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CONFORMAL η-RICCI-YAMABE SOLITONS ON SUBMANIFOLDS OF AN $(\mathcal{LCS})_n$-MANIFOLD ADMITTING A QUARTER-SYMMETRIC METRIC CONNECTION

Sunil Kumar YADAV[1], Abdul HASEEB[2] and Ahmet YILDIZ[3]

[1] Department of Applied Science and Humanities, United College of Engineering & Research, A-31, UPSIDC Institutional Area, Naini-211010, Prayagraj, Uttar Pradesh, INDIA

[2]Department of Mathematics, College of Science, Jazan University, P.O. Box 114, Jazan 45142, KINGDOM of SAUDI ARABIA

[3]Education Faculty, Department of Mathematics, Inonu University, 44280 Malatya, TÜRKİYE

Abstract. This paper presents some results for conformal η-Ricci-Yamabe solitons (CERYS) on invariant and anti-invariant submanifolds of a $(\mathcal{LCS})_n$-manifold admitting a quarter-symmetric metric connection (QSMC). In addition, we developed the characterization of CERYS on $\mathcal{M}$-projectively flat, $\mathcal{Q}$-flat, and concircularly flat anti-invariant submanifolds of a $(\mathcal{LCS})_n$-manifold with respect to the aforementioned connection. Finally, we construct an extensive example that appoints some of our inferences.

1. Background and Motivations

Conformal Ricci flow is defined in a Riemannian n-manifold $(\mathbb{V}, g)$ as a generalisation of classical Ricci flow by [6]

$$\frac{\partial g}{\partial t} = -2(\mathcal{R}ic + \frac{g}{n}) - pg, \quad \tau(g) = -1,$$

where p is called the conformal pressure, g is the Riemannian metric; τ and $\mathcal{R}ic$ denote the scalar curvature and the Ricci tensor of $\mathbb{V}$, respectively.

A conformal Ricci soliton on $(\mathbb{V}, g)$ is defined as follows [2]:

$$\mathfrak{L}_{\mathcal{F}_1} g + 2\mathcal{R}ic = [\frac{1}{n}(pn + 2) - 2\mu]g,$$

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[1] prof_sky16@yahoo.com-Corresponding author; 0000-0001-6930-3585
[2] haseeb@jazanu.edu.sa; malikhaseeb80@gmail.com; 0000-0002-1175-6423
[3] a.yildiz@inonu.edu.tr; 0000-0002-9799-1781.

where $\mu \in \Re$ ($\Re$ is the set of real numbers) and $\mathfrak{L}_{\mathcal{F}_1}$ denotes the Lie-derivative operator along a smooth vector field $\mathcal{F}_1$

A Ricci-Yamabe flow of type (κ, l), which is a scalar combination of Ricci and Yamabe flows, is defined as follows [7]:

$$\frac{\partial}{\partial t} g(t) = 2\kappa \mathcal{R}ic(g(t)) - l\tau(t)g(t), \quad g(0) = g_0,$$

for some scalars κ and l.
A Riemannian manifold is said to have a Ricci-Yamabe solitons of type (κ, l) (briefly, RYS) if [4,29]

$$\mathfrak{L}_{\mathcal{F}_1} g + 2\kappa \mathcal{R}ic + (2\mu - l\tau)g = 0,$$

where $l, \kappa, \mu \in \Re$.
In [30], Zhang et al. studied conformal Ricci-Yamabe soliton (briefly, CRYS), which is defined on $(\mathbb{V}, g)$ by

$$\mathfrak{L}_{\mathcal{F}_1} g + 2\kappa \mathcal{R}ic + [2\mu - l\tau - \frac{1}{n}(pn + 2)]g = 0.$$

In this follow-up, the conformal η-Ricci-Yamabe soliton (briefly, CERYS) on $(\mathbb{V}, g)$ is defined by [28]

$$\mathfrak{L}_{\mathcal{F}_1} g + 2\kappa \mathcal{R}ic + [2\mu - l\tau - \frac{1}{n}(pn + 2)]g + 2\nu\, \eta \otimes \eta = 0, \tag{1}$$

where $l, \kappa, \mu, \nu \in \Re$. If $\mathcal{F}_1 = grad(f)$, then the Equation (1) is called a gradient conformal η-Ricci-Yamabe soliton (briefly, GCERYS) and given by

$$\nabla^2 f + \kappa \mathcal{R}ic + [\mu - \frac{l\tau}{2} - \frac{1}{2}(p + \frac{2}{n})]g + \nu\, \eta \otimes \eta = 0,$$

where $\nabla^2 f$ is said to be the Hessian of f. A CRYS (or GCRYS) is said to be shrinking, steady or expanding if $\mu < 0$, $= 0$ or > 0, respectively. A CERYS (or GCERYS) reduces to (i) CERS if $\kappa = 1$, $l = 0$, (ii) CEYS if $\kappa = 0$, $l = 1$, and (iii) conformal η-Einstein soliton (briefly, CEES) if $\kappa = 1$, $l = -1$.

Shaikh [22] introduced the concept of n-dimensional Lorentzian concircular structure manifold (briefly, $(\mathcal{LCS})_n$-manifold) and demonstrated its existence with several examples [24], which generalises the concept of $\mathcal{LP}$-Sasakian manifolds introduced in [13,14]. We refer to the works [1,10,23] for more extensive studies. Mantica and Molinari [18] recently demonstrated that a $(\mathcal{LCS})_n$-manifold $(n > 3)$ is equal to the GRW spacetime. The authors also examined the applicability of $(\mathcal{LCS})_n$-manifolds in general theory of relativity and cosmology in [3]. Thus the geometry of submanifolds has grown in popularity in modern analysis due to its importance in practical mathematics and theoretical physics.

A linear connection $\bar{\nabla}$ on $(\mathbb{V}, g)$ is said to be a quarter-symmetric connection (briefly, QSC) [8] if its torsion tensor $\bar{\mathcal{T}}$ has the form

$$\bar{\mathcal{T}}(\mathcal{F}_1, \mathcal{F}_2) = \bar{\nabla}_{\mathcal{F}_1} \mathcal{F}_2 - \bar{\nabla}_{\mathcal{F}_2} \mathcal{F}_1 - [\mathcal{F}_1, \mathcal{F}_2] = \mathcal{A}(\mathcal{F}_2)\psi^*(\mathcal{F}_1) - \mathcal{A}(\mathcal{F}_1)\psi^*(\mathcal{F}_2), \tag{2}$$

where $\mathcal{A}$ is a 1-form and ψ^* is a $(1,1)$ type tensor field. If a quarter-symmetric linear connection $\bar{\nabla}$ satisfies the condition

$$(\bar{\nabla}_{\mathcal{F}_1} g)(\mathcal{F}_2, \mathcal{F}_3) = 0,$$

for all $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3 \in \chi(\mathbb{V})$, then $\bar{\nabla}$ is said to be a quarter-symmetric metric connection (briefly, QSMC). If a contact metric manifold admits a QSC, then we take $\mathcal{A}$=η and ψ^*=ϕ and hence (2) takes the form $\bar{\mathcal{T}}(\mathcal{F}_1, \mathcal{F}_2) = \eta(\mathcal{F}_2)\phi(\mathcal{F}_1) - \eta(\mathcal{F}_1)\phi(\mathcal{F}_2)$.

The relation between the Levi-Civita connection ∇ and a QSMC $\bar{\nabla}$ on a contact metric manifold is given by

$$\bar{\nabla}_{\mathcal{F}_1} \mathcal{F}_2 = \nabla_{\mathcal{F}_1} \mathcal{F}_2 - \eta(\mathcal{F}_1)\phi(\mathcal{F}_2).$$

Recently, the QSMC have been studied by many authors such as [9, 12, 19, 31] and many others.

