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SOME PROPERTIES ASSOCIATED WITH THE BESSEL MATRIX FUNCTIONS

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Abstract. The main aim of this work is the development of some interesting properties which are associated with the Bessel matrix functions and its relationship with the hypergeometric matrix functions.

1. Introduction

There has been a significant development in the study of special matrix functions since, in the recent years, the theory of special matrix functions has been a major area of study for mathematicians. These theories have a large study area in terms of both theory and application. By the motivation of such analogues, Bessel matrix functions are discussed by the various authors such as [3, 4, 9, 10, 14, 15, 16, 17, 18, 19]. Our aim here is to present and study of some properties associated with the Bessel matrix functions which we call the matrix functions $\mathbf{R}(A, B, C, z)$ and to derive several other interesting results involving the matrix functions $\mathbf{R}(A, B, C, z)$. We derive a relationship between the hypergeometric matrix function and Bessel matrix functions and are believed to be new. Some of the formulae known so far have been shown to be the necessary consequences of the results of this paper.

In the scalar case, the function $\mathbf{R}(\lambda, \mu, \nu, z)$ associated with the Bessel functions which may be called a generalization of the Bessel functions (see [1, 2, 5, 20]) is defined as

$$\mathbf{R}(\lambda, \mu, \nu, z) = \sum_{k=0}^{\infty} \frac{(-1)^k (\lambda + k + 1)_k z^k}{k! \Gamma(\mu + k + 1) \Gamma(\nu + k + 1)}.$$

1.1. **Preliminaries.** In this subsection, we will give some useful theorems, definitions and lemmas. Throughout this work, for a matrix A in $\mathbb{C}^{N\times N}$, $\sigma(A)$ denotes the set of all the eigenvalues of A and is called its spectrum. Furthermore, I and $\mathbf{O}$ will denote the identity matrix and the null matrix in $\mathbb{C}^{N\times N}$, respectively.

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Theorem 1.1. *(Dunford and Schwartz [7]) If $f(z)$ and $g(z)$ are holomorphic functions defined in an open set Ω of the complex plane, and A, B are matrices in $\mathbb{C}^{N\times N}$ with $\sigma(A)\subset\Omega$ and $\sigma(B)\subset\Omega$, such that $AB = BA$, then*

$$f(A)g(B) = g(B)f(A).$$

Definition 1.1. (Jódar and Cortés [12]) For $A \in \mathbb{C}^{N\times N}$ such that $\sigma(A)$ does not contain 0 or a negative integer ($\sigma(A)\cap\mathbb{Z}^- = \emptyset$ where $\emptyset$ is an empty set), the Pochhammer symbol or shifted factorial is defined by

$$(A)_n = A(A+I)(A+2I)\dots(A+(n-1)I); n \geq 1,\ (A)_0 = I. \tag{1.1}$$

Definition 1.2. (Jódar and Cortés [11]) Let A be a matrix in $\mathbb{C}^{N\times N}$. We say that A is a positive stable matrix if

$$Re(\mu) > 0 \quad \forall\, \mu \in \sigma(A). \tag{1.2}$$

Definition 1.3. (Jódar and Cortés [12]) Let A be a positive stable matrix in $\mathbb{C}^{N\times N}$, then the Gamma matrix functions $\Gamma(A)$ is defined by

$$\Gamma(A) = \int_0^\infty e^{-t}t^{A-I}dt; \quad t^{A-I} = \exp\bigg((A-I)\ln t\bigg). \tag{1.3}$$

Let A be a matrix in $\mathbb{C}^{N\times N}$ such that $A+nI$ is an invertible matrix in $\mathbb{C}^{N\times N}$ for every integer $n\geq 0$, then it follows that (see [13])

$$(A)_n = \Gamma(A+nI)\Gamma^{-1}(A);\ n\geq 1,\ (A)_0 = I, \tag{1.4}$$

where $\Gamma(A)$ is an invertible matrix in $\mathbb{C}^{N\times N}$.

