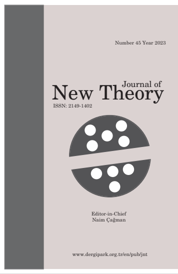
Journal of New Theory

ISSN: 2149-1402

45 (2023) 83-94
Journal of New Theory
https://dergipark.org.tr/en/pub/jnt
Open Access



Screen Semi-Invariant Lightlike Hypersurfaces on Hermite-Like Manifolds

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Article Info
Received: 26 Oct 2023
Accepted: 25 Dec 2023
Published: 31 Dec 2023
doi:10.53570/jnt.1381549
Research Article

Abstract — Hermite-like manifolds, which admit two different, almost complex structures, can be considered a general concept of Hermitian manifolds. Factoring in the effects of these two complex structures on the radical, screen, and transversal spaces, a new classification of lightlike hypersurfaces of Hermite-like manifolds is proposed in the present paper. Moreover, an example of screen semi-invariant lightlike hypersurfaces of Hermite-like manifolds is provided. Besides, some results on these hypersurfaces admitting a statistical structure are obtained. Further, screen semi-invariant lightlike hypersurfaces are investigated on Kaehler-like statistical manifolds. In addition, several characteristics of totally geodesic, mixed geodesic, and totally umbilical screen lightlike hypersurfaces are obtained. Finally, the need for further research is discussed.

Keywords *Hermite-like manifolds, complex structures, lightlike hypersurfaces*

Mathematics Subject Classification (2020) 53C40, 53C50

1. Introduction

Firstly, the concept of Hermite-like manifolds was given by Takano [1, 2]. A different feature of these manifolds that differs from Hermitian manifolds is that even the simplest examples are not found in Euclidean spaces but are found in non-Euclidean spaces. A pseudo-Riemannian manifold $(\widetilde{H}, \widetilde{h})$ with two different almost complex structures J and $J^\star$ providing

$$\widetilde{h}(JZ_1, Z_2) = -\widetilde{h}(Z_1, J^\star Z_2) \tag{1}$$

for any $Z_1, Z_2, \in \Gamma(T\widetilde{H})$ is entitled a Hermite-like manifold. For any Hermite-like manifold, we possess

$$\widetilde{h}(JZ_1, J^\star Z_2) = \widetilde{h}(Z_1, Z_2) \tag{2}$$

If we indite $J = J^\star$ in Equations 1 and 2, then a Hermite-like manifold becomes an almost Hermitian manifold.

Various authors have investigated non-degenerate submanifolds of Hermite-like manifolds [3–5]. Moreover, the authors have researched Riemannian submersions admitting Hermite-like manifolds [6–11]. However, no studies on degenerate submanifolds of Hermite-like manifolds have been published thus far.

In addition to the above facts, lightlike geometry has interesting results thanks to the different ge-

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ometric properties of radical, screen, and transversal distributions. Considering the effects of J and $J^\star$ on the radical, screen, and transversal spaces, new classifications of lightlike hypersurfaces can be identified. With this perspective, we familiarize the impression of screen semi-invariant lightlike hypersurfaces of Hermite-like manifolds and Hermite-like statistical manifolds in this paper.

Section 2 of the handled study presents some basic notions to be used in the following sections. Section 3 provides lightlike hypersurfaces of Hermite-like manifolds. Section 4 analyzes screen semi-invariant lightlike hypersurfaces of Kaehler-like statistical manifolds $(\widetilde{H}, \widetilde{h}, J, \widetilde{D})$.

2. Preliminaries

This section provides some basic properties to be needed in the following sections. For any lightlike hypersurface (H, h) of a pseudo-Riemannian manifold, we invite the radical space at each point $p \in H$ by

$$Rad\ T_pH = \{\xi \in T_pH : h_p(\xi, Z) = 0, \forall Z \in T_pH\}$$

Here, h is the induced degenerate metric from $\widetilde{h}$. The complementary non-degenerate vector bundle of $Rad\ T_pH$ is indicated by $S(TH)$ and we indite

$$TH = Rad\ TH \oplus_{orth} S(TH)$$

There exists a lightlike transversal bundle $ltr\ TH = span\{N\}$ such that we possess

$$\widetilde{h}(Z, N) = \widetilde{h}(N, N) = 0,\ \widetilde{h}(\xi, N) = 1 \tag{3}$$

for any $Z \in \Gamma(S(TH))$. Therefore, the tangent bundle $T\widetilde{H}$ of $\widetilde{H}$ is decomposed as follows:

$$T\widetilde{H} = TH \oplus ltr\ TH = \left\{TH^\perp \oplus ltr(TH)\right\} \oplus_{orth} S(TH) \tag{4}$$

where $\oplus$ indicates the direct sum, not orthogonal. Let $\widetilde{D}^0$ be the Riemannian connection of $(\widetilde{H}, \widetilde{h})$. The Gauss and Weingarten formulas for (H, h) are represented by

$$\begin{aligned} \widetilde{D}^0_{Z_1} Z_2 &= D^0_{Z_1} Z_2 + B^0(Z_1, Z_2)N \\ \widetilde{D}^0_{Z_1} N &= -A^0_N Z_1 + \tau^0(Z_1)N \end{aligned} \tag{5}$$

for any $Z_1, Z_2 \in \Gamma(TH)$. Here, D^0 is the induced connection, B^0 is the second fundamental form, A^0_N is the shape operator, and τ^0 is a $1-$form on $\Gamma(TH)$. We note that D^0 is not a Riemannian connection [12].

