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Hidden Attractors in Chaotic Systems with Nonlinear Functions

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ABSTRACT In the present work, an interesting mini-review of hidden attractors in dynamical systems with associated nonlinear functions is carried out. Chaotic systems with nonlinear functions often possess hidden attractors due to their inherent complexity. These attractors can arise in various mathematical models, such as the Lorenz system, Rössler system, or Chua's circuit. The identification and comprehension of hidden attractors broaden our understanding of complex systems and provide new directions for future study and technological development. The discovery and characterization of hidden attractors in chaotic systems have profound implications for various scientific disciplines, including physics, biology, and engineering.

KEYWORDS
Hidden attractors
Chaotic systems
Nonlinear functions
Fixed points
Numerical simulations

INTRODUCTION

Bifurcation theory deals with the study of how certain behaviors or patterns in a system change as its parameters vary (Dueñas *et al.* 2023). One interesting phenomenon in this theory is the concept of a *hidden oscillation* (Ye and Wang 2023). This refers to a bounded back-and-forth movement that emerges in a system without causing the stationary points (equilibrium states) of the system to become unstable (Djorwe *et al.* 2023).

In nonlinear control theory, we focus on managing systems that do not have a simple linear relationship between their input and output (Gray *et al.* 2023). When we talk about the birth of a hidden oscillation in a time-invariant control system (meaning the system doesn't change over time) with bounded states (the system's variables remain within certain limits), it implies reaching a critical point in the parameter space (Kuznetsov 2020). At this critical point, the stationary states of the system switch from being locally stable (stable in the nearby region) to becoming globally stable (Kuznetsov *et al.* 2020) (stable across the entire system).

In simpler terms, when a system has hidden movements or vibrations that exist within a small part of its overall behavior, and these hidden motions draw in all the nearby movements, we call it a hidden attractor (Djorwe *et al.* 2023). This means that even though these movements may not be obvious at first glance, they have a strong influence on the nearby motions of the system.

The study of hidden attractors gained further momentum in the 21st century, with results obtained by researchers applying advanced analytical and computational techniques to uncover these elusive phenomena (Wang *et al.* 2021; Gong *et al.* 2022; Kuznetsov *et al.* 2023; Zaqueros-Martinez *et al.* 2023). Scientists have explored various mathematical models and physical systems to identify hidden attractors and understand their underlying mechanisms.

From a computational perspective, attractors can be classified into two categories: self-excited attractors and hidden attractors. Self-excited attractors can be easily localized using standard computational procedures and standard analytical procedures (Yang and Lai 2023). These attractors exhibit a transient process where a trajectory, starting from a point on the unstable manifold near an equilibrium, eventually reaches a state of oscillation (Lakshmanan and Rajaseekar 2012). Examples of systems with self-excited attractors include the Lorenz (Dubois *et al.* 2020), Rössler (Rybin *et al.* 2021), and Chua oscillators (Njitacke *et al.* 2020). The presence of self-excited attractors can be readily identified due to the observable oscillatory behavior.

In contrast, hidden attractors pose a greater challenge for localization. In these systems, the basin of attraction does not intersect with any small neighborhoods of equilibria (Cang *et al.* 2019). Hidden attractors can exhibit both chaotic and periodic behavior, such as the coexistence of a stable stationary point and a stable limit

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cycle. Unlike self-excited attractors, the existence of hidden attractors in the phase space is not easily predictable. Therefore, special procedures need to be developed to localize hidden attractors since there are no analogous transient processes leading to their emergence. If a hidden attractor is present in the dynamics of a system and happens to be reached, the system (such as an airplane or electronic circuit) can exhibit quasi-cyclic behavior (Zelinka 2016), which can potentially result in disastrous consequences depending on the nature of the device. Traditionally, dynamical systems without equilibrium points have been considered nonphysical or mathematically incomplete. However, empirical evidence shows that systems can possess hidden dynamical behavior without the presence of an unstable equilibrium state (Dudkowski *et al.* 2016).