2. Preliminaries

Let $\widetilde{\mathbb{V}}$ be an n-dimensional Lorentzian manifold admitting a unit time-like concircular vector field ζ. Then there is

$$g(\zeta, \zeta) = -1.$$

Since ζ is a unit concircular vector field, it follows that there exists a non-zero 1-form η such that for

$$g(\mathcal{F}_1, \zeta) = \eta(\mathcal{F}_1)$$

satisfies [25]

$$(\widetilde{\nabla}_{\mathcal{F}_1} \eta)\mathcal{F}_2 = \alpha[g(\mathcal{F}_1, \mathcal{F}_2) + \eta(\mathcal{F}_1)\eta(\mathcal{F}_2)], \ \alpha \neq 0,$$
$$\widetilde{\nabla}_{\mathcal{F}_1} \zeta = \alpha[\mathcal{F}_1 + \eta(\mathcal{F}_1)\zeta], \ \alpha \neq 0, \tag{3}$$

for $\mathcal{F}_1, \mathcal{F}_2 \in \chi(\widetilde{\mathbb{V}})$, where $\widetilde{\nabla}$ denotes the operator of covariant differentiation with respect to the Lorentzian metric g and α is a non-zero scalar function that satisfies

$$\widetilde{\nabla}_{\mathcal{F}_1} \alpha = (\mathcal{F}_1 \alpha) = d\alpha(\mathcal{F}_1) = \rho\eta(\mathcal{F}_1),$$

ρ being a certain scalar function given by ρ=-$(\zeta\alpha)$. Let us have a look

$$\phi\mathcal{F}_1 = \frac{1}{\alpha}\widetilde{\nabla}_{\mathcal{F}_1} \zeta, \tag{4}$$

then utilizing (3) and (4) we acquire

$$\phi\mathcal{F}_1 = \mathcal{F}_1 + \eta(\mathcal{F}_1)\zeta,$$
$$g(\phi\mathcal{F}_1, \mathcal{F}_2) = g(\mathcal{F}_1, \phi\mathcal{F}_2).$$

Thus the Lorentzian manifold $\widetilde{\mathbb{V}}$ admits the unit time-like concircular vector field ζ, its associated 1-form η and a (1,1) tensor field ϕ is said to be a Lorentzian concircular structure manifold (briefly, $(\mathcal{LCS})_n$-manifold) [17, 22]. Especially, if we take α=1, then we can obtain the $\mathcal{LP}$-Sasakian structure of Matsumoto [13].
In an $(\mathcal{LCS})_n$-manifold, we have [22]:

$$\eta(\zeta) = -1, \ \ \phi \circ \zeta = 0, \ \ \eta(\phi\mathcal{F}_1) = 0, \ \ g(\phi\mathcal{F}_1, \phi\mathcal{F}_2) = g(\mathcal{F}_1, \mathcal{F}_2) + \eta(\mathcal{F}_1)\eta(\mathcal{F}_2),$$

$$\phi^2\mathcal{F}_1 = \mathcal{F}_1 + \eta(\mathcal{F}_1)\zeta,$$
$$\eta(\widetilde{\mathcal{R}}(\mathcal{F}_1,\mathcal{F}_2)\mathcal{F}_3) = (\alpha^2-\rho)[g(\mathcal{F}_2,\mathcal{F}_3)\eta(\mathcal{F}_1) - g(\mathcal{F}_1,\mathcal{F}_3)\eta(\mathcal{F}_2)],$$
$$\widetilde{\mathcal{R}}(\mathcal{F}_1,\mathcal{F}_2)\zeta = (\alpha^2-\rho)[\eta(\mathcal{F}_2)\mathcal{F}_1 - \eta(\mathcal{F}_1)\mathcal{F}_2],$$
$$\widetilde{\mathcal{R}}ic(\mathcal{F}_1,\zeta) = (n-1)(\alpha^2-\rho)\eta(\mathcal{F}_1),$$
$$\widetilde{\mathcal{R}}(\mathcal{F}_1,\mathcal{F}_2)\mathcal{F}_3 = \phi\widetilde{\mathcal{R}}(\mathcal{F}_1,\mathcal{F}_2)\mathcal{F}_3 + (\alpha^2-\rho)[g(\mathcal{F}_2,\mathcal{F}_3)\eta(\mathcal{F}_1) - g(\mathcal{F}_1,\mathcal{F}_3)\eta(\mathcal{F}_2)]\zeta,$$
$$(\widetilde{\nabla}_{\mathcal{F}_1}\phi)\mathcal{F}_2) = \alpha[g(\mathcal{F}_1,\mathcal{F}_2)\zeta + 2\eta(\mathcal{F}_1)\eta(\mathcal{F}_2)\zeta + \eta(\mathcal{F}_2)\mathcal{F}_1],$$

for all $\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3 \in \chi(\widetilde{\mathbb{V}})$.

Let $\mathbb{N}$ be an m-dimensional $(m<n)$ submanifold of an $(\mathcal{LCS})_n$-manifold $\widetilde{\mathbb{V}}$ with induced metric g. Also, let ∇ be the induced connection on the tangent bundle $T\mathbb{N}$ and $\nabla^\perp$ be the induced connection on the normal bundle $T^\perp\mathbb{N}$ of $\mathbb{N}$, respectively. Then the Gauss and Weingarten formulae are respectively given by

$$\widetilde{\nabla}_{\mathcal{F}_1}\mathcal{F}_2 = \nabla_{\mathcal{F}_1}\mathcal{F}_2 + \hbar(\mathcal{F}_1,\mathcal{F}_2), \tag{5}$$

and

$$\widetilde{\nabla}_{\mathcal{F}_1}\mathcal{F}_3 = -\mathcal{A}_{\mathcal{F}_3}\mathcal{F}_1 + \nabla^\perp_{\mathcal{F}_1}\mathcal{F}_3,$$

for all $\mathcal{F}_1,\mathcal{F}_2 \in \chi(\mathbb{N})$ and $\mathcal{F}_3 \in \chi^\perp(\mathbb{N})$, where $\hbar$ and $\mathcal{A}_{\mathcal{F}_3}$ are second fundamental form and the shape operator (corresponding to the normal vector field $\mathcal{F}_3$), respectively for the immersion of $\mathbb{N}$ into $\widetilde{\mathbb{V}}$. The second fundamental form $\hbar$ and the shape operator $\mathcal{A}_{\mathcal{F}_3}$ are related by [26]

$$g(\hbar(\mathcal{F}_1,\mathcal{F}_2),\mathcal{F}_3) = g(\mathcal{A}_{\mathcal{F}_3}\mathcal{F}_1,\mathcal{F}_2),$$

for all $\mathcal{F}_1,\mathcal{F}_2 \in \chi(\mathbb{N})$ and $\mathcal{F}_3 \in \chi^\perp(\mathbb{N})$. We note that $\hbar(\mathcal{F}_1,\mathcal{F}_2)$ is bilinear and since $\nabla_{f\mathcal{F}_1}\mathcal{F}_2{=}f\nabla_{\mathcal{F}_1}\mathcal{F}_2$ for any smooth function f on a manifold, then we have

$$\hbar(f\mathcal{F}_1,\mathcal{F}_2) = f\hbar(\mathcal{F}_1,\mathcal{F}_2).$$

A submanifold $\mathbb{N}$ of an $(\mathcal{LCS})_n$-manifold $\widetilde{\mathbb{V}}$ is said to be totally umbilical if

$$\hbar(\mathcal{F}_1,\mathcal{F}_2) = g(\mathcal{F}_1,\mathcal{F}_2)\mathcal{H}, \tag{6}$$

where $\mathcal{F}_1,\mathcal{F}_2 \in T\mathbb{N}$ and the mean curvature vector $\mathcal{H}$ on $\mathbb{N}$ is given by $\mathcal{H} = \frac{1}{m}\sum_{i=1}^m \hbar(\upsilon_i,\upsilon_i)$, where $\{\upsilon_1,\upsilon_2,....,\upsilon_m\}$ is a local orthonormal frame of vector fields on $\mathbb{N}$. Moreover, if $\hbar(\mathcal{F}_1,\mathcal{F}_2){=}0$ for all $\mathcal{F}_1,\mathcal{F}_2 \in T\mathbb{N}$, then $\mathbb{N}$ is said to be totally geodesic and if $\mathcal{H}{=}0$ then $\mathbb{N}$ is called minimal in $\widetilde{\mathbb{V}}$.

A submanifold $\mathbb{N}$ of $\widetilde{\mathbb{V}}$ is said to be invariant if the structure vector field ζ is tangent to $\mathbb{N}$ at every point of $\mathbb{N}$ and $\phi\mathcal{F}_1$ is tangent to $\mathbb{N}$ for every vector field $\mathcal{F}_1$ tangent to $\mathbb{N}$ at every point of $\mathbb{N}$, i.e., $\phi(T\mathbb{N}) \subset T\mathbb{N}$ at every point of $\mathbb{N}$. Whereas, $\mathbb{N}$ is said to be anti-invariant if for any $\mathcal{F}_1$ tangent to $\mathbb{N}$, $\phi\mathcal{F}_1$ is normal to $\mathbb{N}$, i.e., $\phi(T\mathbb{N}) \subset T^\perp\mathbb{N}$ at every point of $\mathbb{N}$, where $T^\perp\mathbb{N}$ is the normal bundle of $\mathbb{N}$.

