SOME CONTRIBUTIONS TO REGULAR POLYGONS

DENIZ ÖNCEL AND MURAT KIRIŞCI

ABSTRACT. The aim of this work is to use Napoleon's Theorem in different regular polygons, and decide whether we can prove Napoleon's Theorem is only limited with triangles or it could be done in other regular polygons that can create regular polygons.

1. INTRODUCTION

The famous theorem of Napoleon is one of the most interesting assertions from elementary geometry of planar figures. Although over 150 years have passed, mathematicians still continues to be of interest to Napoleon's Theorem as both professional and amateur alike. Because, this theorem is an important and useful tool to expand the mathematical horizon for many mathematicians. In the Euclidean Plane, Napoleon's Theorem is easily proven. A wide variety of proofs of this theorem have been given in a lot of manuscripts.

In this work, we give a well known simple proof of Napoleon's Theorem based on Sine and Cosine Laws. Further, we apply the Theorem to regular polygon such as regular hexagon, square and octagon.

Napoleon's Theorem states the fact that if equilateral triangles are drawn outside of any triangle, the centers of the equilateral triangles will form an equilateral triangle.

We take a main triangle ΔABC, three equilateral triangles, which are outside ΔABC are ΔADB, ΔBCE, ΔAFC. We denote the discovered triangle by ΔPQR. Then, $\widehat{FAC} = \widehat{ACF} = \widehat{CFA} = \widehat{ADB} = \widehat{BAD} = \widehat{DBA} = \widehat{BCE} = \widehat{ECB} = \widehat{CBE} = 60°$ (Figure 1).

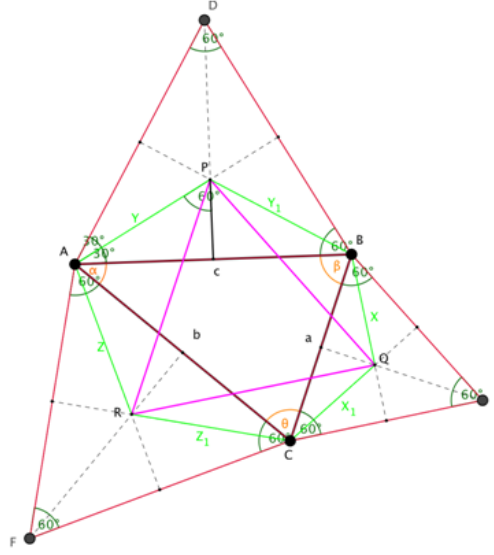We give a well known simple proof of Napoleon's Theorem as follows:

FIGURE 1

Proof. [1] Since ΔECB, ΔFCA, ΔADB are equilateral, then $\widehat{FAC} = \widehat{ACF} = \widehat{CFA} = \widehat{ADB} = \widehat{BAD} = \widehat{DBA} = \widehat{BCE} = \widehat{ECB} = \widehat{CBE} = 60°$. We take $|CQ|$, $|BQ|$, $|BP|$, $|AP|$, $|AR|$, $|CR|$ is an angle bisector. Thus, $\widehat{PAB} = \widehat{PBA} = \widehat{QBC} = \widehat{QCB} = \widehat{RCA} = \widehat{RAC} = 30°$. Isosceles triangles have two equal sides, that is, $x = x_1$, $y = y_1$, $z = z_1$ and so use the fact that the centroid of an equilateral triangle ΔADB, ΔBCE, ΔAFC lies along each median, $2/3$ of the distance from the vertex to the midpoint of the opposite side, $x = x_1 = \frac{2\frac{a\sqrt{3}}{2}}{3} = \frac{a\sqrt{3}}{3}$, $y = y_1 = \frac{c}{\sqrt{3}}$, $z = z_1 = \frac{b}{\sqrt{3}}$. If we use the Cosine formula for ΔPQR, then, we have

$$|RP|^2 = y^2 + z^2 - 2yz\cos(\alpha + 60°)$$
$$|RP|^2 = \left(\frac{c}{\sqrt{3}}\right) + \left(\frac{b}{\sqrt{3}}\right) - 2\frac{c}{\sqrt{3}}\frac{b}{\sqrt{3}}\cos(\alpha + 60°)$$
$$3|RP|^2 = b^2 + c^2 - 2bc\cos(\alpha + 60°)$$
$$3|RP|^2 = b^2 + c^2 - 2bc(\cos\alpha\cos 60° - \sin\alpha\sin 60°)$$
$$3|RP|^2 = b^2 + c^2 - 2bc(\frac{1}{2}\cos\alpha - \frac{\sqrt{3}}{2}\sin\alpha).$$