In short, hidden attractors represent a unique challenge in the study of dynamical systems. Their existence is not easily predictable, and special procedures are required for their localization (Campos *et al.* 2020). While systems with hidden attractors have been viewed as nonphysical in the past, it is now evident that such behavior can occur even without an unstable equilibrium state.

The discovery and characterization of hidden attractors in chaotic systems have profound implications for various scientific disciplines, including physics (Kingni *et al.* 2019; Kuznetsov *et al.* 2023), biology (Chen *et al.* 2020; Lin *et al.* 2020), and engineering (Abdolmohammadi *et al.* 2018; Jasim *et al.* 2021). Hidden attractors are also found in nonlinear systems with applications considering fuzzy control and synchronization, as the works reported by (Tanaka *et al.* 1998; Zaqueros-Martinez *et al.* 2023)

In Section 2 of this short review, addressed the fascinating areas of hidden chaotic attractors in nonlinear dynamical systems. we explored the domain of hidden chaotic attractors without equilibria. Then we discussed hidden chaotic attractors that coexist with equilibria. We also ventured into the realm of hidden chaotic attractors that exhibit extreme multi-stability. Finally, we have studied hidden chaotic attractors with multi-scroll, a class of hidden attractors characterized by their complex multidimensional structure. Section 3, summarises the main discussion on these hidden chaotic attractors, which make a valuable contribution to combining the different studies for a better understanding of nonlinear dynamics.

HIDDEN ATTRACTORS IN NONLINEAR CHAOTIC DYNAMICAL SYSTEMS

In nonlinear chaotic dynamical systems, the region in phase space where a hidden attractor exerts its influence is not connected to any unstable equilibrium point. This phenomenon can be seen in systems where there are either no unstable equilibrium points at all or only one stable equilibrium point, which is a specific instance of having multiple stable equilibrium points. This characteristic defines the nature of hidden attractors in such systems.

In this section, we discuss hidden attractors with different aspects.

Hidden attractor in chaotic dynamical systems

The idea of hidden attractors has been proposed in relation to the identification of unforeseen attractors in Chua's circuit. These unexpected behaviors in the circuit's dynamics have been discussed in various studies (Wang *et al.* 2021; Wu *et al.* 2021; Kuznetsov *et al.* 2023).

We discuss hidden attractors in chaotic dynamical systems with an interesting example of the classical Lorenz system (Munmuangsaen and Srisuchinwong 2018). The classical Lorenz system is explained using three connected equations that represent simple mathematical relationships:

$$\begin{aligned} \dot{x} &= a(y - x), \\ \dot{y} &= -xz + rx - y, \\ \dot{z} &= xy - bz. \end{aligned} \tag{1}$$

Figure 1 shows a newly discovered chaotic attractor represented in red on a coordinate plane (x, y) with parameters $a = 4, r = 29$, and $b = 2$. This attractor is revealed using the starting values $L_1 = (x_0, y_0, z_0) = (5, 5, 5)$. The figure also displays two point attractors, one in blue and the other in pink. These point attractors move towards stable equilibrium points S_2 and S_3, respectively. The blue point attractor starts at $L_2 = (x_0, y_0, z_0) = (0.1, 0, 0)$, while the red point attractor starts and also in 3D is shown in Figure 2. The Runge-Kutta method of order 4 (RK4) is a numerical technique that is used to solve the nonlinear differential equations (ODEs) of system (1) with time step size 0.01 and total number of steps are 2^{16}.

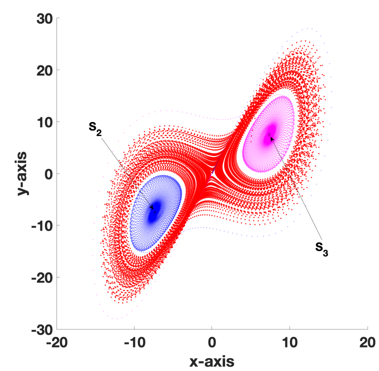


Figure 1 Classical Lorenz system plotted on a coordinate plane (x, y), a novel chaotic attractor, depicted in red, has emerged alongside two distinct point attractors, represented in blue and pink.