If $B^0 = 0$, then a lightlike hypersurface $(H, h, S(TH))$ is called totally geodesic. If there exists a function λ on H satisfying

$$B^0(Z_1, Z_2) = \lambda h(Z_1, Z_2)$$

then $(H, h, S(TH))$ is entitled totally umbilical [13].

Let $(\widetilde{H}, \widetilde{h}, \widetilde{D})$ be a statistical manifold. Then,

$$Z_3 \widetilde{h}(Z_1, Z_2) = \widetilde{h}(\widetilde{D}_{Z_3} Z_1, Z_2) + \widetilde{h}(Z_1, \widetilde{D}^\star_{Z_3} Z_2) \tag{6}$$

and

$$\widetilde{D}^0_{Z_1} Z_2 = \frac{1}{2}(\widetilde{D}_{Z_1} Z_2 + \widetilde{D}^\star_{Z_1} Z_2) \tag{7}$$

The connection $\widetilde{D}^\star$ is entitled the dual of $\widetilde{D}$ [14]. Indicate the Riemannian curvature tensors with regard to connections $\widetilde{D}$ and $\widetilde{D}^\star$ by $\widetilde{R}$ and $\widetilde{R}^\star$, respectively. In this regard,

$$\widetilde{h}(\widetilde{R}^\star(Z_1, Z_2)Z_3, Z_4) = -\widetilde{h}(Z_3, \widetilde{R}(Z_1, Z_2)Z_4) \tag{8}$$

for any $Z_1, Z_2, Z_3, Z_4 \in \Gamma(T\widetilde{H})$ [15]. Equation 8 implies that $\widetilde{R}$ and $\widetilde{R}^\star$ are not symmetric.

Let $(H, h, S(TH))$ be a lightlike hypersurface of $(\widetilde{H}, \widetilde{h}, \widetilde{D})$. The Gauss and Weingarten type formulas with regard to $(\widetilde{D}, \widetilde{D}^\star)$ are formulated by

$$\widetilde{D}_{Z_1} Z_2 = D_{Z_1} Z_2 + B(Z_1, Z_2)N \tag{9}$$

$$\widetilde{D}_{Z_1} N = -A^\star_N Z_1 + \tau^\star(Z_1)N \tag{10}$$

and

$$\widetilde{D}^\star_{Z_1} Z_2 = D^\star_{Z_1} Z_2 + B^\star(Z_1, Z_2)N \tag{11}$$

$$\widetilde{D}^\star_{Z_1} N = -A_N Z_1 + \tau(Z_1)N \tag{12}$$

where $D_{Z_1}Z_2$, $D^\star_{Z_1}Z_2$, $A_N Z_1$, and $A^\star_N Z_1$ are included in $\Gamma(TH)$ and D and $D^\star$ are the induced connections on H.

Suppose that P is the projection mapping from $\Gamma(TH)$ onto $\Gamma(S(TH))$. In this regard,

$$D_{Z_1} PZ_2 = \widetilde{D}_{Z_1} PZ_2 + C(Z_1, PZ_2)\xi \tag{13}$$

and

$$D_{Z_1}\xi = -\widetilde{A}_\xi Z_1 - \tau(Z_1)\xi \tag{14}$$

where $\widetilde{D}_{Z_1}PZ_2$ and $\widetilde{A}_\xi Z_1$ are included in $\Gamma(S(TH))$. Then,

$$B(Z_1, Z_2) = \widetilde{h}(\widetilde{D}_{Z_1} Z_2, \xi), \quad \tau^\star(Z_1) = \widetilde{h}(\widetilde{D}_{Z_1} N, \xi) \tag{15}$$

and

$$B^\star(Z_1, Z_2) = \widetilde{h}(\widetilde{D}^\star_{Z_1} Z_2, \xi), \quad \tau(Z_1) = \widetilde{h}(\widetilde{D}^\star_{Z_1} N, \xi) \tag{16}$$

Similarly, in view of Equations 13 and 14, we indite

$$D^\star_{Z_1} PZ_2 = \widetilde{D}^\star_{Z_1} PZ_2 + C^\star(Z_1, PZ_2)\xi \tag{17}$$

and

$$D^\star_{Z_1}\xi = -\widetilde{A}^\star_\xi Z_1 - \tau^\star(Z_1)\xi \tag{18}$$

where $\widetilde{D}^\star_{Z_1}PZ_2$ and $\widetilde{A}^\star_\xi Z_1$ are included in $\Gamma(S(TH))$ [16]. Using Equations 11-18, the following relations are provided:

$$B(Z_1, Z_2) = h(\widetilde{A}^\star_\xi Z_1, Z_2) + B^\star(Z_1, \xi)\widetilde{h}(Z_2, N) \tag{19}$$

and

$$B^\star(Z_1, Z_2) = h(\widetilde{A}_\xi Z_1, Z_2) + B(Z_1, \xi)\widetilde{h}(Z_2, N) \tag{20}$$