Now we recall the following results:

Lemma 1. *[11] On an $(\mathcal{LCS})_n$-manifold $\widetilde{\mathbb{V}}$ with a QSMC $\bar{\widetilde{\nabla}}$, we have*

(i) $\bar{\widetilde{\nabla}}_{\mathcal{F}_1}\mathcal{F}_2 = \widetilde{\nabla}_{\mathcal{F}_1}\mathcal{F}_2 + \eta(\mathcal{F}_2)\phi\mathcal{F}_1 - g(\phi\mathcal{F}_1, \mathcal{F}_2)\zeta,$

$$
\begin{aligned}
(ii) \quad \bar{\mathcal{R}}(\mathcal{F}_1, \mathcal{F}_2)\mathcal{F}_3 &= \widetilde{\mathcal{R}}(\mathcal{F}_1, \mathcal{F}_2)\mathcal{F}_3 + (2\alpha - 1)[g(\phi\mathcal{F}_1, \mathcal{F}_3)\phi\mathcal{F}_2 - g(\phi\mathcal{F}_2, \mathcal{F}_3)\phi\mathcal{F}_1] \\
&+ \alpha[\eta(\mathcal{F}_2)\mathcal{F}_1 - \eta(\mathcal{F}_1)\mathcal{F}_2]\eta(\mathcal{F}_3) + \alpha[g(\mathcal{F}_2, \mathcal{F}_3)\eta(\mathcal{F}_1) - g(\mathcal{F}_1, \mathcal{F}_3)]\zeta,
\end{aligned}
$$

$$
\begin{aligned}
(iii) \quad \bar{\mathcal{R}}ic(\mathcal{F}_2, \mathcal{F}_3) &= \widetilde{\mathcal{R}}ic(\mathcal{F}_2, \mathcal{F}_3) + (\alpha - 1)g(\mathcal{F}_2, \mathcal{F}_3) + (n\alpha - 1)\eta(\mathcal{F}_2)\eta(\mathcal{F}_3) \\
&-(2\alpha - 1)\varepsilon g(\phi\mathcal{F}_2, \mathcal{F}_3),
\end{aligned}
$$

where $\bar{\mathcal{R}}$, $\bar{\mathcal{R}}ic$ are the curvature and the Ricci tensors of $\widetilde{\mathbb{V}}$ with respect to $\bar{\widetilde{\nabla}}$ and $\varepsilon = \text{trace}\phi$.

3. Cerys on Submanifolds of $(\mathcal{LCS})_n$-Manifolds

Let $(g, \zeta, \mu, \kappa, l)$ be a CERYS on submanifold $\mathbb{N}$ of an $(\mathcal{LCS})_n$-manifold $\widetilde{\mathbb{V}}$. Then in view of (1) we obtain

$$
\begin{aligned}
\mathfrak{L}_\zeta g(\mathcal{F}_2, \mathcal{F}_3) &= -2\kappa\mathcal{R}ic(\mathcal{F}_2, \mathcal{F}_3) - [2\mu - l\tau - \frac{1}{n}(pn + 2)]g(\mathcal{F}_2, \mathcal{F}_3) \\
&-2\nu\eta(\mathcal{F}_2)\eta(\mathcal{F}_3).
\end{aligned}
\tag{7}
$$

With the help of (4) and (5) one can get

$$
\alpha\phi\mathcal{F}_1 = \widetilde{\nabla}_{\mathcal{F}_1}\zeta = \nabla_{\mathcal{F}_1}\zeta + \hbar(\mathcal{F}_1, \zeta). \tag{8}
$$

If $\mathbb{N}$ is invariant in $\widetilde{\mathbb{V}}$, then $\phi\mathcal{F}_1, \zeta \in T\mathbb{N}$. So from (8) we yields

$$
(i) \;\; \alpha\phi\mathcal{F}_1 = \nabla_{\mathcal{F}_1}\zeta, \;\; (ii) \;\; \hbar(\mathcal{F}_1, \zeta) = 0. \tag{9}
$$

Using (9)(i) in (7), we obtain

$$
\mathcal{R}ic(\mathcal{F}_2, \mathcal{F}_3) = -\frac{1}{\kappa}[\mu + \alpha - \frac{l\tau}{2} - \frac{1}{2n}(pn + 2)]g(\mathcal{F}_2, \mathcal{F}_3) - \frac{(\nu + \alpha)}{\kappa}\eta(\mathcal{F}_2)\eta(\mathcal{F}_3), \tag{10}
$$

where $\mathfrak{L}_\zeta g(\mathcal{F}_2, \mathcal{F}_3) = 2\alpha[g(\mathcal{F}_2, \mathcal{F}_3) + \eta(\mathcal{F}_2)\eta(\mathcal{F}_3)]$.

Also, with the help of (9)(ii), we get from (6) that $\eta(\mathcal{E})\mathcal{H} = 0 \implies \mathcal{H} = 0$. So, we obtain the result:

Theorem 1. *If $(g, \zeta, \mu, \nu, \kappa, l)$ be a CERYS on an invariant submanifold $\mathbb{N}$ of an $(\mathcal{LCS})_n$-manifold $\widetilde{\mathbb{V}}$, then $\mathbb{N}$ is an η-Einstein manifold and also minimal in $\widetilde{\mathbb{V}}$.*

Also, we have

$$
\mathcal{R}(\mathcal{F}_2, \mathcal{F}_3)\zeta = \nabla_{\mathcal{F}_2}\nabla_{\mathcal{F}_3}\zeta - \nabla_{\mathcal{F}_3}\nabla_{\mathcal{F}_2}\zeta - \nabla_{[\mathcal{F}_2, \mathcal{F}_3]}\zeta = (\alpha^2 - \rho)[\eta(\mathcal{F}_3)\mathcal{F}_2 - \eta(\mathcal{F}_2)\mathcal{F}_3],
$$

which by using (9)(i), we lead to

$$
\mathcal{R}ic(\mathcal{F}_2, \zeta) = (m - 1)(\alpha^2 - \rho)\eta(\mathcal{F}_2), \text{ for all } \mathcal{F}_2. \tag{11}
$$

By fixing $\mathcal{F}_3$=ζ in (10) and using (11), we get

$$\mu = \nu - \kappa(m-1)(\alpha^2 - \rho) + \frac{l\tau}{2} + \frac{1}{2}(p + \frac{2}{n}).$$

As consequence, we can make the following claim:

Theorem 2. *If $(g, \zeta, \mu, \nu, \kappa, l)$ be a CERYS on an invariant submanifold $\mathbb{N}$ of an $(\mathcal{LCS})_n$-manifold $\widetilde{\mathbb{V}}$, then the CERYS reduces to*
(i) CERS if $\mu = \nu - (m-1)(\alpha^2 - \rho) + \frac{1}{2}(p + \frac{2}{n})$,
(ii) CEYS if $\mu = \nu + \frac{\tau}{2} + \frac{1}{2}(p + \frac{2}{n})$,
(iii) CEES if $\mu = \nu - (m-1)(\alpha^2 - \rho) - \frac{\tau}{2} + \frac{1}{2}(p + \frac{2}{n})$.

Corollary 1. *An η-Yamabe soliton on an invariant submanifold $\mathbb{N}$ of an $(\mathcal{LCS})_n$-manifold $\widetilde{\mathbb{V}}$ of type $(0,1)$, is contracting, stable or increasing accordingly as $\tau < -2\nu$, $\tau = -2\nu$, or $\tau > -2\nu$, respectively.*

Corollary 2. *An η-Ricci soliton on an invariant submanifold $\mathbb{N}$ of an $(\mathcal{LCS})_n$-manifolds $\widetilde{\mathbb{V}}$ of type $(1,0)$, is contracting, stable or increasing accordingly as $\nu < (m-1)(\alpha^2 - \rho)$, $\nu = (m-1)(\alpha^2 - \rho)$ or $\nu > (m-1)(\alpha^2 - \rho)$, provided $\alpha^2 \neq \rho$.*

Corollary 3. *An η-Einstein soliton on an invariant submanifold $\mathbb{N}$ of an $(\mathcal{LCS})_n$-manifolds $\widetilde{\mathbb{V}}$ of type $(1,-1)$, is contracting, stable or increasing accordingly as $\tau > 2[\nu - (m-1)(\alpha^2 - \rho)]$, $\tau = 2[\nu - (m-1)(\alpha^2 - \rho)]$ or $\tau < 2[\nu - (m-1)(\alpha^2 - \rho)]$, provided $\alpha^2 \neq \rho$.*