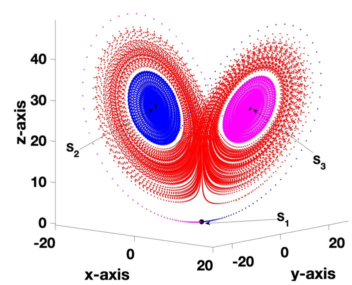


Figure 2 Classical Lorenz system plotted on coordinates (x, y, z), a novel chaotic attractor, depicted in red, has emerged alongside two distinct point attractors, represented in blue and pink.

Hidden chaotic attractors without equilibria

Hidden chaotic attractors without equilibria are a fascinating phenomenon in the field of nonlinear dynamics. Unlike well-studied chaotic systems with equilibria (such as the Lorenz system or the Rössler system), these attractors do not have stable fixed points.

Instead, they exhibit chaotic behavior with no underlying stable states. The discovery and study of such systems have challenged traditional notions of chaotic dynamics.

As an example, the Sprott case D system pioneered the investigation of a dynamical system that does not have equilibria, along with its various modifications (Wei 2011). The following is the system that has hidden chaotic attractors without equilibria/fixed points, as depicted in Figure 3.

$$\begin{aligned} \dot{x} &= -y, \\ \dot{y} &= x + z, \\ \dot{z} &= 3y^2 + xz. \end{aligned} \tag{2}$$

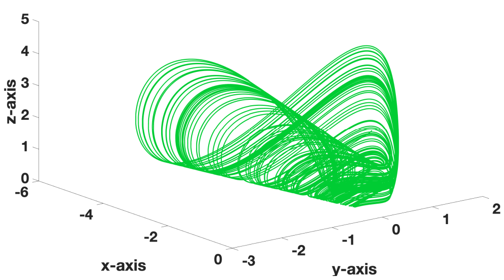


Figure 3 Hidden chaotic attractor of the system (2) with no equilibria in system plotted on coordinates (x, y, z) with the initial value (-1.6, 0.82, 1.9).

The chaotic system (2) has a single equilibrium point at $O(0,0,0)$. If we analyze the linearized version of the system at this equilibrium point, the characteristic values $(\lambda_1, \lambda_2, \lambda_3)$ of the Jacobian matrix are $\lambda_1 = 0$ and $\lambda_{2,3} = \pm i$.

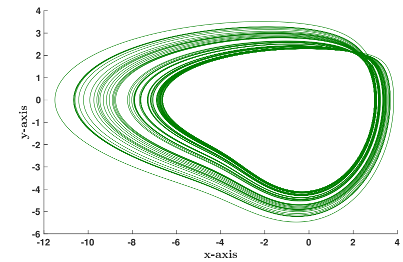
Jafari and Sprott conducted a mathematical exploration to identify the most basic three-dimensional chaotic systems with hidden attractor without equilibria (Jafari *et al.* 2013). The following is the mathematical modeling and depicted in Figure 4.

The Runge-Kutta method of order 4 (RK4) is used for systems (2) & (3) with time step size 0.01 and total number of steps are 2^{16}.

$$\begin{aligned} \dot{x} &= -y, \\ \dot{y} &= -x + z, \\ \dot{z} &= -0.8x^2 + z^2 + 2. \end{aligned} \tag{3}$$

This system incorporated quadratic nonlinear and the absence of equilibria.

Many other researchers worked on hidden attractors that are chaotic systems with no equilibrium. Pham et al. discussed a novel autonomous system with a hidden attractor there is no equilibrium point in this system (Pham *et al.* 2017). Although their proposed system is simple with six terms, it exhibits complex behavior. Moving forward, Lai et al. have created a novel chaotic system and designed both the model and the circuit itself (Lai *et al.* 2020). This system behavuniquely way, it does not follow the usual patterns, and it has a hidden attractor with no equilibrium. Furthermore, Nag and Ghosh have developed an innovative 3D system that has some unique features (Nag Chowdhury and Ghosh 2020). In this



