + \operatorname{Re}\left(\sum_{n=1}^{\infty} a_{\frac{n}{2}} z^{\frac{n}{2}}\right) \\ &= 1 + \operatorname{Re}\left(\sum_{n=1}^{\infty} |a_{\frac{n}{2}}| r^{\frac{n}{2}} e^{i\pi}\right) \\ &= 1 - \sum_{n=1}^{\infty} |a_{\frac{n}{2}}| r^{\frac{n}{2}} > \beta. \end{aligned} \tag{2.17}$$

This shows us the inequality (2.15) for $r \to 1$. Further, it is clear that $q(z)$ given by (2.16) satisfies the equality in (2.15).

Taking $\beta = -1$ in Theorem 2.2, we have

Corollary 2.3 *Let $q(z)$ be given by* (1.7) *with* (2.14)*. Then $q(z) \in \mathcal{Q}(-1)$ if and only if*

$$\sum_{n=1}^{\infty} |a_{\frac{n}{2}}| \leqq 2. \tag{2.18}$$

The equality in (2.18) *is satisfied for*

$$q(z) = 1 + \sum_{n=1}^{\infty} \frac{2e^{i(\pi-\frac{n}{2}\theta)}}{n(n+1)} z^{\frac{n}{2}}. \tag{2.19}$$

Furthermore, letting $\beta = \dfrac{1}{2}$ in Theorem 2.2, we obtain

Corollary 2.4 *Let $q(z)$ be given by* (1.7) *with* (2.14)*. Then $q(z) \in \mathcal{Q}(\frac{1}{2})$ if and only if*

$$\sum_{n=1}^{\infty} |a_{\frac{n}{2}}| \leqq \frac{1}{2}. \tag{2.20}$$

The equality in (2.20) *holds true for*

$$q(z) = 1 + \sum_{n=1}^{\infty} \frac{e^{i(\pi - \frac{n}{2}\theta)}}{2n(n+1)} z^{\frac{n}{2}}. \tag{2.21}$$

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