In particular, if $\mathbb{N}$ is an anti-invariant submanifold on $\widetilde{\mathbb{V}}$. Then for any $\mathcal{F}_1 \in T\mathbb{N}$ and $\phi\mathcal{F}_1 \in T^{\perp}\mathbb{N}$, we get from (8) that $\nabla_{\mathcal{F}_1}\zeta$=0, $\hbar(\mathcal{F}_1, \zeta)$=$\alpha\phi\mathcal{F}_1$. Thus, $\mathfrak{L}_\zeta g(\mathcal{F}_1, \mathcal{F}_2)$=0, that is, ζ is a Killing vector field (briefly, KVF) and in this case from (7), we have

$$\mathcal{R}ic(\mathcal{F}_2, \mathcal{F}_3) = -\frac{1}{\kappa}[\mu - \frac{l\tau}{2} - \frac{1}{2}(p + \frac{2}{n})]g(\mathcal{F}_2, \mathcal{F}_3) - \frac{\nu}{\kappa}\eta(\mathcal{F}_2)\eta(\mathcal{F}_3). \tag{12}$$

This results in the following outcomes:

Theorem 3. *If $(g, \zeta, \mu, \nu, \kappa, l)$ be a CERYS on an anti-invariant submanifold $\mathbb{N}$ of an $(\mathcal{LCS})_n$-manifolds $\widetilde{\mathbb{V}}$, then $\mathbb{N}$ is an η-Einstein and ζ is a KVF.*

Again, for an anti-invariant submanifold $\mathbb{N}$ of $\widetilde{\mathbb{V}}$, we have $\mathcal{R}(\mathcal{F}_2, \mathcal{F}_3)\zeta$=0 and hence $\mathcal{R}ic(\mathcal{F}_2, \zeta)$=0. Also, from (12) we obtain $\mathcal{R}ic(\mathcal{F}_2, \zeta) = -\frac{1}{\kappa}[\mu - \frac{l\tau}{2} - \frac{1}{2}(p + \frac{2}{n}) - \nu]\eta(\mathcal{F}_1)$. So, we get $\mu = \frac{l\tau}{2} + \frac{1}{2}(p + \frac{2}{n}) + \nu$. Thus, we have finalized the result:

Corollary 4. *A CERYS of type (κ, l) on an anti-invariant submanifold $\mathbb{N}$ of an $(\mathcal{LCS})_n$-manifold $\widetilde{\mathbb{V}}$ is contracting, stable or increasing accordingly as $\tau < \frac{-1}{l}[2\nu + (p + \frac{2}{n})]$, $\tau = \frac{-1}{l}[2\nu + (p + \frac{2}{n})]$ or $\tau > \frac{-1}{l}[2\nu + (p + \frac{2}{n})]$.*

4. Cerys on Submanifolds of $(\mathcal{LCS})_n$-Manifolds Admitting $\bar{\bar{\nabla}}$

Assume that $(g,\zeta,\mu,\nu,\kappa,l)$ be a CERYS on a submanifold $\mathbb{N}$ of an $(\mathcal{LCS})_n$-manifold $\widetilde{\mathbb{V}}$ in view of QSMC $\bar{\bar{\nabla}}$. Then from (1) we obtain

$$\begin{aligned}\bar{\mathfrak{L}}_{\mathcal{F}_1} g(\mathcal{F}_2,\mathcal{F}_3) &= -2\kappa\bar{\mathcal{R}}ic(\mathcal{F}_2,\mathcal{F}_3) - [2\mu - l\bar{\tau} - \frac{1}{n}(pn+2)]g(\mathcal{F}_2,\mathcal{F}_3) \\ &\quad -2\nu\eta(\mathcal{F}_2)\eta(\mathcal{F}_3) = 0.\end{aligned} \tag{13}$$

In view of QSMC $\bar{\nabla}$, the second fundamental form $\bar{\hbar}$ on $\mathbb{N}$ is given by

$$\bar{\bar{\nabla}}_{\mathcal{F}_1}\mathcal{F}_2 = \bar{\nabla}_{\mathcal{F}_1}\mathcal{F}_2 + \bar{\hbar}(\mathcal{F}_1,\mathcal{F}_2). \tag{14}$$

Using Lemma 2.1(i) and (5) in (14), we lead to

$$\bar{\nabla}_{\mathcal{F}_1}\mathcal{F}_2 + \bar{\hbar}(\mathcal{F}_1,\mathcal{F}_2) = \nabla_{\mathcal{F}_1}\mathcal{F}_2 + \hbar(\mathcal{F}_1,\mathcal{F}_2) + \eta(\mathcal{F}_2)\phi\mathcal{F}_1 - g(\phi\mathcal{F}_1,\mathcal{F}_2)\xi. \tag{15}$$

We suppose that $\mathbb{N}$ is invariant in $\widetilde{\mathbb{V}}$, then $\phi\mathcal{F}_1, \xi \in T\mathbb{N}$. Thus from (15) we have

$$\bar{\nabla}_{\mathcal{F}_1}\mathcal{F}_2 = \nabla_{\mathcal{F}_1}\mathcal{F}_2 + \eta(\mathcal{F}_2)\phi\mathcal{F}_1 - g(\phi\mathcal{F}_1,\mathcal{F}_2)\zeta, \tag{16}$$

which means $\mathbb{N}$ admits QSME $\bar{\bar{\nabla}}$. Also, in view of (9)(i), it follows that $\bar{\nabla}_{\mathcal{F}_1}\zeta$=$(\alpha - 1)\phi\mathcal{F}_1$ and hence

$$\bar{\mathfrak{L}}_{\mathcal{F}_1} g(\mathcal{F}_2,\mathcal{F}_3) = 2(\alpha-1)[g(\mathcal{F}_2,\mathcal{F}_3) + \eta(\mathcal{F}_2)\eta(\mathcal{F}_3)]. \tag{17}$$

Let $\bar{\mathcal{R}}$ be the curvature tensor of submanifold $\mathbb{N}$ with respect to the QSMC $\bar{\bar{\nabla}}$. Then we get

$$\begin{aligned}\bar{\mathcal{R}}(\mathcal{F}_1,\mathcal{F}_2),\mathcal{F}_3 &= \widetilde{\mathcal{R}}(\mathcal{F}_1,\mathcal{F}_2)\mathcal{F}_3 + (2\alpha-1)[g(\phi\mathcal{F}_1,\mathcal{F}_3)\phi\mathcal{F}_2 - g(\phi\mathcal{F}_2,\mathcal{F}_3)\phi\mathcal{F}_1)] \\ &\quad +\alpha[\eta(\mathcal{F}_2)\mathcal{F}_1 - \eta(\mathcal{F}_1)\mathcal{F}_2]\eta(\mathcal{F}_3) \\ &\quad +\alpha[g(\mathcal{F}_2,\mathcal{F}_3)\eta(\mathcal{F}_1) - g(\mathcal{F}_1,\mathcal{F}_3)\eta(\mathcal{F}_2)]\zeta,\end{aligned} \tag{18}$$

where $\bar{\mathcal{R}}(\mathcal{F}_1,\mathcal{F}_2)\mathcal{F}_3$=$\bar{\bar{\nabla}}_{\mathcal{F}_1}\bar{\bar{\nabla}}_{\mathcal{F}_2}\mathcal{F}_3 - \bar{\bar{\nabla}}_{\mathcal{F}_2}\bar{\bar{\nabla}}_{\mathcal{F}_1}\mathcal{F}_3 - \bar{\bar{\nabla}}_{[\mathcal{F}_1,\mathcal{F}_2]}\mathcal{F}_3$.
On contracting (18), we obtain

$$\begin{aligned}\bar{\mathcal{R}}ic(\mathcal{F}_2,\mathcal{F}_3) &= \widetilde{\mathcal{R}}ic(\mathcal{F}_2,\mathcal{F}_3) + [\alpha(1-2\varepsilon)+\varepsilon]g(\mathcal{F}_2,\mathcal{F}_3) \\ &\quad +[\alpha(m-2\varepsilon)+\varepsilon-1]\eta(\mathcal{F}_2)\eta(\mathcal{F}_3).\end{aligned} \tag{19}$$

In view of (17) and (19), equation (13) reduces to

$$\begin{aligned}\widetilde{\mathcal{R}}ic(\mathcal{F}_2,\mathcal{F}_3) &= -\frac{1}{\kappa}\big[\mu - \frac{l\bar{\tau}}{2} - \frac{1}{2n}(pn+2) + (\alpha-1) + \kappa\{\alpha(1-2\varepsilon)+\varepsilon\}\big]g(\mathcal{F}_2,\mathcal{F}_3) \\ &\quad -\big[\kappa\{\alpha(m-2\varepsilon)+\varepsilon-1\} + \alpha - 1 + \nu\big]\eta(\mathcal{F}_2)\eta(\mathcal{F}_3).\end{aligned}$$