If we apply the Law of Cosine to ΔABC and the Law of Sine for the area of ΔABC, then, we can write $a^2 = b^2 + c^2 - 2bc\cos\alpha$ and $2(Area\ of\ \ \Delta ABC) = bc\sin\alpha$, respectively. Substituting these statement into the $3|RP|^2 = b^2 + c^2 - 2bc(\frac{1}{2}\cos\alpha - \frac{\sqrt{3}}{2}\sin\alpha)$ gives

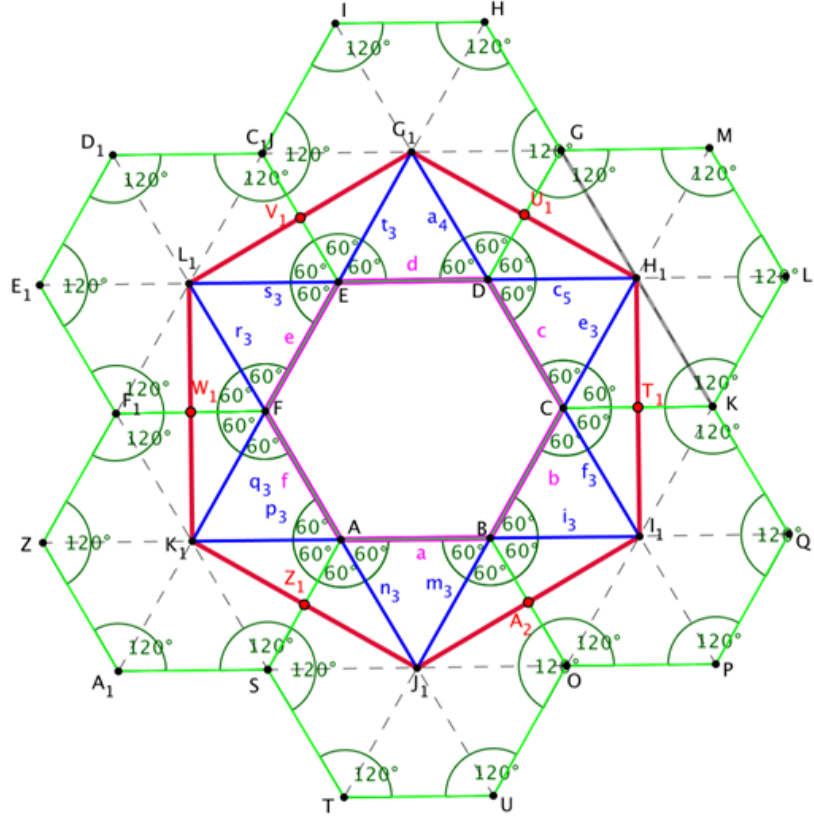$$3|RP|^2 = \frac{1}{2}\left(a^2 + b^2 + c^2\right) + 2\sqrt{3}(Area\ of\ \ \Delta ABC).$$

FIGURE 2

In the same idea, we can compute the length of $|PQ|$ and $|RQ|$. Since, the ΔAPB, ΔBQC, ΔCRA are isosceles triangles, we say $\alpha = \beta = \theta$ and also $|RQ| = |PQ| = |RP|$. Then, it follows that the ΔPQR connecting the three centroids is equilateral.

$\square$

2. MAIN RESULTS

In this section, we will apply Napoleon's Theorem to some regular polygon and prove as in Section 1.

2.1. **Application to Regular Hexagon.** Consider the regular hexagon $ABCDEF$ and the six regular hexagons, which are outside the main regular hexagon(Figure 2).