Definition 1.4. (Sastre and Jódar [14]) Let us take $A \in \mathbb{C}^{N\times N}$ satisfying the condition

$$\mu \text{ is not a negative integer for all eigenvalues } \mu \in \sigma(A). \tag{1.5}$$

Then the Bessel matrix functions $J_A(z)$ of the first kind of order A is defined as follows:

$$\begin{aligned} J_A(z) &= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\Gamma^{-1}(A+(k+1)I)\left(\frac{z}{2}\right)^{A+2kI}\\ &= \left(\frac{z}{2}\right)^{A}\Gamma^{-1}(A+I)\ {}_0F_1\left(-;A+I;-\frac{z^2}{4}\right);\ |z|<\infty;\ |arg(z)|<\pi, \end{aligned} \tag{1.6}$$

and the modified Bessel matrix functions $I_A(x)$ has been defined in the form:

$$\begin{aligned} \mathbf{I}_A(z) &= \sum_{k=0}^{\infty}\frac{1}{k!}\Gamma^{-1}(A+(k+1)I)\left(\frac{z}{2}\right)^{A+2kI}\\ &= \left(\frac{z}{2}\right)^{A}\Gamma^{-1}(A+I)\ {}_0F_1\left(-;A+I;\frac{z^2}{4}\right);\ |z|<\infty;\ |arg(z)|<\pi. \end{aligned} \tag{1.7}$$

Definition 1.5. (Jódar and Cortés [12]) Let A and B be positive stable matrices in $\mathbb{C}^{N\times N}$, then the Beta matrix functions $\mathbf{B}(A,B)$ is defined as:

$$\mathbf{B}(A,B) = \int_0^1 t^{A-I}(1-t)^{B-I}dt. \tag{1.8}$$

Lemma 1.1. [11] *If A, B and $A+B$ are positive stable matrices in $\mathbb{C}^{N\times N}$ such that $AB=BA$ and $A+nI$, $B+nI$ and $A+B+nI$ are invertible matrices for all $n\geq 0$, then we have*

$$\boldsymbol{B}(A,B)=\Gamma(A)\Gamma(B)\Gamma^{-1}(A+B). \tag{1.9}$$

Theorem 1.2. [4] *If A and B are matrices in $\mathbb{C}^{N\times N}$ satisfying the conditions $Re(\mu)\notin\mathbb{Z}^-$ for $\mu\in\sigma(A)$, $Re(\nu)\notin\mathbb{Z}^-$ for $\nu\in\sigma(B)$, $Re(\alpha)\notin\mathbb{Z}^-$ for $\alpha\in\sigma(A-B)$ and $AB=BA$, then we have*

$$J_A(z)=2\Gamma^{-1}(A-B)\bigg(\frac{z}{2}\bigg)^{A-B}\int_0^1(1-t^2)^{A-B-I}t^{B+I}\ J_B(zt)dt. \tag{1.10}$$

Theorem 1.3. [8] *Let A and B be matrices in $\mathbb{C}^{N\times N}$ such that A, B and $B-A$ are positive stable matrices with $AB=BA$ and $B+kI$ is an invertible matrix for all $k\geq 0$. Then, for all natural number n the following identity holds*

$$ {}_2F_1\bigg(-nI,A;B;1\bigg)=(B-A)_n[(B)_n]^{-1}. \tag{1.11}$$

Lemma 1.2. *For $n\geq 0$ and $k\geq 0$, the following relation is given by Defez and Jódar in* [6]*:*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}A(k,n-k) \tag{1.12}$$

where $A(k,n)$ is a matrix in $\mathbb{C}^{N\times N}$.

2. Some properties of the Bessel matrix functions

In this section, we give some new properties and formulas for the Bessel matrix functions.

Theorem 2.1. *Let A, B, C_1, C_2 and C_3 be commutative matrices $\mathbb{C}^{N\times N}$ such that C_1+kI, C_2+kI and C_3+kI are invertible matrices for all integers $k\geq 0$ and $Re(\mu)>0$ for all $\mu\in\sigma(A)$. Then*

$$\begin{aligned}{}_2F_3\bigg(A,B;C_1,C_2,C_3;\frac{w}{z}\bigg)=&z^A\Gamma^{-1}(A)\int_0^\infty e^{-zt}t^{A-I}\\&\times\ {}_1F_3\bigg(B;C_1,C_2,C_3;wt\bigg)dt,\quad z\neq 0.\end{aligned} \tag{2.1}$$

Proof. Taking $A=A+kI$ and $t=zt$ in (1.3), we get the integral

$$\Gamma(A+kI)z^{-A-kI}=\int_0^\infty e^{-z\ t}t^{A+(k-1)I}dt;\quad z\neq 0. \tag{2.2}$$

Using (2.2), into the right side of equation (2.1), we can write

$$\begin{aligned}&z^A\Gamma^{-1}(A)\sum_{k=0}^{\infty}\frac{w^k}{k!}(B)_k[(C_1)_k]^{-1}[(C_2)_k]^{-1}[(C_3)_k]^{-1}\Gamma(A+kI)z^{-A-kI}\\&=\sum_{k=0}^{\infty}\frac{w^k}{k!}(A)_k(B)_k[(C_1)_k]^{-1}[(C_2)_k]^{-1}[(C_3)_k]^{-1}z^{-k}=\ {}_2F_3\bigg(A,B;C_1,C_2,C_3;\frac{w}{z}\bigg).\end{aligned}$$

This completes the proof. $\square$

In a similar manner as in the proof of Theorem 2.1, one can easily obtain the next result.