In view of Equations 19 and 20,

$$B(Z_1, \xi) + B^\star(Z_1, \xi) = 0, \quad h(A_N Z_1 + A^\star_N Z_1, Z_2) = 0, \quad \text{and} \quad C(Z_1, PZ_2) = h(A_N Z_1, PZ_2) \tag{21}$$

As a result of Equation 21, we obtain that B and $B^\star$ do not vanish on the radical space [17, 18].

A lightlike hypersurface of a statistical manifold is entitled *i.* totally geodesic with regard to $\widetilde{D}$ if $B = 0$,

ii. totally geodesic with regard to $\widetilde{D}^\star$ if $B^\star = 0$,

iii. totally tangential umbilical about $\widetilde{D}$ if there exists a smooth function k such that $B(Z_1, Z_2) = kh(Z_1, Z_2)$,

and

iv. totally tangential umbilical with respect to $\widetilde{D}^\star$ if there exists a smooth function $k^\star$ such that $B^\star(Z_1, Z_2) = k^\star h(Z_1, Z_2)$ [17].

3. Lightlike Hypersurfaces of Hermite-like Manifolds

This section presents lightlike hypersurfaces of Hermite-like manifolds.

Definition 3.1. [2] A Hermite-like manifold is called a Hermite-like statistical manifold if there is a linear connection $\widetilde{D}$ providing Equations 6 and 7. A Hermite-like statistical manifold is specified by $(\widetilde{H}, \widetilde{h}, J, \widetilde{D})$.

Definition 3.2. [2] A Hermite-like statistical manifold $(\widetilde{H}, \widetilde{h}, J, \widetilde{D})$ is entitled a Kaehler-like statistical manifold if $\widetilde{D}J = 0$. For each Kaehler-like statistical manifold $(\widetilde{H}, \widetilde{h}, J, \widetilde{D})$, $\widetilde{D}^\star J^\star = 0$.

We define semi-invariant lightlike hypersurfaces inspiring [19–24] as follows:

Definition 3.3. A lightlike hypersurface $(H, h, S(TH))$ is called screen semi-invariant if $J(Rad\,TH)$ and $J(ltr\ TH)$ are included in $S(TH)$.

In view of Equation 1, if $(H, h, S(TH))$ is a screen semi-invariant lightlike hypersurface, then $J^\star(Rad\,TH)$ and $J^\star(ltr\ TH)$ are included in $S(TH)$.

Example 3.4. Let $(\widetilde{H}, \widetilde{h})$ be a $6-$dimensional pseudo-Riemannian manifold with a pseudo-Riemannian metric $\widetilde{h}$ provided by

$$\widetilde{h} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Define almost complex structures

$$J = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and

$$J^\star = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

Then, Equation 1 is satisfied. Therefore, $(\widetilde{H}, \widetilde{h}, J)$ is an example of Hermite-like manifold.

Let $\{\partial_1, \partial_2, \partial_3, \partial_4, \partial_5, \partial_6\}$ be the standard frame field on $\Gamma(TH))$. Denote $\widetilde{\nabla}_{\partial_i}\partial_j = \sum_{k=1}^{8} \Gamma_{ij}^{k}\partial_k$ and $\widetilde{\nabla}^{\star}_{\partial_i}\partial_j = \sum_{k=1}^{8} \Gamma_{ij}^{\star k}\partial_k$, for $i, j \in \{1, \ldots, 8\}$. From Equation 6, $\Gamma_{ij}^{k}\, \widetilde{h}(\partial_k, \partial_k) + \Gamma_{ik}^{\star j}\, \widetilde{h}(\partial_j, \partial_j) = 0$. Considering this fact, $\widetilde{\nabla}_{\partial_1}\partial_1 = \partial_1$, $\widetilde{\nabla}_{\partial_1}\partial_2 = \partial_2$, $\widetilde{\nabla}^{\star}_{\partial_1}\partial_1 = -\partial_1$, and $\widetilde{\nabla}^{\star}_{\partial_1}\partial_2 = -\partial_2$ and the other terms of $\widetilde{\nabla}$ and $\widetilde{\nabla}^{\star}$ vanish. Then, $(\widetilde{H}, \widetilde{h}, J)$ becomes a Kaehler-like statistical manifold.