Each interior of this hexagon is $120°$, because the sum of the interior angles of any hexagon is $720°$. Now, we divide regular hexagons into six pieces by using its diagonals. It will make six equilateral triangles, which are all $60°$. Then, $\widehat{G_1ED} = \widehat{G_1DE} =< \widehat{H_1DC} = \widehat{H_1CD} = \widehat{I_1CB} = \widehat{I_1BC} = \widehat{J_1BA} = \widehat{J_1AB} =$

$\widehat{K_1AF} = \widehat{K_1FA} = \widehat{L_1EF} = \widehat{L_1FE} = 60°.$

The diagonals of regular hexagon are equal. Hence half of the diagonals are equal too. Hence, $c_5 = e_3$, $m_3 = n_3$, $f_3 = i_3$, $t_3 = a_4$, $r_3 = s_3$, $q_3 = p_3$. Also, $a = b = c = d = e$, since regular hexagon has equal sides. We know that the equilateral triangles have equal sides. From this, we can write $c_5 = e_3 = c$, $f_3 = i_3 = b$, $m_3 = n_3 = a$, $q_3 = p_3 = f$, $r_3 = s_3 = e$ and $t_3 = a_4 = d$. In that case, we obtain $\Delta CH_1I_1 = \Delta BI_1J_1 = \Delta AK_1J_1 = \Delta FL_1K_1 = \Delta EG_1L_1 = \Delta DH_1G_1$ and $|H_1I_1| = |I_1J_1| = |J_1K_1| = |K_1L_1| = |L_1G_1| = |G_1H_1|$, as we desired.

The ratio between the lengths of edges of the main hexagon and the discovered/new hexagon:

Let's think that one side of the main hexagon is x, $(x \in \mathbb{Z})$.

$$\frac{\sin 60°}{x} = \frac{\sin 60°}{|CH_1|}$$
$$x = |CH_1|$$
$$\Delta H_1DC \text{ is isosceles triangle since} |CH_1| = |DH_1|.$$

Also, ΔH_1DC and ΔI_1CB are equal hence $|DH_1| = |BI_1|$, the edges $c = c_5 = e_3 = x$. Hence, from the special triangle $(30 - 60 - 90)$ in the ΔCH_1T_1

$$\frac{\sin 90°}{x} = \frac{\sin 60°}{|H_1T_1|}$$
$$\frac{\sqrt{3}}{2}x = |H_1T_1|.$$

Since $|H_1I_1|$ is the double of $|H_1T_1|$ because the edges see the same degree, $60°$

$$|H_1I_1| = \sqrt{3}x.$$

The one side of the main hexagon is x then, one side of the discovered hexagon need to be $\sqrt{3}x$, since the ratio between sides won't change no matter the size of the hexagon because same sine law will be applied with the same, equal angles.

2.2. **Application to Square.** The main square is $ABCD$ and the four squares, which are outside the main square(Figure 3).

Each interior of a square is $90°$, because the sum of the interior angles of any quadrilateral is $360°$. DG and CH, BJ and CI, BK and AL, MA and ND are the bisector of the squares. Then, $\widehat{ODC} = \widehat{OCD} =< \widehat{PCB} = \widehat{PBC} = \widehat{QBA} = \widehat{QAB} = \widehat{H_1AD} = \widehat{H_1DA} = \widehat{ODC} = \widehat{OCD} = 45°.$