Theorem 2.2. *Let A, C_1, C_2 and C_3 be commutative matrices $\mathbb{C}^{N\times N}$ such that C_1+kI, C_2+kI and C_3+kI are invertible matrices for all integers $k\geq 0$ and $Re(\mu)>0$ for all $\mu\in\sigma(A)$ and $Re(z)>0$, then*

$$\begin{aligned}{}_2F_3\left(A, A+\frac{1}{2}I; C_1, C_2, C_3; \frac{w^2}{z^2}\right) =&z^{2A}\Gamma^{-1}(2A)\int_0^\infty e^{-zt}t^{2A-I}\\ &\times\ {}_0F_3\left(-; C_1, C_2, C_3; \frac{1}{4}w^2t^2\right)dt, \quad z\neq 0.\end{aligned} \tag{2.3}$$

The expression to be derived here are the relationship between the hypergeometric and Bessel matrix functions and are believed to be new.

Theorem 2.3. *Let $2A-I$ and $2A-2I$ be matrices in $\mathbb{C}^{N\times N}$ satisfying the condition (1.5), then the hypergeometric matrix function ${}_0F_3$ and Bessel matrix functions satisfy the interesting results*

$$\begin{aligned}{}_0F_3(-;\frac{1}{2}I, A, A+\frac{1}{2}I; z) = &\frac{1}{2}\Gamma(2A)(2\sqrt[4]{z})^{I-2A}\\ &\times\left[\boldsymbol{I}_{2A-I}(4\sqrt[4]{z})+J_{2A-I}(4\sqrt[4]{z})\right],\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}{}_0F_3(-;\frac{3}{2}I, A, A+\frac{1}{2}I; z) = &\frac{1}{2}\Gamma(2A)(2\sqrt[4]{z})^{-2A}\\ &\times\left[\boldsymbol{I}_{2A-2I}(4\sqrt[4]{z})-J_{2A-2I}(4\sqrt[4]{z})\right].\end{aligned} \tag{2.5}$$

Proof. The series ${}_0F_1(-;A;z)$ converges absolutely and can therefore be separated into two series consisting of even and add terms:

$$\begin{aligned}{}_0F_1(-;A;z) =& \sum_{n=0}^\infty [(A)_n]^{-1}\frac{z^n}{n!}\\ =&\sum_{n=0}^\infty [(\frac{1}{2}A)_n]^{-1}[(\frac{1}{2}(A+I))_n]^{-1}[(\frac{1}{2}I)_n]^{-1}\frac{1}{n!}\left(\frac{z^2}{16}\right)^n\\ &+zA^{-1}\sum_{n=0}^\infty [(\frac{1}{2}A+I)_n]^{-1}[(\frac{1}{2}(A+I))_n]^{-1}[(\frac{3}{2}I)_n]^{-1}\frac{1}{n!}\left(\frac{z^2}{16}\right)^n,\end{aligned}$$

where A is an invertible matrix in $\mathbb{C}^{N\times N}$ for all integers $n\geq 0$.

Hence

$$\begin{aligned}{}_0F_1(-;A;z) = &{}_0F_3\left(-;\frac{1}{2}I, \frac{1}{2}A, \frac{1}{2}(A+I); \frac{z^2}{16}\right)\\ &+zA^{-1}\ {}_0F_3\left(-;\frac{3}{2}I, \frac{1}{2}(A+I), \frac{1}{2}A+I; \frac{z^2}{16}\right).\end{aligned} \tag{2.6}$$