Regard as a hypersurface of $(\widetilde{H}, \widetilde{h}, J)$ described by

$$H = \left\{(z_i)_{i\in\{1,2,3,4,5,6\}} : z_1 = z_3\right\}$$

In this case, the induced metric h becomes

$$h = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

By a straightforward computation,

$$Rad\ TH = \text{span}\,\{\xi = \partial_1 + \partial_3\}$$

$$ltr\ TH = \text{span}\left\{N = -\frac{1}{2}(\partial_1 + \partial_3)\right\}$$

and

$$S(TH) = \text{span}\,\{e_1 = \partial_1, e_2 = \partial_4, e_3 = \partial_5, e_4 = \partial_6\}$$

Therefore,

$$J\xi = e_2 + e_4, \quad JN = \frac{1}{2}(e_2 + e_4), \quad J^{\star}\xi = -(e_2 + e_4), \quad \text{and} \quad J^{\star}N = \frac{1}{2}(e_2 - e_4)$$

which indicate that $(H, h, S(TH))$ is screen semi-invariant.

Let $(H, h, S(TH))$ be a screen semi-invariant lightlike hypersurface of $(\widetilde{H}, \widetilde{h}, J)$. In this regard,

$$JN = \alpha, \quad J^{\star}N = \alpha^{\star}, \quad J\xi = \beta, \quad \text{and} \quad J^{\star}\xi = \beta^{\star} \tag{22}$$

where α, $\alpha^{\star}$, β, and $\beta^{\star}$ are included in $\Gamma(S(TH))$. For each $Z \in \Gamma(TH)$,

$$JZ = \psi Z + w^{\star}(Z)\xi + \eta^{\star}(Z)N \tag{23}$$

and

$$J^{\star}Z = \psi^{\star}Z + w(Z)\xi + \eta(Z)N \tag{24}$$

where ψ and $\psi^{\star}$ are projections from $\Gamma(TH)$ onto $\Gamma(S(TH))$ and w, $w^{\star}$, η, and $\eta^{\star}$ are 1−forms described by

$$w(Z) = h(Z, \alpha), \quad w^{\star}(Z) = h(Z, \alpha^{\star})$$

and

$$\eta(Z) = h(Z, \beta), \quad \eta^{\star}(Z) = h(Z, \beta^{\star})$$

for all $Z \in \Gamma(TH)$.

Proposition 3.5. Let $(H, h, S(TH))$ be a screen semi-invariant lightlike hypersurface of $(\widetilde{H}, \widetilde{h}, J)$.

Then, the following equations hold:

$$\eta^\star(\psi Z) = 0 \quad \text{and} \quad \eta(\psi^\star Z) = 0$$

for all $Z \in \Gamma(TH)$. In particular,

$$w^\star(\psi Z) = 0 \quad \text{and} \quad w(\psi^\star Z) = 0$$

for all $Z \in \Gamma(S(TH))$.

PROOF.
Using Equations 22-24,

$$-Z = J^2 Z = J(\psi Z) + w^\star(Z)J\xi + \eta^\star(Z)JN$$

and

$$-Z = \psi^2 Z + w^\star(\psi Z)\xi + \eta^\star(\psi Z)N + w^\star(Z)\beta + \eta^\star(Z)\alpha \tag{25}$$

Investigating the tangential and transversal sides of Equation 25, $\eta^\star(\psi Z) = 0$.

If Z is included in $\Gamma(S(TH))$, then $w^\star(\psi Z) = 0$. Applying $(J^\star)^2 = -\mathbf{I}_{n+2}$ and a similar technique as in the proof of Equation 25,

$$-Z = (\psi^\star)^2 Z + w(\psi^\star Z)\xi + \eta(\psi^\star Z)N + w(Z)\beta^\star + \eta(Z)\alpha^\star \tag{26}$$

which indicates $\eta(\psi^\star Z) = 0$, for all $Z \in \Gamma(TH)$. If $Z \in \Gamma(S(TH))$, then $w(\varphi^\star Z) = 0$ from Equation 26. $\square$

Using Equations 25 and 26, the following results are obtained.

Proposition 3.6. For any screen semi-invariant lightlike hypersurface $(M, g, S(TH))$ of $(\widetilde{H}, \widetilde{h}, J)$, the following relations occur, for all $Z \in \Gamma(TH)$,

$$\psi^2 Z = -PZ - w^\star(Z)\beta - \eta^\star(Z)\alpha$$

$$(\psi^\star)^2 Z = -PZ - w(Z)\beta^\star - \eta(Z)\alpha^\star$$

and

$$w^\star(\psi Z) = w(\psi^\star Z)$$

Proposition 3.7. For any screen semi-invariant lightlike hypersurface $(M, g, S(TH))$, the following relations occur, for all $Z_1, Z_2 \in \Gamma(TH)$,

$$\widetilde{h}(\psi Z_1, Z_2) + \eta^\star(Z_2)\widetilde{h}(Z_2, N) = \widetilde{h}(Z_1, \psi^\star Z_2) + \eta(Z_2)\widetilde{h}(Z_1, N)$$

and

$$\widetilde{h}(\psi Z_1, \psi^\star Z_2) = -\widetilde{h}(Z_1, Z_2) - w^\star(Z_1)\eta(Z_2) - \eta^\star(Z_1)w(Z_2)$$

In particular, the relation

$$h(\psi Z_1, Z_2) = h(Z_1, \psi^\star Z_2)$$

is valid, for all $Z_1, Z_2 \in \Gamma(S(TH))$.