The diagonals of square are equal, so half of the diagonals are equal too. From this statement, $u = w$, $o = p_1$, $q_1 = d_2$, $e_2 = q$. Then, we can write the isosceles triangles $\Delta ODC = \Delta PDB = \Delta BQA = \Delta AH_1D = \Delta DOC$. Since, square has equal sides, then $g = f = e = d$. Using the special triangle (45-45-90), $\frac{g\sqrt{2}}{2} = w$, $\frac{f\sqrt{2}}{2} = o$, $\frac{e\sqrt{2}}{2} = d_2$, $\frac{d\sqrt{2}}{2} = q$ and $\frac{g\sqrt{2}}{2} = w = o = d_2 = q$. Thus, $|OP| = |PQ| = |QH_1| = |H_1O|$, as we desired.
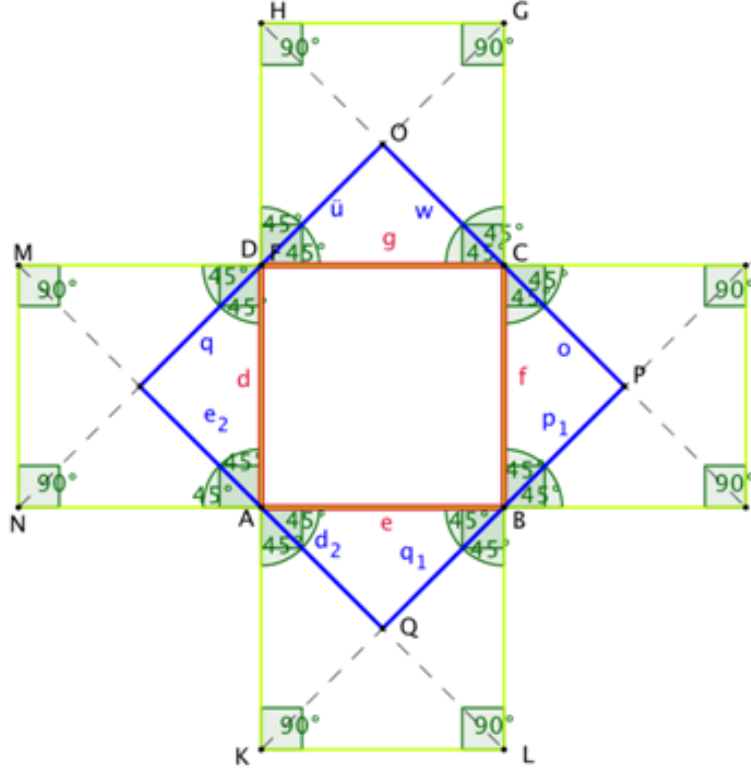
FIGURE 3

The ratio between the lengths of edges of the main square and the discovered/new square:

Let's think that one side of the main square is x, $(x \in \mathbb{Z})$.

$$\frac{\sin 90°}{x} = \frac{\sin 45°}{|OC|}$$

$$\frac{\sqrt{2}}{2}x = |OC|.$$

Since $|OP|$ is the double of $|DC|$ because of edges $w = o$, $|OP| = \sqrt{2}x$. The one side of the main square is x, then one side of the discovered square need to be $\sqrt{2}x$ since the ratio between the sides will not change no matter the size of the square is because same sine law will be applied with the same angles.

2.3. **Application to Octagon.** The main octagon is $ABCDEFGH$ and the eight octagons, which are outside the main octagon(Figure 4).

Now, we know that, the sum of the interior angles of any octagon is $1080°$ and the diagonals are bisectors of the octagon, which are $67,5°$. Thus, we obtain that each of interior angle of a regular octagon is $135°$. And also, we can say that $\widehat{I_2FE} = \widehat{I_2EF} =< \widehat{J_2ED} =< \widehat{J_2DE} =< \widehat{K_2CD} =< \widehat{K_2DC} =< \widehat{L_2CB} =<$
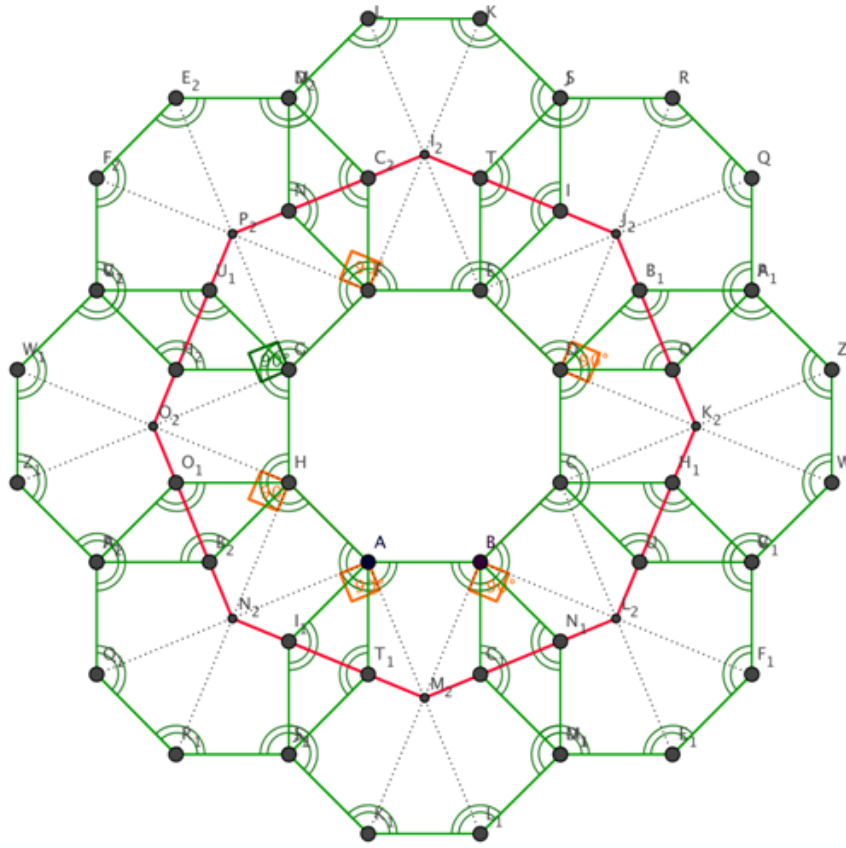
FIGURE 4