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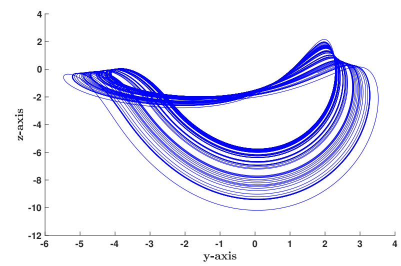

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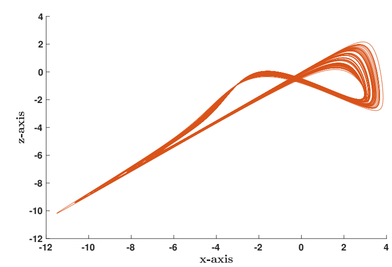


(c)

Figure 4 Hidden attractor of the system (3) without fixed-point (a) in xy-plane, (b) in yz-plane and (c) in xz-plane with the initial value (0, 2.3, 0).

system, there are certain hidden attractors with no equilibrium that cannot be predicted or tracked by conventional methods. The behavior of the system can be seen as a slow and steady trend by looking at its changes over time.

Hidden chaotic attractors with equilibria

The study of hidden chaotic attractors with equilibria remains a vibrant area of research. These systems exhibit a combination of stable equilibria and chaotic behavior.

We consider Wang and Chen's work as an example (Wang and Chen 2012). They introduced a chaotic system that operates in three dimensions. Within this system, there is a unique equilibrium point $p^\star = (0.25, 0.0625, -0.096)$ as shown in Figure 5 & 6.

$$\begin{aligned} \dot{x} &= yz + 0.006, \\ \dot{y} &= x^2 - y, \\ \dot{z} &= 1 - 4x. \end{aligned} \tag{4}$$

M. Molaie found twenty-three systems that have hidden attractors with one equilibrium point (Molaie *et al.* 2013). We found another fascinating example from one of those three-dimensional nonlinear systems. Following is the system and It is illustrated in Figure 7. The Runge-Kutta method of order 4 (RK4) is used to solve the

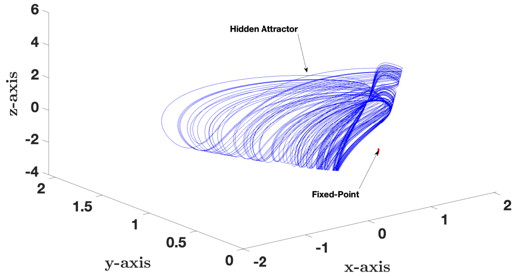
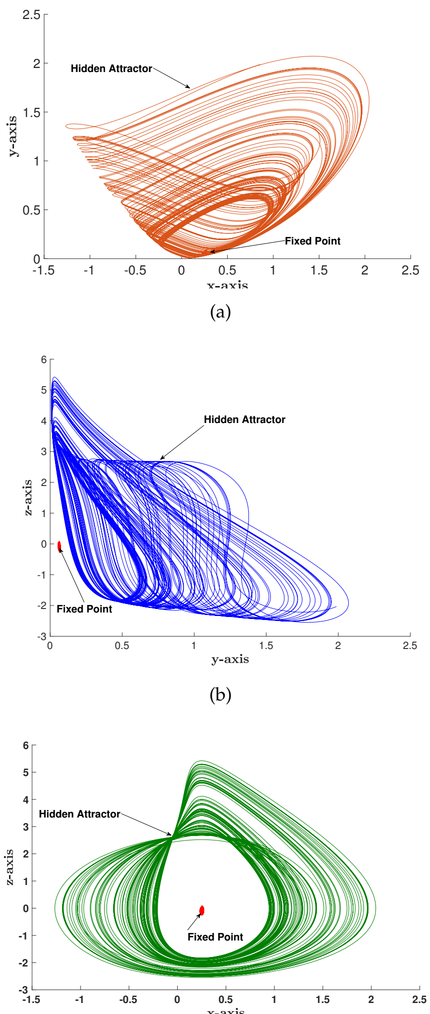


Figure 5 Hidden attractor of the system (4) with stable fixed-point in 3D with the initial value (0, 0, 0).