Adding or subtracting the same equation with z replaced by $-z$

$$
\begin{aligned}
{}_0F_1(-;A;-z) = {}_0F_3\Big(-;\frac{1}{2}I,\frac{1}{2}A,\frac{1}{2}(A+I);\frac{z^2}{16}\Big)\\
- zA^{-1}\,{}_0F_3\Big(-;\frac{3}{2}I,\frac{1}{2}(A+I),\frac{1}{2}A+I;\frac{z^2}{16}\Big)
\end{aligned}
$$

and making suitable change or notation, we find

$$
{}_0F_3\Big(-;\frac{1}{2}I,A,A+\frac{1}{2}I;z\Big) = \frac{1}{2}\,{}_0F_1(-;2A;4\sqrt{z}) + \frac{1}{2}\,{}_0F_1(-;2A;-4\sqrt{z}), \tag{2.7}
$$

and

$$
\begin{aligned}
{}_0F_3\Big(-;\frac{3}{2}I,A,A+\frac{1}{2}I;z\Big) =& \frac{1}{8\sqrt{z}}(2A-I)\Big[\,{}_0F_1(-;2A-I;4\sqrt{z})\\
&- \,{}_0F_1(-;2A-I;-4\sqrt{z})\Big].
\end{aligned} \tag{2.8}
$$

Nota that the right hand side of the equation (2.8) has singularities at $z=0$ and at $A=\frac{1}{2}I$ which are removed by requiring continuity at these values.

From (2.6), we obtain the Bessel matrix functions representations

$$
\begin{aligned}
J_A(z) = \Big(\frac{z}{2}\Big)^{A}\Gamma^{-1}(A+I)\,{}_0F_3\Big(-;\frac{1}{2}I,\frac{1}{2}(A+I),\frac{1}{2}A+I;\frac{z^4}{256}\Big)\\
-\Big(\frac{z}{2}\Big)^{A+2I}\Gamma^{-1}(A+2I)\,{}_0F_3\Big(-;\frac{3}{2}I,\frac{1}{2}A+I,\frac{1}{2}(A+3I);\frac{z^4}{256}\Big),
\end{aligned} \tag{2.9}
$$

and

$$
\begin{aligned}
\mathbf{I}_A(z) = \Big(\frac{z}{2}\Big)^{A}\Gamma^{-1}(A+I)\,{}_0F_3\Big(-;\frac{1}{2}I,\frac{1}{2}(A+I),\frac{1}{2}A+I;\frac{z^4}{256}\Big)\\
+\Big(\frac{z}{2}\Big)^{A+2I}\Gamma^{-1}(A+2I)\,{}_0F_3\Big(-;\frac{3}{2}I,\frac{1}{2}A+I,\frac{1}{2}(A+3I);\frac{z^4}{256}\Big)
\end{aligned} \tag{2.10}
$$

Adding and subtracting these equations, we find the inverse relations

$$
{}_0F_3\Big(-;\frac{1}{2}I,A,A+\frac{1}{2}I;z\Big) = \frac{1}{2}\Gamma(2A)(2\sqrt[4]{z})^{I-2A}\Big[\mathbf{I}_{2A-I}(4\sqrt[4]{z}) + J_{2A-I}(4\sqrt[4]{z})\Big],
$$

and

$$
{}_0F_3\Big(-;\frac{3}{2}I,A,A+\frac{1}{2}I;z\Big) = \frac{1}{2}\Gamma(2A)(2\sqrt[4]{z})^{-2A}\Big[\mathbf{I}_{2A-2I}(4\sqrt[4]{z}) - J_{2A-2I}(4\sqrt[4]{z})\Big].
$$

Thus proof of the theorem is completed. □

Theorem 2.4. *If A and B are matrices in $C^{N\times N}$ satisfying the condition (1.5), then we have the product of a series representing Bessel matrix functions*

$$
\begin{aligned}
J_A(z)J_B(z) = \sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\Big(\frac{z}{2}\Big)^{A+B+2mI}\Gamma^{-1}(A+(m+1)I)\\
\times\Gamma^{-1}(B+(m+1)I)(A+B+(m+1)I)_m.
\end{aligned} \tag{2.11}
$$

Proof. The coefficient of $(-1)^m\left(\frac{1}{2}z\right)^{A+B+2mI}$ in the product of the two absolutely convergent series

$$\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\Gamma^{-1}(A+(m+1)I)\left(\frac{z}{2}\right)^{A+2mI}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\Gamma^{-1}(B+(n+1)I)\left(\frac{z}{2}\right)^{B+2nI}$$