The proof is obvious by utilizing Equations 23 and 24 in Equation 1.

4. Screen Semi-Invariant Lightlike Hypersurface of Kaehler-like Statistical Manifolds

This section analyzes screen semi-invariant lightlike hypersurfaces of Kaehler-like statistical manifolds $(\widetilde{H}, \widetilde{h}, J, \widetilde{D})$.

Proposition 4.1. Let $(H, h, S(TH))$ be a screen semi-invariant lightlike hypersurface of $(\widetilde{H}, \widetilde{h}, J, \widetilde{D})$. Then, the following relations occur, for all $Z \in \Gamma(TH)$,

$$\widetilde{D}_Z\alpha - \eta^\star(D_Z\alpha)\alpha = \psi A_N Z \tag{27}$$

and

$$\eta^\star(D_Z\alpha) = -\tau^\star(Z) \tag{28}$$

Proof.
Considering $(\widetilde{H}, \widetilde{h}, J)$ is a Kaehler-like statistical manifold,

$$-\widetilde{D}_Z N = \widetilde{D}_Z J\alpha = J\widetilde{D}_Z\alpha \tag{29}$$

Using Equations 10 and 29,

$$J\widetilde{D}_Z\alpha = A_N Z - \tau^\star(Z)N \tag{30}$$

From Equation 9 in Equation 30,

$$\psi D_Z\alpha + w^\star(D_Z\alpha)\xi + \eta^\star(D_Z\alpha)N + B(Z,\alpha)\alpha = A_N Z - \tau^\star(Z)N \tag{31}$$

Because $\psi\alpha = 0$ and investigating the tangential and transversal sides of Equation 31, Equation 28 is obtained and

$$\psi D_Z\alpha + w^\star(D_Z\alpha)\xi + B(Z,\alpha)\alpha = A_N Z \tag{32}$$

From Equation 32,

$$\psi^2 D_Z\alpha + w^\star(D_Z\alpha)\psi\xi = \psi A_N Z \tag{33}$$

Using Equations 22, 25, and 33, Equation 27 is obtained. □

Definition 4.2. Let $(\widetilde{H}, \widetilde{h})$ be pseudo-Riemannian manifold and $\widetilde{D}^{\widetilde{H}}$ indicate a linear connection on $(\widetilde{H}, \widetilde{h})$. A vector field v on $\widetilde{H}$ is entitled torse-forming with regard to $\widetilde{D}^{\widetilde{H}}$ if the following circumstance is provided, for each $Z \in \Gamma(\widetilde{H})$,

$$\widetilde{D}^{\widetilde{H}}_Z v = \gamma Z + \varphi(Z)v$$

where φ is a linear form and γ is a function [25]. A torse-forming vector field is entitled

i. torqued if $\varphi(v) = 0$,

ii. concircular if $\varphi = 0$,

iii. concurrent if $\gamma = 1$ and $\varphi = 0$,

and

iv. reccurrent if $\gamma = 0$.

In view of Proposition 4.1 and Definition 4.2, the following are obtained.

Corollary 4.3. If $(H, h, S(TH))$ is totally geodesic with regard to $\widetilde{D}$, then there is no less than one vector field lying on $\Gamma(S(TH))$, reccurrent with regard to D.

Corollary 4.4. If α is a torse-forming vector field with regard to D, then $(H, h, S(TH))$ can not be totally geodesic with regard to $\widetilde{D}$.

PROOF.
Assume that α is torse-forming with regard to D. If we indite Equation 5 in Equation 27, then

$$PA_N Z = \alpha\psi Z \tag{34}$$

for each vector field Z, orthogonal to α and β. From Equation 34, if H is totally geodesic with regard to $\widetilde{D}$, then $\psi Z = 0$. This contradicts the fact that $(H, h, S(TH))$ is screen semi-invariant. □

Corollary 4.5. If α is parallel with regard to D (or $\widetilde{D}$), then the shape operator takes the following format:

$$A_N Z = w^\star(A_N Z)\beta + \eta^\star(A_N Z)\alpha$$

Corollary 4.6. There does not exist any totally umbilical semi-invariant lightlike hypersurface of $(\widetilde{H}, \widetilde{h}, J, \widetilde{D})$ admitting a parallel vector field $JN = \alpha$ with regard to D (or $\widetilde{D}$).