Thus, we state:

Theorem 4. *Let $(g,\zeta,\mu,\nu,\kappa,l)$ be a CERYS on an invariant submanifold $\mathbb{N}$ of an $(\mathcal{LCS})_n$-manifold $\widetilde{\mathbb{V}}$ with respect to QSMC $\bar{\bar{\nabla}}$. If $\bar{\nabla}$ be the induced connection on $\mathbb{N}$ from the connection $\bar{\bar{\nabla}}$, then $\mathbb{N}$ is an η-Einstein manifold.*

Next, if $\mathbb{N}$ is anti-invariant submanifold on $\widetilde{\mathbb{V}}$ as per $\bar{\bar{\nabla}}$, then from (15), we get $\bar{\nabla}_{\mathcal{F}_1}\zeta$=0 and hence we find $\bar{\mathfrak{L}}_\zeta g(\mathcal{F}_2,\mathcal{F}_3)$=0. So from (13) we leads to the outcome:

Theorem 5. *Let $(g,\zeta,\mu,\nu,\kappa,l)$ be a CERYS on an anti-invariant submanifold $\mathbb{N}$ of an $(\mathcal{LCS})_n$-manifold $\widetilde{\mathbb{V}}$ admits QSMC $\bar{\bar{\nabla}}$. Then $\mathbb{N}$ is η-Einstein with respect to induced Riemannian connection.*

Corollary 5. *There does not exist a CEYS on an invariant $(or, anti-invariant)$ submanifold $\mathbb{N}$ of an $(\mathcal{LCS})_n$-manifold $\widetilde{\mathbb{V}}$ with respect to the QSMC $\bar{\bar{\nabla}}$.*

5. Cerys on $\mathcal{M}$-Projectively Flat Anti-Invariant Submanifolds Admitting $\bar{\bar{\nabla}}$

The $\mathcal{M}$-projective curvature tensor $\mathcal{M}^\flat$ of rank three on $(\mathbb{N}^n,g)$ is given by [5,20]

$$\begin{aligned}\mathcal{M}^\flat(\mathcal{F}_1,\mathcal{F}_2)\mathcal{F}_3 &= \mathcal{R}(\mathcal{F}_1,\mathcal{F}_2)\mathcal{F}_3 - \frac{1}{2(n-1)}[\mathcal{R}ic(\mathcal{F}_2,\mathcal{F}_3)\mathcal{F}_1 - \mathcal{R}ic(\mathcal{F}_1,\mathcal{F}_3)\mathcal{F}_2] \\ &\quad -\frac{1}{2(n-1)}[g(\mathcal{F}_2,\mathcal{F}_3)\mathcal{Q}\mathcal{F}_1 - g(\mathcal{F}_1,\mathcal{F}_3)\mathcal{Q}\mathcal{F}_2] \end{aligned} \tag{20}$$

for all smooth vectors fields $\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3\in\chi(\mathbb{N})$, where $\mathcal{Q}$ is the Ricci operator.

We suppose that, $\mathbb{N}$ is $\mathcal{M}$-projectively flat with respect to QSMC $\bar{\bar{\nabla}}$, i.e., $\mathcal{M}^\flat(\mathcal{E},\mathcal{F})\mathcal{G}=0$, then from (20) we have

$$\begin{aligned}\bar{\mathcal{R}}(\mathcal{F}_1,\mathcal{F}_2)\mathcal{F}_3 &= \frac{1}{2(n-1)}[\bar{\mathcal{R}}ic(\mathcal{F}_2,\mathcal{F}_3)\mathcal{F}_1 - \bar{\mathcal{R}}ic(\mathcal{F}_1,\mathcal{F}_3)\mathcal{F}_2] \\ &\quad +\frac{1}{2(n-1)}[g(\mathcal{F}_2,\mathcal{F}_3)\bar{\mathcal{Q}}\mathcal{F}_1 - g(\mathcal{F}_1,\mathcal{F}_3)\bar{\mathcal{Q}}\mathcal{F}_2],\end{aligned}$$

which implies that

$$\bar{\mathcal{R}}ic(\mathcal{F}_2,\mathcal{F}_3) = \frac{\bar{\tau}}{n}g(\mathcal{F}_2,\mathcal{F}_3). \tag{21}$$

With the help of (21) and Lemma 2.1 (iii), we obtain

$$\begin{aligned}\widetilde{\mathcal{R}}ic(\mathcal{F}_2,\mathcal{F}_3) &= [\frac{\bar{\tau}}{n} + \varepsilon(2\alpha-1) + (1-\alpha)]g(\mathcal{F}_2,\mathcal{F}_3) \\ &+ [\varepsilon(2\alpha-1) - (n\alpha-1)]\eta(\mathcal{F}_2)\eta(\mathcal{F}_3).\end{aligned} \tag{22}$$

Putting $\mathcal{F}_3$=ζ in (22) and then multiplying both sides by 2κ, we get

$$2\kappa\widetilde{\mathcal{R}}ic(\mathcal{F}_2,\zeta) = [\frac{2\kappa\bar{\tau}}{n} + 2\kappa\alpha(n-1)]\eta(\mathcal{F}_2). \tag{23}$$

Next, let $(g,\zeta,\mu,\nu,\kappa,l)$ be a CERYS on $\mathbb{N}$ and $\mathbb{N}$ is anti-invariant, then from (1), we lead to

$$2\kappa\widetilde{\mathcal{R}}ic(\mathcal{F}_2,\mathcal{F}_3) = -[2\mu - l\tau - \frac{1}{n}(pn+2)]g(\mathcal{F}_2,\mathcal{F}_3) - 2\nu\eta(\mathcal{F}_2)\eta(\mathcal{F}_3). \tag{24}$$

Again setting $\mathcal{F}_3$=ζ in (24), we have

$$2\kappa\widetilde{\mathcal{R}}ic(\mathcal{F}_2,\zeta)=[-2\mu+l\tau+\frac{1}{n}(pn+2)+2\nu]\eta(\mathcal{F}_2). \tag{25}$$

Equating (23) and (25), we get

$$\mu=-\frac{\kappa\bar{\tau}}{n}-\kappa\alpha(n-1)+\frac{l\tau}{2}+\frac{1}{2n}(pn+2)+\nu. \tag{26}$$

We assert the outcome:

Theorem 6. *If an anti-invariant submanifold* $\mathbb{N}$ *of an* $(\mathcal{LCS})_n$*-manifold* $\widetilde{\mathbb{V}}$ *is* $\mathcal{M}$*-projectively flat with respect to QSMC* $\bar{\tilde{\nabla}}$*, then the CERYS of type* (κ,l) *on* $\mathbb{N}$ *is contracting, stable or increasing accordingly as*

$$-\frac{\kappa\bar{\tau}}{n}-\kappa\alpha(n-1)+\frac{l\tau}{2}+\frac{1}{2n}(pn+2)+\nu\lesseqqgtr 0.$$

It is clear, from (26) that, if $\kappa=0$, then $\mu=\frac{l\tau}{2}+\frac{1}{2n}(pn+2)+\nu$ and if $l=0$, then $\mu=-\frac{\kappa\bar{\tau}}{2}-\kappa\alpha(n-1)+\frac{1}{2n}(np+2)+\nu$. Thus, we state:

Corollary 6. *If an anti-invariant submanifold* $\mathbb{N}$ *of an* $(\mathcal{LCS})_n$*-manifold* $\widetilde{\mathbb{V}}$ *is* $\mathcal{M}$*-projectively flat with respect to QSMC* $\bar{\tilde{\nabla}}$*, then the CEYS of type* $(0,1)$ *on* $\mathbb{N}$ *is contracting, stable or increasing accordingly as* $\tau<-\frac{1}{n}[n(p+2\nu)+2]$*,* $\tau=-\frac{1}{n}[n(p+2\nu)+2]$*, or* $\tau>-\frac{1}{n}[n(p+2\nu)+2]$*, respectively.*