is

$$\begin{aligned}&\sum_{n=0}^{m}\frac{\Gamma^{-1}(A+(n+1)I)\Gamma^{-1}(B+(m-n+1)I)}{n!(m-n)!}\\ =&\frac{(-1)^m}{m!}\Gamma^{-1}(A+(m+1)I)\Gamma^{-1}(B+(m+1)I)\sum_{n=0}^{m}\frac{m!(-A-mI)_{m-n}(-B-mI)_n}{n!(m-n)!}\\ =&\frac{(-1)^m}{m!}\Gamma^{-1}(A+(m+1)I)\Gamma^{-1}(B+(m+1)I)(-A-B-2mI)_m\\ =&\frac{1}{m!}\Gamma^{-1}(A+(m+1)I)\Gamma^{-1}(B+(m+1)I)(A+B+(m+1)I)_m,\end{aligned}$$

the proof is completed. □

Theorem 2.5. *If A and B are matrices in $C^{N\times N}$ satisfying condition (1.5), then we have the product of Bessel matrix functions*

$$\begin{aligned}J_A(az)J_B(bz)=&\left(\frac{bz}{2}\right)^{B}\Gamma^{-1}(B+I)\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\left(\frac{az}{2}\right)^{A+2mI}\\ &\times\Gamma^{-1}(A+(m+1)I)\ {}_2F_1\left(-mI,-A-mI;B+I;\frac{b^2}{a^2}\right),\end{aligned}\tag{2.12}$$

where a, b are arbitrary constants and $a\neq 0$.

Proof. If we multiply the series for $J_A(az)$ and $J_B(bz)$, we obtain an expansion in which the coefficient of $(-1)^m a^A b^B\left(\frac{1}{2}z\right)^{A+B+2mI}$ is

$$\begin{aligned}&\sum_{n=0}^{m}\frac{a^{2m-2n}b^{2n}\Gamma^{-1}(B+(n+1)I)\Gamma^{-1}(A+(m-n+1)I)}{n!(m-n)!}\\ =&\frac{a^{2m}}{m!}\Gamma^{-1}(A+(m+1)I)\Gamma^{-1}(B+I)\ {}_2F_1\left(-mI,-A-mI;B+I;\frac{b^2}{a^2}\right).\end{aligned}$$

Then, the proof is finished. □

Corollary 2.1. *For the special case $a=b=2$ in Theorem 2.5, we have the interest relation*

$$\begin{aligned}J_A(2z)J_B(2z)=&\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}z^{A+B+2mI}(B+A+(m+1)I)_m\\ &\times\Gamma^{-1}(A+(m+1)I)\Gamma^{-1}(B+(m+1)I).\end{aligned}\tag{2.13}$$

Proof. If we take $a = b = 2$ in (2.12) and using (1.11), we obtain

$$J_A(2z)J_B(2z) = \Gamma^{-1}(B+I)\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}z^{A+B+2mI}\Gamma^{-1}(A+(m+1)I)$$
$$\times\ {}_2F_1\Big(-mI, -A-mI; B+I; 1\Big) = \Gamma^{-1}(B+I)\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}z^{A+B+2mI}$$
$$\times\,\Gamma^{-1}(A+(m+1)I)(B+A+(m+1)I)_m[(B+I)_m]^{-1} = \sum_{m=0}^{\infty}\frac{(-1)^m}{m!}$$
$$z^{A+B+2mI}\Gamma^{-1}(A+(m+1)I)\Gamma^{-1}(B+(m+1)I)(B+A+(m+1)I)_m.$$

Hence, the proof is finished. □

3. Some properties associated with the Bessel matrix functions

In this section, we define the associated with the Bessel matrix functions and obtain some of their significant properties.

Definition 3.1. Let A, B and C be matrices in $\mathbb{C}^{N\times N}$ satisfying the condition

$$A+(k+1)I, B+(k+1)I \text{ and } C+(k+1)I \text{ are invertible matrices for all integers } k \geq -1, \tag{3.1}$$

and these matrices are commutative, we define the associated with the Bessel matrix functions $\mathbf{R}(A, B, C, z)$ by the series

$$\mathbf{R}(A, B, C, z) = \sum_{k=0}^{\infty}\frac{(-1)^k z^k}{k!}(A+(k+1)I)_k \times\Gamma^{-1}(B+(k+1)I)\Gamma^{-1}(C+(k+1)I). \tag{3.2}$$

Theorem 3.1. *The matrix functions $\mathbf{R}(A, B, C, z)$ given by (3.2) have the hypergeometric matrix functions representation*

$$\mathbf{R}(A, B, C, z) = \Gamma^{-1}(B+I)\Gamma^{-1}(C+I) \times\ {}_2F_3\Big(\frac{1}{2}(A+I), \frac{1}{2}A+I; A+I, B+I, C+I; -4z\Big). \tag{3.3}$$

Proof. From (1.4), we can rewrite the formula

$$(A+(k+1)I)_k = (A+I)_{2k}[(A+I)_k]^{-1} \tag{3.4}$$

and

$$(A+I)_{2k} = 2^{2k}\Big(\frac{1}{2}(A+I)\Big)_k\Big(\frac{1}{2}A+I\Big)_k. \tag{3.5}$$

On using (3.4), (3.5) and (3.2) in (3.3) the desired result follows directly. □

From (3.3), we have the following result.