Contemplate the following distributions:

$$\mathbb{D}_1 = span\{\beta, \beta^\star\} \text{ and } \mathbb{D}_2 = span\{\alpha, \alpha^\star\}$$

Hence, there exists a $(n-4)-$dimensional pseudo-Riemannian distribution $\mathbb{D}$ in $S(TH)$ such that

$$S(TH) = \mathbb{D} \oplus_{orth} \{\mathbb{D}_1 \oplus \mathbb{D}_2\}$$

Therefore, from Equations 3 and 4,

$$TH = \mathbb{D} \oplus_{orth} \{\mathbb{D}_1 \oplus \mathbb{D}_2\} \oplus_{orth} Rad(TH)$$

and

$$T\widetilde{H} = \mathbb{D} \oplus_{orth} \{\mathbb{D}_1 \oplus \mathbb{D}_2\} \oplus_{orth} \{Rad\ TH \oplus ltr\ TH\}$$

From the above verities, $\mathbb{D}$ is invariant with respect to J and $J^\star$. Suppose that

$$\widetilde{\mathbb{D}} = \mathbb{D} \oplus_{orth} Rad\ TH \oplus_{orth} J(Rad\ TH) \oplus_{orth} J^\star(Rad\ TH)$$

Hence, $\widetilde{\mathbb{D}}$ is invariant with respect to J and $J^\star$.

Theorem 4.7. Let $(H, h, S(TH))$ be a screen semi-invariant lightlike hypersurface of $(\widetilde{H}, \widetilde{h}, J, \widetilde{D})$. Then, the following assertions are equivalent:

i) $\widetilde{\mathbb{D}}$ is integrable with regard to D.

ii) The equality

$$B(Z_1, tZ_2) = B(Z_2, tZ_1)$$

is valid, for all $Z_1, Z_2 \in \Gamma(\widetilde{\mathbb{D}})$.

iii) The equality

$$h(\widetilde{A}^\star_\xi Z_1, tZ_2) - h(\widetilde{A}^\star_\xi Z_2, tZ_1) = B(Z_1, \xi)\widetilde{h}(tZ_2, N) - B(Z_2, \xi)\widetilde{h}(tZ_1, N)$$

is valid, for all $Z_1, Z_2 \in \Gamma(\widetilde{\mathbb{D}})$, where $tZ_1 = \psi Z_1 + w^\star(Z_1)\xi$.

PROOF.
Since $(\widetilde{H}, \widetilde{h}, J, \widetilde{D})$ is a Kaehler-like statistical manifold, it is obvious that

$$\widetilde{D}_{Z_1} JZ_2 = J\widetilde{D}_{Z_1} Z_2 \tag{35}$$

for all $Z_1, Z_2 \in \Gamma(\widetilde{\mathbb{D}})$. If Z_2 is perpendicular to α and $\alpha^\star$, then

$$\widetilde{D}_{Z_1} J Z_2 = \widetilde{D}_{Z_1}(\psi Z_2 + w^\star(Z_2)\xi) \tag{36}$$

From Equations 9 and 36,

$$\widetilde{D}_{Z_1} J Z_2 = \widetilde{D}_{Z_1}\psi Z_2 + B(Z_1, \psi Z_2)N + Z_1\left[w^\star(Z_2)\right]\xi + w^\star(Z_2)D_{Z_1}\xi + w^\star(Z_2)B(Z_1,\xi)N \tag{37}$$

Combining Equations 9 and 23,

$$J\widetilde{D}_{Z_1} Z_2 = \psi D_{Z_1} Z_2 + w^\star(D_{Z_1} Z_2)\xi + \eta^\star(D_{Z_1} Z_2)N + B(Z_1, Z_2)\alpha \tag{38}$$

Taking into account of Equations 35, 37, and 38,

$$\begin{aligned}\widetilde{D}_{Z_1}\psi Z_2 + B(Z_1, \psi Z_2)N + Z_1\left[w^\star(Z_2)\right]\xi + w^\star(Z_2)D_{Z_1}\xi + w^\star(Z_2)B(Z_1,\xi)N = \psi D_{Z_1} Z_2 + w^\star(D_{Z_1} Z_2)\xi \\ +\eta^\star(D_{Z_1} Z_2)N + B(Z_1, Z_2)\alpha\end{aligned} \tag{39}$$

Altering the position of Z_1 and Z_2 in Equation 39,

$$\begin{aligned}\widetilde{D}_{Z_2}\psi Z_1 + B(Z_2, \psi Z_1)N + Y\left[w^\star(Z_1)\right]\xi + w^\star(Z_1)D_{Z_2}\xi + w^\star(Z_1)B(Z_2,\xi)N = \psi D_{Z_2} Z_1 + w^\star(D_{Z_2} Z_1)\xi \\ +\eta^\star(D_{Z_2} Z_1)N + B(Z_2, Z_1)\alpha\end{aligned} \tag{40}$$

If we subtract Equations 39 and 40 side to side,

$$\eta^\star(D_{Z_1} Z_2) - \eta^\star(D_{Z_2} Z_1) = B(Z_1, \psi Z_2) - B(Z_2, \psi Z_1) + w^\star(Z_2)B(Z_1,\xi) - w^\star(Z_1)B(Z_2,\xi)$$

which shows

$$B(Z_1, tZ_2) - B(Z_2, tZ_1) = \eta^\star([Z_1, Z_2]) \tag{41}$$

Taking into consideration of Equation 41, $B(Z_1, tZ_2) = B(Z_2, tZ_1)$ is provided for all $Z_1, Z_2 \in \Gamma(\widetilde{\mathbb{D}})$ if and only if $[Z_1, Z_2] \in \Gamma(\widetilde{\mathbb{D}})$. Hence, $(i) \Leftrightarrow (ii)$. From Equations 19 and 41, $(ii) \Leftrightarrow (iii)$.