$$\widehat{L_2BC} =< \widehat{BM_2A} =< \widehat{AM_2B} =< \widehat{AN_2H} =< \widehat{HN_2A} =< \widehat{HO_2C} =< \widehat{CO_2H} =<$$
$$\widehat{CP_2F} =< \widehat{FP_2C}.$$

Diagonals of regular octagon are equal. So half of the diagonals are equal too. Therefore, $|I_2F| = |I_2E| = |J_2E| = |J_2D| = |K_2D| = |K_2C| = |L_2C| = |L_2B| = |BM_2| = |AM_2| = |N_2A| = |N_2H| = |O_2H| = |O_2C| = |P_2C| = |P_2F|$. Since they see the same angle, which is $45°$, then $|CF| = |FE| = |ED| = |DC| = |CB| = |BA| = |AH| = |HC|$. The triangles $\Delta AM_2B = \Delta BL_2C = \Delta CK_2D = \Delta DJ_2F = \Delta FI_2E = \Delta CP_2F = \Delta HO_2C = \Delta AN_2H$ are isosceles triangles. This step completes the proof.

The ratio between the lengths of edges of the main octagon and the discovered/new octagon:

Assume that one side of the main regular octagon is x, $(x \in \mathbb{Z})$.

$$\frac{\sin 45°}{x} = \frac{\sin 67,5°}{|K_2C|}$$
$$1,31x = |K_2C|.$$

Also $|K_2C|$ and $|CL_2|$ are equal because they see the same angle which is $67,5°$. If we say the edges which see $45°$ will be x then the edges which see $90°$ will be $\sqrt{2}x$.

$$|K_2C| = |CL_2| = 1,31x$$
$$|K_2L_2| = 1,85x.$$

The one side of the main octagon is x then, one side of the discovered octagon need to be $1.85x$ since the ratio between sides won't change no matter the size of the hexagon because same sine law will be applied with the same, equal angles.

3. Conclusion

In this work, we have investigated how Napoleon's Theorem is applied to regular polygons. Firstly, we gave a simple proof of the Theorem using Sine and Cosine Laws. Further, we applied the Theorem to regular polygons such as hexagon, square and octagon, with the same idea. We draw a triangle, which is not equilateral, and then draw equilateral triangles outside the triangle. We got the centroids of the equilateral triangle, unit them with lines.

In our application on hexagon, we found that the main regular hexagon was x and the discovered square was $\sqrt{3}x$. Hence, the main regular hexagon would every time be nearly twice as small as the new regular hexagon. We did the same ratios in the application of square. This times the main square was x, and the discovered square was $\sqrt{2}x$. Hence, this time the main square was one and a half smaller than the new square. And also, we used the octagon. Thus, the regular polygons length ratios which are the main and discovered polygon, was transferred in graphs and seen that there was a strong correlation between them.

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DEPARTMENT OF MATHEMATICAL EDUCATION, HASAN ALI YÜCEL EDUCATION FACULTY, ISTANBUL UNIVERSITY, VEFA, 34470, FATIH, ISTANBUL, TURKEY
E-mail address: `denizoncel844@gmail.com`

DEPARTMENT OF MATHEMATICAL EDUCATION, HASAN ALI YÜCEL EDUCATION FACULTY, ISTANBUL UNIVERSITY, VEFA, 34470, FATIH, ISTANBUL, TURKEY
E-mail address: `mkirisci@hotmail.com, murat.kirisci@istanbul.edu.tr`