Theorem 3.2. *Let A, B and C be matrices in $\mathbb{C}^{N\times N}$ such that $A+(k+1)I$, $B+(k+1)I$ and $C+(k+1)I$ are invertible matrices for all integers $k \geq -1$. Then, the associated with the Bessel matrix functions $\mathbf{R}(A, B, C, z)$ is an entire function.*

The connection between the matrix functions $\mathbf{R}(A, B, C, z)$ and Bessel matrix functions is evident from the following formulas:

Corollary 3.1. *Let B and C be matrices in $\mathbb{C}^{N\times N}$ satisfying the condition (1.5), then we have the identity*

$$z^{B+C}\mathbf{R}(B+C,B,C,z^2)=J_B(2z)J_C(2z). \tag{3.6}$$

where B, C and $B+C$ are matrices in $\mathbb{C}^{N\times N}$ satisfying the condition (3.1).

Proof. In Theorem 3.1, putting $A=B+C$, $z=z^2$ and using definition (3.2), and properties of hypergeometric matrix functions (1.6), we have the desired identity. □

Corollary 3.2. *Let $2A$ be a matrix in $\mathbb{C}^{N\times N}$ satisfying the condition (1.5). Then we can deduce as follows:*

$$\mathbf{R}\bigg(2A,A,A-\frac{1}{2}I,z^2\bigg)=\frac{1}{\sqrt{\pi}}z^{-2A}J_{2A}(4z). \tag{3.7}$$

where $2A$, A and $A\frac{1}{2}I$ are matrices in $\mathbb{C}^{N\times N}$ satisfying the condition (3.1).

Proof. In Theorem 3.1, replacing A by $2A$ and taking $B=A$, $C=A-\frac{1}{2}I$ and $z=z^2$ with the help of (1.6), we obtain (3.7). □

Theorem 3.3. *Let A, B and C be matrices in $\mathbb{C}^{N\times N}$ satisfying the condition (3.1), then the $\mathbf{R}(A,B,C,z)$ is a solution of the matrix differential equation of the fourth order*

$$\begin{aligned}&\Big[\theta(\theta I+A)(\theta I+B)(\theta I+C)\\ &+z(2\theta\, I+A+I)(2\theta\, I+A+2I)\Big]\,\mathbf{R}(A,B,C,z)=\boldsymbol{0},\end{aligned} \tag{3.8}$$

where $\theta=z\frac{d}{dz}$ is the differential operator.

Proof. The proof of this theorem follows immediately on using the definition and properties of the hypergeometric matrix functions. □

Now we give various types of integral representations for the matrix functions $\mathbf{R}(A,B,C,z)$. For this purpose, we state the following results.

Theorem 3.4. *Let A, B and C be matrices in $\mathbb{C}^{N\times N}$ satisfying the condition (3.1) with $Re(\mu)>-1$ for all $\mu\in\sigma(A)$ and if these matrices be commutative, we have the formula*

$$\begin{aligned}\mathbf{R}(A,B,C,z)=&\Gamma^{-1}\bigg(\frac{1}{2}(A+I)\bigg)\Gamma^{-1}(B+I)\Gamma^{-1}(C+I)\int_0^\infty e^{-t}t^{\frac{1}{2}A-\frac{1}{2}I}\\ &\times\ {}_1F_3\bigg(\frac{1}{2}A+I;A+I,B+I,C+I;-4zt\bigg)dt.\end{aligned} \tag{3.9}$$

Proof. Starting from the right hand side of (3.9) and using (2.1), (2.2) and (3.2), this theorem can be easily proved. □