Corollary 7. *If an anti-invariant submanifold* $\mathbb{N}$ *of an* $(\mathcal{LCS})_n$*-manifold* $\widetilde{\mathbb{V}}$ *is* $\mathcal{M}$*-projective flat with respect to QSMC* $\bar{\tilde{\nabla}}$*, then the CERS of type* $(1,0)$ *on* $\mathbb{N}$ *is contracting, stable or increasing accordingly as*

$$-\frac{\bar{\tau}}{2}-\alpha(n-1)+\frac{1}{2n}(np+2)+\nu\lesseqqgtr 0.$$

Again taking $\mathcal{F}_2$=$\mathcal{F}_3$=υ_i, $i\,(1\leq i\leq n)$ in (1) and using (21), we have

$$\bar{\mathfrak{L}}_{\mathcal{F}_1}g(\upsilon_i,\upsilon_i)+\left\{\frac{2\kappa\bar{\tau}}{n}+2\mu-l\tau-\frac{1}{n}(pn+2)\right\}g(\upsilon_i,\upsilon_i)+2\nu\eta(\upsilon_i)\eta(\upsilon_i)=0,$$

which leads to

$$div(\mathcal{F}_1)+\left\{\kappa\bar{\tau}+n\mu-\frac{ln\tau}{2}-\frac{1}{2}(pn+2)\right\}-\nu=0. \tag{27}$$

If $\mathcal{F}_1$ is solenoidal, then $div(\mathcal{F}_1)$=0 and hence (27) reduces to

$$\mu=(\frac{p}{2}+\frac{1}{n})+\frac{l\tau}{2}-\frac{\kappa\bar{\tau}}{2}+\frac{\nu}{n}.$$

Again, if $\mathcal{F}_1$=$grad(f)$, then the equation (27) becomes

$$\nabla^2 f=-\kappa\bar{\tau}-n\mu+\frac{ln\tau}{2}+\frac{1}{2}(pn+2)+\nu. \tag{28}$$

As a result, we may state:

Theorem 7. *Let the metric g of an $\mathcal{M}$-projectively flat anti-invariant submanifold $\mathbb{N}$ of an $(\mathcal{LCS})_n$-manifold $\widetilde{\mathbb{V}}$ with respect to QSMC $\bar{\bar{\nabla}}$ be a CERYS of type (κ, l), where $\mathcal{F}_1$=grad(f) then (28) holds.*

Corollary 8. *Let the metric g of an $\mathcal{M}$-projectively flat anti-invariant submanifold $\mathbb{N}$ of an $(\mathcal{LCS})_n$-manifold $\widetilde{\mathbb{V}}$ with respect to QSMC $\bar{\bar{\nabla}}$ be a CERYS of type (κ, l). Then the vector field $\mathcal{F}_1$ is solenoidal iff*

$$\mu = \frac{1}{2}(p + \frac{2}{n}) + \frac{l\tau}{2} - \frac{\kappa\bar{\tau}}{n} + \frac{\nu}{n}.$$

6. Cerys on Pseudo-Projectively Flat Anti-Invariant Submanifolds Admitting $\bar{\bar{\nabla}}$

The pseudo-projective curvature tensor $\widetilde{\mathcal{P}}$ of rank three on $(\mathbb{N}^n, g)$ is given by [21]

$$\begin{aligned} \widetilde{\mathcal{P}}(\mathcal{F}_1, \mathcal{F}_2)\mathcal{F}_3 &= \sigma\mathcal{R}(\mathcal{F}_1, \mathcal{F}_2)\mathcal{F}_3 + \varsigma[\mathcal{R}ic(\mathcal{F}_2, \mathcal{F}_3)\mathcal{F}_1 - \mathcal{R}ic(\mathcal{F}_1, \mathcal{F}_3)\mathcal{F}_2] \\ &\quad +\varrho\tau[g(\mathcal{F}_2, \mathcal{F}_3)\mathcal{F}_1 - g(\mathcal{F}_1, \mathcal{F}_3)\mathcal{F}_2], \end{aligned} \tag{29}$$

for all smooth vectors fields $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3 \in \chi(\mathbb{N})$, where σ, ς, ϱ are non-zero constants related by $\varrho = -\frac{1}{n}(\frac{\sigma}{n-1} + \varsigma)$.

Let $(\mathbb{N}^n, g)$ is pseudo-projectively flat with respect to QSMC $\bar{\bar{\nabla}}$, then from (29), we yields

$$\begin{aligned} \sigma\bar{\mathcal{R}}(\mathcal{F}_1, \mathcal{F}_2)\mathcal{F}_3 &= -\varsigma[\bar{\mathcal{R}}ic(\mathcal{F}_2, \mathcal{F}_3)\mathcal{F}_1 - \bar{\mathcal{R}}ic(\mathcal{F}_1, \mathcal{F}_3)\mathcal{F}_2] \\ &\quad - \varrho\bar{\tau}[g(\mathcal{F}_2, \mathcal{F}_3)\mathcal{F}_1 - g(\mathcal{F}_1, \mathcal{F}_3)\mathcal{F}_2], \end{aligned}$$

which is equivalent to

$$[\sigma + \varsigma(n-1)]\bar{\mathcal{R}}ic(\mathcal{F}_2, \mathcal{F}_3) = -\varrho\bar{\tau}(n-1)g(\mathcal{F}_2, \mathcal{F}_3). \tag{30}$$

Using (30) in Lemma 2.1-(iii), we obtain

$$\begin{aligned} \widetilde{\mathcal{R}}ic(\mathcal{F}_2, \mathcal{F}_3) &= \big[\frac{-\varrho\bar{\tau}(n-1)}{\{\sigma + \varsigma(n-1)\}} + \varepsilon(2\alpha - 1) - (\alpha - 1)\big]g(\mathcal{F}_2, \mathcal{F}_3) \\ &\quad -[(n\alpha - 1) - \varepsilon(2\alpha - 1)]\eta(\mathcal{F}_2)\eta(\mathcal{F}_3). \end{aligned} \tag{31}$$

By fixing $\mathcal{G} = \xi$ in (31) and then multiplying both sides by 2κ, we have

$$2\kappa\widetilde{\mathcal{R}}ic(\mathcal{F}_2, \zeta) = [\frac{-2\kappa\varrho\bar{\tau}(n-1)}{\{\sigma + \varsigma(n-1)\}} + 2\alpha\kappa(n-1)]\eta(\mathcal{F}_2). \tag{32}$$

In view of (25) and (32), we get

$$\mu = \frac{\kappa\varrho\bar{\tau}(n-1)}{\{\sigma - \varsigma(1-n)\}} + \frac{l\tau}{2} + (\frac{p}{2} + \frac{1}{n}) + \alpha\kappa(1-n) + \nu.$$

Accordingly, as the Section 5, we claim:

Theorem 8. *If an anti-invariant submanifold $\mathbb{N}$ of an $(\mathcal{LCS})_n$-manifold $\widetilde{\mathbb{V}}$ is pseudo-projectively flat with respect to QSMC $\bar{\bar{\nabla}}$, then the CERYS of type (κ, l) on $\mathbb{N}$ is contracting, stable or increasing accordingly as*

$$\frac{\kappa\varrho\bar{\tau}(n-1)}{\{\sigma-\varsigma(1-n)\}}+\alpha\kappa(1-n)+\frac{l\tau}{2}+(\frac{p}{2}+\frac{1}{n})+\nu \lesseqqgtr 0.$$

Corollary 9. *If an anti-invariant submanifold $\mathbb{N}$ of an $(\mathcal{LCS})_n$-manifold $\widetilde{\mathbb{V}}$ is pseudo-projectively flat admits QSMC $\bar{\bar{\nabla}}$, then the CEYS of type $(0,1)$ on $\mathbb{N}$ is contracting, stable or increasing accordingly as $\tau < -[(p+\frac{2}{n})+2\nu]$, $\tau = -[(p+\frac{2}{n})+2\nu]$ or $\tau > -[(p+\frac{2}{n})+2\nu]$.*

Corollary 10. *If an anti-invariant submanifold $\mathbb{N}$ of an $(\mathcal{LCS})_n$-manifold $\widetilde{\mathbb{V}}$ is pseudo-projectively flat admits QSMC $\bar{\bar{\nabla}}$, then the CERYS of type $(1,0)$ on $\mathbb{N}$ is contracting, stable or increasing accordingly as*

$$\frac{\varrho\bar{\tau}(n-1)}{\{\sigma-\varsigma(1-n)\}}+\alpha(1-n)+(\frac{p}{2}+\frac{1}{n})+\nu \lesseqqgtr 0.$$