□

From Theorem 4.7, the following results are obtained.

Corollary 4.8. If $(H, h, S(TH))$ is totally geodesic with regard to D, then $\widetilde{\mathbb{D}}$ is integrable with regard to D.

Corollary 4.9. If $(H, h, S(TH))$ is totally umbilical with regard to D, then $\widetilde{\mathbb{D}}$ is not integrable with regard to D.

An analogous to Theorem 4.7 is as follows:

Theorem 4.10. For any screen semi-invariant lightlike hypersurface $(H, h, S(TH))$, the following assertions are equivalent:

i) $\widetilde{\mathbb{D}}$ is integrable with regard to $D^\star$.

ii) The equality

$$B^\star(Z_1, t^\star Z_2) = B^\star(Z_2, t^\star Z_1)$$

is valid, for all $Z_1, Z_2 \in \Gamma(\widetilde{\mathbb{D}})$.

iii) The equality

$$h(\widetilde{A}_\xi Z_1, t^\star Z_2) - h(A^\star_\xi Z_2, t^\star Z_1) = B^\star(Z_1,\xi)\widetilde{h}(t^\star Z_2, N) - B^\star(Z_2,\xi)\widetilde{h}(t^\star Z_1, N)$$

is valid, for all $Z_1, Z_2 \in \Gamma(\widetilde{\mathbb{D}})$, where $t^\star Z_1 = \psi^\star Z_1 + w(Z_1)\xi$.

Theorem 4.11. $(H, h, S(TH))$ is mixed geodesic with regard to $\widetilde{D}$ if and only if $A^\star_N Z$ is included in $\mathbb{D}_1^\perp$, for all $Z \in \Gamma(\widetilde{\mathbb{D}})$.

Proof.
Assume that $(H, h, S(TH))$ is mixed geodesic with regard to $\widetilde{D}$. In view of Equations 9 and 10,

$$\widetilde{D}_Z\alpha = D_Z\alpha + B(Z,\alpha)N \tag{42}$$

and

$$J\widetilde{D}_Z N = -tA^\star_N Z - \eta^\star(A^\star_N Z)N + \tau^\star(Z)\alpha \tag{43}$$

for all $Z \in \Gamma(\widetilde{\mathbb{D}})$. Since $(\widetilde{H}, \widetilde{h}, J)$ is a Kaehler-like statistical manifold, we derive the following relation using Equations 42 and 43:

$$0 = B(Z,\alpha) = -\eta^\star(A^\star_N Z) = -\widetilde{g}(JA^\star_N Z, \xi)$$

Hence, $h(A^\star_N Z, \beta^\star) = 0$. With similar arguments,

$$\widetilde{D}_Z\alpha^\star = D_Z\alpha^\star + B(Z,\alpha^\star)N \tag{44}$$

and

$$J^\star\widetilde{D}_Z N = -h^\star A^\star_N Z - \eta(A^\star_N Z)N + \tau^\star(Z)\alpha^\star \tag{45}$$

From Equations 44 and 45,

$$0 = B(Z,\alpha^\star) = -\eta(A^\star_N Z) = -\widetilde{g}(J^\star A^\star_N Z, \xi)$$

Hence, $h(A^\star_N Z, \beta) = 0$. Therefore, $A^\star_N Z$ is included in $\mathbb{D}_1^\perp$ for all $Z \in \Gamma(\widetilde{\mathbb{D}})$. The proof of converse is clear. □

With a similar method of Theorem 4.11, the following result is obtained.

Theorem 4.12. $(H, h, S(TH))$ is mixed geodesic with regard to $\widetilde{D}^\star$ if and only if $A_N Z$ is included in $\mathbb{D}_1^\perp$, for all $Z \in \Gamma(\widetilde{\mathbb{D}})$.

5. Conclusion

This study investigates the geometry of screen semi-invariant lightlike hypersurfaces, where almost complex structures J and includes $J^\star$ in the screen distribution. With this perspective, new types of lightlike hypersurfaces can be introduced. For example, the cases where almost complex structures J and $J^\star$ are invariant or anti-invariant in the radical space or invariant and anti-invariant on the screen space can be examined. Thus, the problem of the existence of new kinds of lightlike hypersurfaces for almost Hermite-like manifolds and Kaehler-like statistical manifolds arises in the future.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

Acknowledgement

This work was supported by the Scientific and Technological Research Council of Turkey (TÜBİTAK), Project number: 122F326.

References

[1] K. Takano, *Statistical Manifolds with Almost Contact Structures and Its Statistical Submersions*, Journal of Geometry 85 (1-2) (2006) 171–187.