(a)

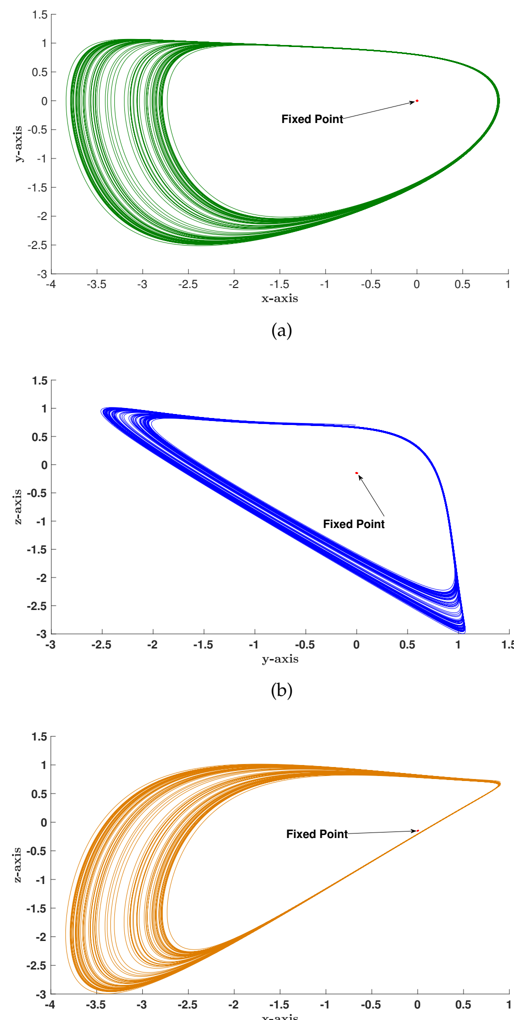
(b)

(c)

Figure 6 Hidden attractor of the system (4) with stable fixed-point in different phase spaces with the initial value (0, 0, 0). (a) in xy-axis, (b) in yz-axis, (c) in xz-axis.

nonlinear system (4) & (5) with time step size 0.01 and total number of steps are 2^{16}.

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x + yz, \\ \dot{z} &= 2x - 2z + y^2 - 0.3. \end{aligned} \tag{5}$$



(a)

(b)

(c)

Figure 7 Hidden attractor of the system (5) with fixed-point in different phase spaces have initial value (0.9, 0, 0.7). (a) in xy-axis, (b) in yz-axis, (c) in xz-axis.

Other chaotic systems with equilibrium points were also explained.

Gong et al. have developed a chaotic system that generates both four-wing and single-wing hidden patterns, with only one stable node-focus equilibrium point (Gong *et al.* 2020). In addition, Cao and Zhao presented a unique chaotic system that exists in four dimensions and exhibits various interesting behaviors (Cao and Zhao 2021). The proposed system is characterized by three quadratic nonlinearity terms and exhibits various types of hidden attractors with equilibrium points. Further, Islam et al. studied a three-dimensional chaotic system that makes a hidden chaotic attractor with a line equilibrium in which a single non-bifurcation parameter is used to control the amplitude and frequency (Islam *et al.* 2022).

Hidden chaotic attractor with extreme multi-stability

The study of multistability in the context of hidden chaotic attractors is crucial. Multistability with hidden attractors means that a system may have more than one stable state and that these stable states may not be directly observable or predictable without a detailed understanding of the underlying dynamics of the system.

The discovery of the hidden chaotic attractor with extreme multistability is a proof of the elusive nature of complex dynamical systems. It emerged in the late 20th century as researchers delved deeper into nonlinear dynamics.