Next, we replace $\mathcal{F}_2$=$\mathcal{F}_3$=υ_i $i(1 \leq i \leq n)$ in (1) we have

$$\begin{aligned}\bar{\mathfrak{L}}_{\mathcal{F}_1}g(\upsilon_i,\upsilon_i) &= \left\{\frac{2\kappa\varrho\bar{\tau}(n-1)}{\sigma+\varsigma(n-1)}+2\kappa\{\alpha(1-2\varepsilon)+\varepsilon\}-\{2\mu-l\tau-\frac{1}{n}(pn+2)\}\right\}g(\upsilon_i,\upsilon_i)\\ &- [2\nu-2\kappa\{\alpha(m-2\varepsilon)+\varepsilon-1\}]\eta(\upsilon_i)\eta(\upsilon_i),\end{aligned}$$

which implies that

$$\begin{aligned}div(\mathcal{F}_1) &= \left\{\frac{n\kappa\varrho\bar{\tau}(n-1)}{\sigma+\varsigma(n-1)}+n\kappa\{\alpha(1-2\varepsilon)+\varepsilon\}-\{n\mu-\frac{nl\tau}{2}-\frac{1}{2}(pn+2)\}\right\}\\ &- [\nu-\kappa\{\alpha(m-2\varepsilon)+\varepsilon-1\}]. \end{aligned} \tag{33}$$

If $\mathcal{F}_1$ is solenoidal, then $div(\mathcal{F}_1)$=0 and hence equation (33) reduces to

$$\begin{aligned}\mu &= \left[\frac{\kappa\varrho\bar{\tau}(n-1)}{\sigma+\varsigma(n-1)}+\frac{l\tau}{2}+\frac{1}{2n}(pn+2)+\kappa\{\alpha(1-2\varepsilon)+\varepsilon\}\right]\\ &- \frac{1}{n}[\nu-\kappa\{\alpha(m-2\varepsilon)+\varepsilon-1\}]. \end{aligned} \tag{34}$$

Again, if $\mathcal{F}_1$=$grad(f)$, then the equation (33) becomes

$$\begin{aligned}\nabla^2 f &= \frac{n\kappa\varrho\bar{\tau}(n-1)}{\sigma-\varsigma(n-1)}+n\kappa\{\alpha(1-2\varepsilon)+\varepsilon\}-n\mu+\frac{nl\tau}{2}+\frac{1}{2}(pn+2)\\ &- [\nu-\kappa\{\alpha(m-2\varepsilon)+\varepsilon-1\}]. \end{aligned} \tag{35}$$

Thus, we assert:

Theorem 9. *Let the metric g of a pseudo-projectively flat anti-invariant submanifold $\mathbb{N}$ of an $(\mathcal{LCS})_n$-manifold $\widetilde{\mathbb{V}}$ with respect to QSMC $\bar{\bar{\nabla}}$ be a CERYS of type (κ, l), where $\mathcal{F}_1$=$grad(f)$, then (35) holds.*

Corollary 11. *Let the metric g of a pseudo-projectively flat anti-invariant submanifold $\mathbb{N}$ of an $(\mathcal{LCS})_n$-manifold $\widetilde{\mathbb{V}}$ with respect to QSMC $\bar{\bar{\nabla}}$ be a CERYS of type (κ, l), then the vector field $\mathcal{F}_1$ is solenoidal iff the relation (34) holds.*

7. Cerys on $\mathcal{Q}$ Flat Anti-Invariant Submanifolds Admitting $\bar{\bar{\nabla}}$

A curvature tensor of type $(1,3)$ on $(\mathbb{N}^n, g)(n > 2)$ is denoted by $\mathcal{Z}$ and defined by

$$\mathcal{Z}(\mathcal{F}_1, \mathcal{F}_2)\mathcal{F}_3 = \mathcal{R}(\mathcal{F}_1, \mathcal{F}_2)\mathcal{F}_3 - \frac{\psi}{n-1}[g(\mathcal{F}_2, \mathcal{F}_3)\mathcal{F}_1 - g(\mathcal{F}_1, \mathcal{F}_3)\mathcal{F}_2], \tag{36}$$

where ψ can be any scalar function. This type of tensor $\mathcal{Z}$ is known as a $\mathcal{Q}$-curvature tensor [15, 16]. If $\psi = \frac{\tau}{n}$, then the $\mathcal{Q}$ curvature tensor is reduced to the concircular curvature tensor.
Let the submanifold $\mathbb{N}$ be $\mathcal{Q}$-flat with respect to $\bar{\bar{\nabla}}$, i.e., $\bar{\mathcal{Z}}(\mathcal{F}_1, \mathcal{F}_2)\mathcal{F}_3 = 0$. Then from (36), we have

$$\bar{\mathcal{R}}(\mathcal{F}_1, \mathcal{F}_2)\mathcal{F}_3 = \frac{\psi}{n-1}[g(\mathcal{F}_2, \mathcal{F}_3)\mathcal{F}_1 - g(\mathcal{F}_1, \mathcal{F}_3)\mathcal{F}_2],$$

which implies that

$$\bar{\mathcal{R}}ic(\mathcal{F}_2, \mathcal{F}_3) = \psi g(\mathcal{F}_2, \mathcal{F}_3). \tag{37}$$

With the help of (9) and Lemma 2.1-(iii), we obtain

$$\begin{aligned} \widetilde{\mathcal{R}}ic(\mathcal{F}_2, \mathcal{F}_3) &= [\psi + \varepsilon(2\alpha - 1) + (1 - \alpha)]g(\mathcal{F}_2, \mathcal{F}_3) \\ &\quad - [n\alpha - 1 + \varepsilon(1 - 2\alpha)]\eta(\mathcal{F}_2)\eta(\mathcal{F}_3). \end{aligned} \tag{38}$$

After taking $\mathcal{F}_3 = \zeta$ in (38) and then multiplying both sides by 2κ we lead to

$$2\kappa\widetilde{\mathcal{R}}ic(\mathcal{F}_2, \zeta) = 2\kappa[\psi + \alpha(n-1)]\eta(\mathcal{F}_2). \tag{39}$$

Equating (25) and (39), we find

$$\mu = \frac{1}{2}(p + \frac{2}{n}) + \frac{l\tau}{2} - \kappa[\psi + \alpha(n-1)] + \nu. \tag{40}$$

Thus, likewise section 6 we bring the outcome:

Theorem 10. *If an anti-invariant submanifold $\mathbb{N}$ of an $(\mathcal{LCS})_n$-manifold $\widetilde{\mathbb{V}}$ is $\mathcal{Q}$-flat with respect to QSMC $\bar{\bar{\nabla}}$, then the CERYS of type (κ, l) on $\mathbb{N}$ is contracting, stable or increasing accordingly as*

$$\frac{1}{2}(p + \frac{2}{n}) + \frac{l\tau}{2} - \kappa[\psi + \alpha(n-1)] + \nu \lesseqqgtr 0.$$

As a result of the aforementioned theorem, we have the following result:

Corollary 12. *If an anti-invariant submanifold $\mathbb{N}$ of an $(\mathcal{LCS})_n$-manifold $\widetilde{\mathbb{V}}$ is concircularly flat with respect to QSMC $\bar{\tilde{\nabla}}$, then the CERYS of type (κ, l) on $\mathbb{N}$ is contracting, stable or increasing accordingly as*

$$\tau \lesseqqgtr \frac{1}{(nl-2\kappa)}[2\kappa\alpha n(n-1)-(np+2)-2n\nu].$$

Also, from (40), if $\kappa = 0$, $l = 1$, then $\mu = \frac{\tau}{2} + \frac{1}{2}(p + \frac{2}{n}) + \nu$, and if $l = 0$, $\kappa = 1$,then $\mu = \frac{1}{2}(p + \frac{2}{n}) - [\psi - \alpha(1-n)] + \nu$. Thus, we state the results:

Corollary 13. *If an anti-invariant submanifold $\mathbb{N}$ of an $(\mathcal{LCS})_n$-manifold $\widetilde{\mathbb{V}}$ is concircularly flat with respect to QSMC $\bar{\tilde{\nabla}}$, then the CEYS of type $(0,1)$ on $\mathbb{N}$ is contracting, stable or increasing accordingly as $\tau < -[(p+\frac{2}{n})+2\nu]$, $\tau = -[(p+\frac{2}{n})+2\nu]$ or $\tau > -[(p+\frac{2}{n})+2\nu]$, respectively.*