Theorem 3.5. *Suppose that A, B and C are matrices in $\mathbb{C}^{N\times N}$ satisfying the condition (3.1) and $Re(\mu)>-1$ for all $\mu\in\sigma(A)$, the integral representation for* *the matrix functions* $\mathbf{R}(A,B,C,z)$ *holds:*

$$\begin{aligned}\mathbf{R}(A,B,C,z^2)=&\Gamma^{-1}(A+I)\Gamma^{-1}(B+I)\Gamma^{-1}(C+I)\int_0^{\infty}e^{-t}t^{A}\\&\times\ {}_0F_3\Big(-;A+I,B+I,C+I;-z^2t^2\Big)dt.\end{aligned}\tag{3.10}$$

Proof. Applying (3.2) and (2.3) and using the Gamma matrix function (1.3), we get (3.10). □

Corollary 3.3. *Let A be a matrix in $\mathbb{C}^{N\times N}$ satisfying the condition $Re(\mu)>-1$ for every $\mu\in\sigma(A)$, we have*

$$\begin{aligned}\mathbf{R}\Big(A,\frac{1}{2}(A-I),\frac{1}{2}A,z^2\Big)&=\frac{1}{\sqrt{\pi}}2^{A}\Gamma^{-1}(A+I)\Gamma^{-1}(A+I)\int_0^{\infty}e^{-t}t^{A}\\\times\ {}_0F_1\Big(-;A+I;2zt\Big)\,{}_0F_1&\Big(-;A+I;-2zt\Big)dt.\end{aligned}\tag{3.11}$$

where A,$\frac{1}{2}(AI)$ and $\frac{1}{2}A$ are matrices in $\mathbb{C}^{N\times N}$ satisfying the condition (3.1).

Proof. By (1.3) and (1.12) into the equation (3.11) follows directly. □

Corollary 3.4. *If A is a matrix in $\mathbb{C}^{N\times N}$ satisfying the condition $Re(\mu)\notin\mathbb{Z}^-$ for $\mu\in\sigma(A)$, then we obtain*

$$\begin{aligned}\mathbf{R}\Big(2A,A,A-\frac{1}{2}I,z^2\Big)=&\frac{2}{\sqrt{\pi}}\Gamma^{-1}(A)\Big(\frac{1}{2}z\Big)^{-A}\\&\times\int_0^1(1-t^2)^{A-I}t^{A+I}\ J_A(4zt)dt.\end{aligned}\tag{3.12}$$

where $2A$,A and $A-\frac{1}{2}I$ are matrices in $\mathbb{C}^{N\times N}$ satisfying the condition (3.1).

Proof. Applying (1.6) to (3.3), the proof follows. □

Theorem 3.6. *If A, B and C are matrices in $\mathbb{C}^{N\times N}$ satisfying the condition (3.1). Then it can be also established that*

$$\begin{aligned}\mathbf{R}(A,B,C,z)=&\frac{1}{2\pi\ i}\int_{-\infty}^{0}e^{t}t^{-C-I}\ {}_2F_2\Big(\frac{1}{2}A+\frac{1}{2}I,\frac{1}{2}A+I;\\&A+I,B+I;-\frac{4z}{t}\Big)dt;|arg(t)|\leq\pi.\end{aligned}\tag{3.13}$$

Proof. From (1.3), we have

$$\Gamma^{-1}(C+(k+1)I)=\frac{1}{2\pi\ i}\int_{-\infty}^{0}t^{-C-(k+1)I}e^{t}dt.\tag{3.14}$$

By using (3.14) and (3.2), we can prove (3.13). □

Next, we discuss some infinite integrals involving the matrix functions $\mathbf{R}(A,B,C,z)$.

Theorem 3.7. *If A, B, C and $C+D$ are matrices in $\mathbb{C}^{N\times N}$ satisfying the condition (3.1) and let D be a positive stable matrix in $\mathbb{C}^{N\times N}$. The integral representation for the matrix functions $\mathbf{R}(A,B,C,zt)$ is valid:*

$$\int_0^1\mathbf{R}(A,B,C,zt)t^{C}(1-t)^{D-I}dt=\Gamma(D)\mathbf{R}(A,B,C+D,z),\tag{3.15}$$

where $Re(\alpha) > -1$ for all the eigenvalues of $\alpha \in \sigma(C)$ and $Re(\beta) > 0$ for every eigenvalue of $\beta \in \sigma(D)$.