[2] K. Takano, *Statistical Manifolds with Almost Complex Structures*, in: H. Shimada (Ed.), Proceedings of the 45th Symposium on Finsler Geometry, Tokyo, 2011, pp. 54–57.

[3] M. Aquib, *Some Inequalities for Statistical Submanifolds of Quaternion Kaehler-like Statistical Space Forms*, International Journal of Geometric Methods in Modern Physics 16 (08) (2019) 1950129 16 pages.

[4] M. Aquib, M. H. Shahid, *Generalized Normalized δ-Casorati Curvature for Statistical Submanifolds in Quaternion Kaehler-like Statistical Space Forms*, Journal of Geometry 109 (1) (2018) Article Number 13 13 pages.

[5] H. Aytimur, M. Kon, A. Mihai, C. Özgür, K. Takano, *Chen Inequalities for Statistical Submanifolds of Kähler-like Statistical Manifolds*, Mathematics 7 (12) (2019) Article ID 1202 19 pages.

[6] H. Aytimur, C. Özgür, *On Cosymplectic-like Statistical Submersions*, Mediterranean Journal of Mathematics 16 (2019) 1–14.

[7] S. Kazan, K. Takano, *Anti-invariant Holomorphic Statistical Submersions*, Results in Mathematics 78 (4) (2023) Article Number 128 18 pages.

[8] S. Kazan, *Anti-Invariant $\xi^{\perp}$-Cosymplectic-like Statistical Submersions*, Thermal Science 26 (4) (2022) 2991–3001.

[9] A. N. Siddiqui, S. Uddin, M. H. Shahid, *B.-Y. Chen's Inequality for Kähler-like Statistical Submersions*, International Electronic Journal of Geometry 15 (2) (2022) 277–286.

[10] M. D. Siddiqi, A. N. Siddiqui, F. Mofarreh, H. Aytimur, *A Study of Kenmotsu-like Statistical Submersions*, Symmetry 14 (8) (2022) Article ID 1681 13 pages.

[11] A. D. Vilcu, G. E. Vilcu, *Statistical Manifolds with Almost Quaternionic Structures and Quaternionic Kähler-like Statistical Submersions*, Entropy 17 (9) (2015) 6213–6228.

[12] K. L. Duggal, B. Şahin, Differential Geometry of Lightlike Submanifolds, Birkhäuser, Basel, 2011.

[13] K. L. Duggal, D. H. Jin, *Totally Umbilical Lightlike Submanifolds*, Kodai Mathematical Journal 26 (2003) 49–68.

[14] S. Amari, Differential-Geometrical Methods in Statistics, Springer, New York, 1985.

[15] H. Furuhata, *Hypersurfaces in Statistical Manifolds*, Differential Geometry and its Applications 27 (3) (2009) 420–429.

[16] O. Bahadır, *On Lightlike Geometry of Indefinite Sasakian Statistical Manifolds*, AIMS Mathematics 6 (11) (2021) 12845–12862.

[17] O. Bahadır, M. M. Tripathi, *Geometry of Lightlike Hypersurfaces of a Statistical Manifold*, WSEAS Transactions on Mathematics 22 (2023) 466–474.

[18] O. Bahadır, A. N. Siddiqui, M. Gülbahar, A. H. Alkhaldi, *Main Curvatures Identities on Lightlike Hypersurfaces of Statistical Manifolds and Their Characterizations*, Mathematics 10 (13) (2022) Article ID 2290 18 pages.

[19] B. E. Acet, S. Yüksel Perktaş, E. Kılıç, *On Lightlike Geometry of Para-Sasakian Manifolds*, The Scientific World Journal 2014 (2014) Article ID 696231 12 pages.

[20] B. E. Acet, *Lightlike Hypersurfaces of Metallic Semi-Riemannian Manifolds*, International Journal of Geometric Methods in Modern Physics 15 (12) (2018) Article ID 1850201 16 pages.

[21] E. Kılıç, B. Şahin, S. Keleş, *Screen Semi Invariant Lightlike Submanifolds of Semi-Riemannian Product Manifolds*, International Electronic Journal of Geometry 4 (2) (2011) 120–135.

[22] S: Y. Perktaş, E. Kılıç, B. E. Acet, *Lightlike Hypersurfaces of a Para-Sasakian Space Form*, Gulf Journal of Mathematics 2 (2) (2014) 7–18.

[23] N. Ö. Poyraz, E. Yaşar, *Lightlike Hypersurfaces of a Golden Semi-Riemannian Manifold*, Mediterranean Journal of Mathematics 14 (2017) 1–20.

[24] N. Ö. Poyraz, *Screen Semi-Invariant Lightlike Submanifolds of Golden Semi-Riemannian Manifolds*, International Electronic Journal of Geometry 14 (1) (2021) 207–216.

[25] K. Yano, *On the Torse-Forming Directions in Riemannian Spaces*, Proceedings of the Imperial Academy Tokyo 20 (6) (1944) 340–345.