The fascinating example of such type of work derived by Jafari (Jafari *et al.* 2018). They created a unique chaotic system with five dimensions. It's special because it has a hidden attractor and shows extreme multi-stability. These traits are quite rare in existing studies. The following is the mathematical model and illustration shown in Figures 8 & 9. To solve this nonlinear system (6) & (7) numerically with time step size 0.01 and total number of steps are 2^{16} the Runge-Kutta method of order 4 (RK4) is used.

$$
\begin{aligned}
\dot{x} &= y, \\
\dot{y} &= z, \\
\dot{z} &= w, \\
\dot{w} &= 4v + 1.7xz + 0.5xw, \\
\dot{v} &= y^2 + 1.1xy + xz.
\end{aligned}
\tag{6}
$$

Let us consider another system as an example by Khalaf, A. J. M. (Khalaf *et al.* 2020):

$$
\begin{aligned}
\dot{x} &= y, \\
\dot{y} &= z, \\
\dot{z} &= w, \\
\dot{w} &= -0.16w^2 - 0.86w + v + 3.35xz - 0.36yz, \\
\dot{v} &= 1.09y^2 - 0.96y + 1.09xz - 1.92zw.
\end{aligned}
\tag{7}
$$

Khalaf Analyzed a new 5D chaotic system that reveals hidden attractors with extreme multi-stability which is the modification of Jafari (Jafari *et al.* 2018) work shown in Figure 10.

In recent findings, researchers worked with chaotic systems that have hidden attractors with extreme multi-stability in nonlinear dynamics. Ahmadi et al. presented a rare chaotic system with extreme multistability and a unique equilibrium line (Ahmadi *et al.* 2020). Such systems are exceptionally rare. This newly developed chaotic system falls into the category of dynamical systems with hidden attractors. Its complete dynamical properties have been thoroughly investigated. This discovery expands our understanding of the hidden chaotic system's behavior. Additionally, Huang et al., derived a novel four-dimensional chaotic system from a known three-dimensional chaotic system that exhibits extreme multi-stability with an equilibrium point along a line (Huang *et al.* 2022). This system can generate innumerable symmetric and homogeneous attractors.

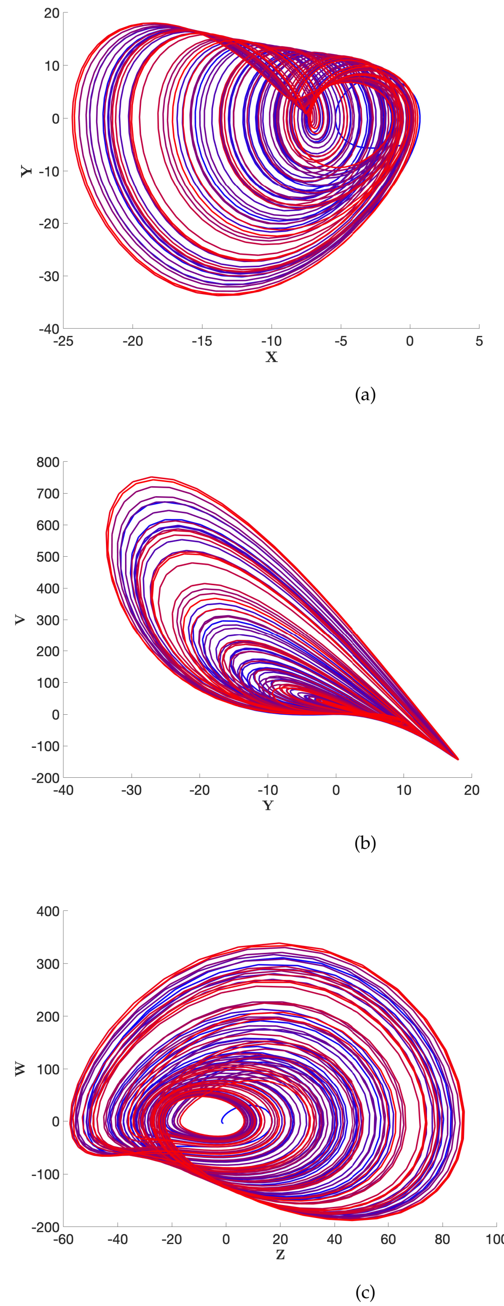


Figure 8 Strange attractor of the System (6) displays distinct shapes in three distinct projections when started from initial conditions (0,-5,-1,-4, 0).

Multi-scroll hidden chaotic attractors in nonlinear dynamics

Multi-scroll hidden chaotic attractors are a fascinating phenomenon in nonlinear dynamical systems. Unlike traditional attractors, these possess multiple basins of attraction, leading to complex, unpredictable trajectories.