Proof. Let us consider the left hand side of (3.15). From (1.8) and (1.9), we have

$$\begin{aligned}\int_0^1 t^{C+kI}(1-t)^{D-I}dt &= \mathbf{B}(C+(k+1)I, D)\\ &= \Gamma(C+(k+1)I)\Gamma(D)\Gamma^{-1}(C+D+(k+1)I),\end{aligned} \tag{3.16}$$

from (3.16) and (3.2), we find

$$\begin{aligned}&\Gamma(D)\sum_{k=0}^{\infty}\frac{(-1)^k z^k}{k!}(A+(k+1)I)_k\Gamma^{-1}(B+(k+1)I)\Gamma^{-1}(C+D+(k+1)I)\\ &= \Gamma(D)\,\mathbf{R}(A,B,C+D,z),\end{aligned}$$

which gives (3.15). Hence the proof is established. □

Theorem 3.8. *Let A, B and C be matrices in $\mathbb{C}^{N\times N}$ satisfying the condition (3.1). The associated with the Bessel matrix functions satisfy the properties:*

$$\begin{aligned}(i)\ &\int_0^{\infty} e^{-Pt}t^{D}\mathbf{R}(A,B,C,zt)dt = \Gamma(D+I)\Gamma^{-1}(B+I)\Gamma^{-1}(C+I)\\ &\times P^{-D-I}\ {}_3F_3\left(\frac{1}{2}(A+I),\frac{1}{2}A+I,D+I;A+I,B+I,C+I;-\frac{4z}{P}\right),\end{aligned} \tag{3.17}$$

where $Re(\alpha) > 0$ for all the eigenvalues of $\alpha \in \sigma(P)$ and $Re(\beta) > -1$ for every eigenvalue of $\beta \in \sigma(D)$.

$$\begin{aligned}(ii)\ &\int_0^{\infty} e^{-Pt}t^{2D-I}\mathbf{R}(A,B,C,z^2t^2)dt = \Gamma(2D)\Gamma^{-1}(B+I)\Gamma^{-1}(C+I)\\ &\times P^{-2D}\ {}_4F_3\left(\frac{1}{2}(A+I),\frac{1}{2}A+I,D,D+\frac{1}{2}I;A+I,B+I,C+I;\right.\\ &\left.-\frac{16z^2}{P^2}\right);\ Re(P) > 4|Re(z)| \text{ and } Re(D) > 0,\end{aligned} \tag{3.18}$$

where $Re(\alpha) > 4|Re(z)|$ for all the eigenvalues of $\alpha \in \sigma(P)$ and $Re(\beta) > 0$ for every eigenvalue of $\beta \in \sigma(D)$.

Proof. (i) To prove (i), take $A = D + kI$ and $tI = Pt$ in equation (1.3) and use (3.2).

(ii) Take $A = 2D + 2kI$ and $tI = Pt$ in equation (1.3) and use (3.2), which completes of proof of (ii). □

Theorem 3.9. *Suppose that A, $\frac{1}{2}A$, $\frac{1}{2}(A-I)$, $A+B$ and $B+C$ are matrices in $\mathbb{C}^{N\times N}$ satisfying the condition (3.1). The integral representation for the matrix functions $\mathbf{R}(A,B,C,z)$ also holds:*

$$\begin{aligned}&\int_0^1 \mathbf{R}\left(A,\frac{1}{2}A,\frac{1}{2}(A-I),\frac{1}{4}zt\right)\mathbf{R}\left(B,\frac{1}{2}B,\frac{1}{2}(B-I),\frac{1}{4}zt\right)t^{B}(1-t)^{C-I}dt\\ &= \frac{1}{\sqrt{\pi}}\Gamma(C)2^{A+B}\mathbf{R}(A+B,A,B+C,z),\end{aligned} \tag{3.19}$$

where $Re(\beta) > -1$ for every eigenvalue of $\beta \in \sigma(B)$ and $Re(\alpha) > 0$ for all the eigenvalues of $\alpha \in \sigma(C)$.

Proof. Starting from the left hand side of the equation in (3.19) and using Beta matrix function (3.16), (1.12) and (3.2), theorem can be proved. □

Finally, let A, B and C are matrices in $\mathbb{C}^{N\times N}$ satisfying the condition (3.1). The relationship to be derived is of further interest for the extended modified Bessel and the extended Tricomi matrix functions by the relations link with extended Bessel matrix functions:

$$\mathbf{I}(A,B,C,z) = i^{-A}\ \mathbf{R}(A,B,C,iz) \tag{3.20}$$

and

$$\mathbf{C}(A,B,C,z) = z^{-\frac{1}{2}A}\ \mathbf{R}(A,B,C,2\sqrt{z}). \tag{3.21}$$

This is an open problem for future studies. Further applications will be discussed in a forthcoming paper.

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