An interesting example of multi-scroll hidden attractors has been derived by Xiaoyu Hu (Hu *et al.* 2017). They proposed the following novel 5-dimensional chaotic system in which hidden multi-scroll attractors and hidden multi-wing attractors can be observed at different phase levels as shown in Figure 11. The same as the previous RK4 method is used to solve the system (8)

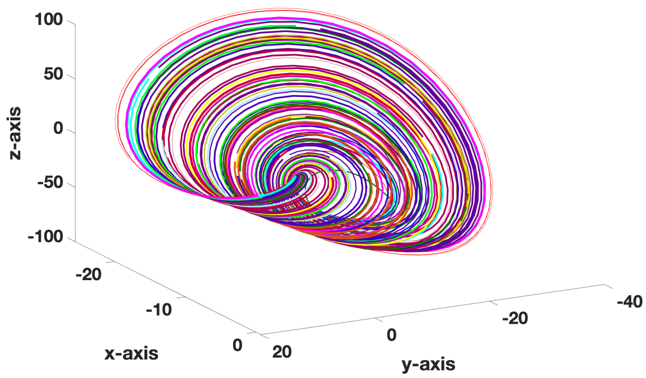
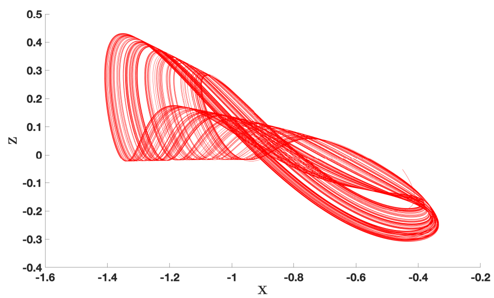
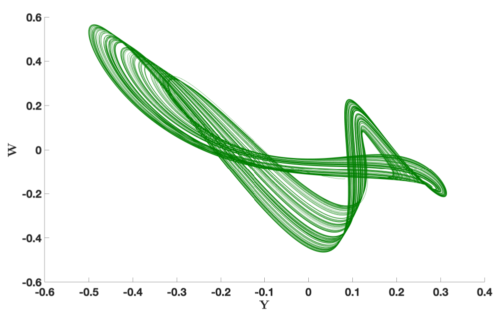


Figure 9 Strange attractor of the System (6) in 3D has initial conditions (0,-5,-1,-4, 0).



(a)



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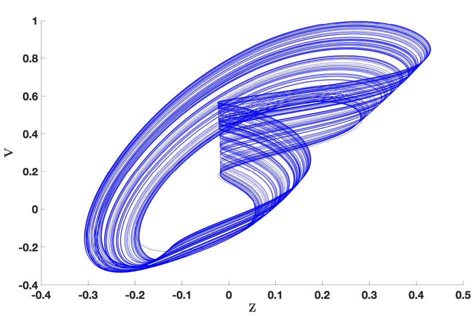
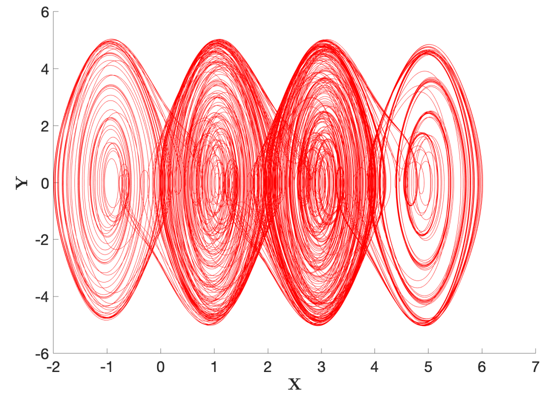


(c)

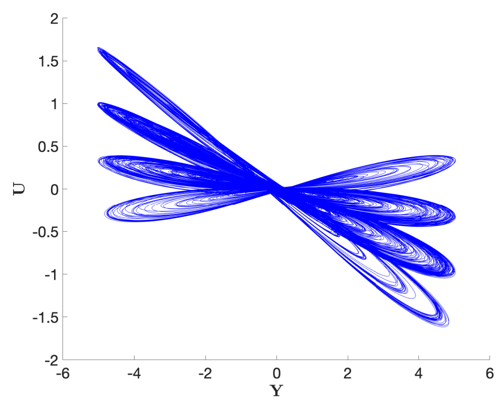
Figure 10 Visualize chaotic trajectories of system (7) with initial conditions (-1.44, 0.57, -0.82, -1.62, -0.75) through phase portrait projections of strange attractors. (a) in XZ, (b) in YW, (c) in ZV.

numerically.

$$
\begin{aligned}
\dot{x} &= ay, \\
\dot{y} &= by - z + csin(2\pi dx), \\
\dot{z} &= y - ez, \\
\dot{u} &= -xy - (g + h\phi^2)u + k, \\
\dot{\phi} &= u.
\end{aligned} \tag{8}
$$



(a)



(b)

Figure 11 Hidden attractors of the system (8) with initial values $(x_0, y_0, z_0, u_0, \phi_0 = 0.2, 0, 0, 0.2)$, fascinating dynamics develop over a transient simulation period of 3000-time units. In (a) $x - y$ phase plane reveals the presence of 4 scroll hidden attractors, while in (b) $y - u$ phase plane reveals the presence of eight butterfly wings hidden attractors.

In this given scenario, the system parameters have been defined as follows: a = 0.25, b = 0.4, c = 2, d = 0.5, e = 0.5, g = 15, h = 0.01, and k = 0.05.

Some notable researchers have contributed to the study of multi-scroll hidden chaotic attractors. Escalante and Campos explored hidden attractors in addition to self-excitation (Escalante-González and Campos-Cantón 2019). First, a double-scroll attractor is generated from two equilibria connected by heteroclinic orbits. Hidden attractors arise when trajectories resembling these trajectories break apart on a larger scale, increasing the complexity of the system. Pulido et al. have developed a method to generate dynamical systems with unique patterns, which they call bidirectional multiscroll hidden attractors (Pulido-Luna *et al.* 2021). These special attractors arise from piecewise linear systems, starting from their rest states, and can have both unidirectional (1D) and bidirectional (2D) lattice multiscroll patterns. This method opens up exciting possibilities for the design of complex and fascinating dynamical systems. In addition, Escalante and Campos have developed a method to generate complex systems with multiple hidden attractors (Escalante-González and Campos 2022). They use a nonlinear function to generate multiple self-excited attractors at specific points. Each pair of self-excited attractors leads to a hidden attractor, and these pairs combine to form larger hidden attractors. The number of self-excited attractors determines how many nested hidden attractors are created. These researchers have made significant strides in uncovering and studying multi-scroll hidden chaotic attractors, advancing our understanding of complex dynamical systems.

CONCLUSION

In this short review paper, we have addressed the fascinating area of hidden attractors in chaotic systems characterized by nonlinear functions. Through an extensive survey of various papers, we have discussed a variety of hidden chaotic attractors, each with particular features and behaviors.

First, we explored the realm of hidden chaotic attractors without equilibria. These fascinating phenomena challenge conventional wisdom and show that chaotic behavior can manifest in systems without stable equilibrium points. Moving forward, we have examined hidden chaotic attractors that coexist with equilibria. Furthermore, we have ventured into the realm of hidden chaotic attractors that exhibit extreme multiple stability. These systems exhibit a remarkable richness of dynamical behavior. This phenomenon has significant implications in the areas of control and synchronization, as it introduces a variety of possible states that the system can assume under different conditions. Lastly, we explored multi-scroll hidden chaotic attractors, a class of attractors characterized by their complex, multi-dimensional structure.

Collectively, this review underscores the profound importance of hidden attractors in nonlinear dynamics. These elusive phenomena challenge our conventional understanding of chaotic systems and offer new perspectives and avenues for research. Moreover, the diversity of hidden attractors discussed in this review provides fertile ground for further exploration and application in various scientific and engineering domains.

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Availability of data and material

Not applicable.

Conflicts of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

Ethical standard

The authors have no relevant financial or non-financial interests to disclose.

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