Corollary 14. *If an anti-invariant submanifold $\mathbb{N}$ of an $(\mathcal{LCS})_n$-manifold $\widetilde{\mathbb{V}}$ is concircularly flat with respect to QSMC $\bar{\tilde{\nabla}}$, then the CERS of type $(1,0)$ on $\mathbb{N}$ is contracting, stable or increasing accordingly as*

$$(\frac{p}{2}+\frac{1}{n}) - \kappa[\psi - \alpha(1-n)] + \nu \lesseqqgtr 0.$$

Finally, using (37) in (1) and replacing $\mathcal{F}_2 = \mathcal{F}_3 = \upsilon_i, i(1 \le i \le n)$, we get

$$\begin{aligned}\bar{\mathfrak{L}}_{\mathcal{F}_1} g(\upsilon_i,\upsilon_i) &= -\left\{2\mu - l\tau - \frac{1}{n}(pn+2) + 2\kappa\psi - 2\kappa\{\alpha(1-2\varepsilon)+\varepsilon\}\right\} g(\upsilon_i,\upsilon_i)\\ &\quad -[2\nu - 2\kappa\{\alpha(m-2\varepsilon)+\varepsilon-1\}]\eta(\upsilon_i)\eta(\upsilon_i),\end{aligned}$$

it leads to the conclusion that

$$\begin{aligned}div(\mathcal{F}_1) &= -[n\mu - \frac{nl\tau}{2} - \frac{1}{2}(pn+2) + n\kappa\psi - n\kappa\{\alpha(1-2\varepsilon)+\varepsilon\}]\\ &\quad -[\nu - \kappa\{\alpha(m-2\varepsilon)+\varepsilon-1\}].\end{aligned} \tag{41}$$

If $\mathcal{F}_1$ is solenoidal, then $div(\mathcal{F}_1)=0$ and hence (41) reduces to

$$\mu = \frac{l\tau}{2} + \frac{1}{2n}(pn+2) - \psi\kappa + \kappa\{\alpha(1-2\varepsilon)+\varepsilon\} - \frac{1}{n}[\nu - \kappa\{\alpha(m-2\varepsilon)+\varepsilon-1\}]. \tag{42}$$

Again, if $\mathcal{F}_1 = grad(f)$, then the equation (41) becomes

$$\begin{aligned}\nabla^2 f &= [-n\mu + \frac{nl\tau}{2} + \frac{1}{2}(pn+2) - n\kappa\psi + n\kappa\{\alpha(1-2\varepsilon)+\varepsilon\}]\\ &\quad -[\nu - \kappa\{\alpha(m-2\varepsilon)+\varepsilon-1\}].\end{aligned} \tag{43}$$

Theorem 11. *If the metric g of a $\mathcal{Q}$-flat anti-invariant submanifold $\mathbb{N}$ of an $(\mathcal{LCS})_n$-manifold $\widetilde{\mathbb{V}}$ with respect to QSMC $\bar{\tilde{\nabla}}$ be a CERYS of type (κ, l), where $\mathcal{F}_1 = grad(f)$, then (43) holds.*

Corollary 15. *Let the metric g of a $\mathcal{Q}$-flat anti-invariant submanifold $\mathbb{N}$ of an $(\mathcal{LCS})_n$-manifold $\widetilde{\mathbb{V}}$ with respect to QSMC $\bar{\tilde{\nabla}}$ be a CERYS of type (κ, l). Then the vector field $\mathcal{F}_1$ is solenoidal iff the relation (42) holds.*

8. Harmonic Aspect of CERYS on Anti-Invariant Submanifolds Admitting $\tilde{\bar{\nabla}}$

Taking a look at a function $f:\mathbb{N} \to \Re$. We say that f harmonic if $\nabla^2 f=0$, where ∇^2 is the Lalplacian operator on $\mathbb{N}$ [27]. Since, $\zeta=grad(f)$. Then, utilizing Theorems 7, 9, and 11, we convey the following outcomes:

Theorem 12. *If the metric g of an $\mathcal{M}$-projectively flat anti-invariant submanifold $\mathbb{N}$ of an $(\mathcal{LCS})_n$-manifold $\widetilde{\mathbb{V}}$ admits a CERYS of type (κ, l) with respect to QSMC $\tilde{\bar{\nabla}}$ and $\mathcal{F}_1=grad(f)$. If f is a harmonic function on $\mathbb{N}$, then the soliton is increasing, stable, or contracting*
(i) $\tau > \frac{2}{nl}[\kappa\bar{\tau} - \frac{1}{2}(pn+2) - \nu]$,
(ii) $\tau > \frac{2}{nl}[\kappa\bar{\tau} - \frac{1}{2}(pn+2) - \nu]$, *or*
(iii) $\tau > \frac{2}{nl}[\kappa\bar{\tau} - \frac{1}{2}(pn+2) - \nu]$, *respectively.*

Proof. With the help of (28), We may just accomplish the needed results. □

Theorem 13. *If the metric g of a pseudo-projectively flat anti-invariant submanifold $\mathbb{N}$ of an $(\mathcal{LCS})_n$-manifold $\widetilde{\mathbb{V}}$ admits a CERYS of type (κ, l) with respect to QSMC $\tilde{\bar{\nabla}}$ and $\mathcal{F}_1=grad(f)$. If f is a harmonic on $\mathbb{N}$, then the soliton is growing, stable, or collapsing*
(i) $\tau > \frac{-1}{l}\left[\frac{2\kappa\varrho\bar{\tau}(n-1)}{(\sigma+\varsigma(n-1))} + 2\kappa\{\alpha(1-2\varepsilon)+\varepsilon\} + (p+\frac{2}{n}) - \frac{2}{n}[\nu - \kappa\{\alpha(m-2\varepsilon)+\varepsilon-1\}\right]$,
(ii) $\tau = \frac{-1}{l}\left[\frac{2\kappa\varrho\bar{\tau}(n-1)}{(\sigma+\varsigma(n-1))} + 2\kappa\{\alpha(1-2\varepsilon)+\varepsilon\} + (p+\frac{2}{n}) - \frac{2}{n}[\nu - \kappa\{\alpha(m-2\varepsilon)+\varepsilon-1\}\right]$, *or*
(iii) $\tau < \frac{-1}{l}\left[\frac{2\kappa\varrho\bar{\tau}(n-1)}{(\sigma+\varsigma(n-1))} + 2\kappa\{\alpha(1-2\varepsilon)+\varepsilon\} + (p+\frac{2}{n}) - \frac{2}{n}[\nu - \kappa\{\alpha(m-2\varepsilon)+\varepsilon-1\}\right]$, *respectively.*

Proof. We arrive at our conclusions using the equation (35). □

Theorem 14. *If the metric g of a $\mathcal{Q}$-flat anti-invariant submanifold $\mathbb{N}$ of an $(\mathcal{LCS})_n$-manifold $\widetilde{\mathbb{V}}$ admits a CERYS of type (κ, l) with respect to QSMC $\tilde{\bar{\nabla}}$ and $\mathcal{F}_1=grad(f)$. If f is a harmonic on $\mathbb{N}$, then the soliton is growing, stable, or collapsing*
(i) $\tau > -\frac{2}{l}[\frac{1}{2}(p+\frac{2}{n}) - \kappa\psi + \kappa\{\alpha(1-2\varepsilon)+\varepsilon\} - \frac{1}{n}[\nu - \kappa\{\alpha(m-2\varepsilon)-1\}]]$,
(ii) $\tau = -\frac{2}{l}[\frac{1}{2}(p+\frac{2}{n}) - \kappa\psi + \kappa\{\alpha(1-2\varepsilon)+\varepsilon\} - \frac{1}{n}[\nu - \kappa\{\alpha(m-2\varepsilon)-1\}]]$,
(iii) $\tau < -\frac{2}{l}[\frac{1}{2}(p+\frac{2}{n}) - \kappa\psi + \kappa\{\alpha(1-2\varepsilon)+\varepsilon\} - \frac{1}{n}[\nu - \kappa\{\alpha(m-2\varepsilon)-1\}]]$, *respectively.*

Proof. By virtue of equation (43) we may simply obtain the desired outcome. □

9. Example

We define $\widetilde{\mathbb{V}}^5=\{(r,s,t,u,v) \in \Re^5 : u \neq 0\}$, where $\{\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4, \upsilon_5\}$ being standard coordinates of linearly independent vector fields of $\widetilde{\mathbb{V}}^5$ given by

$$\upsilon_1 = e^u\frac{\partial}{\partial r} + e^u s\frac{\partial}{\partial t},\ \upsilon_2 = \frac{\partial}{\partial s},\ \upsilon_3 = \frac{\partial}{\partial t} = \zeta,\ \upsilon_4 = \frac{\partial}{\partial u} + e^u v\frac{\partial}{\partial t},\ \upsilon_5 = \frac{\partial}{\partial v}.$$

Also, the metric g of $\widetilde{\mathbb{V}}^5$ has the following relations

$$g(\upsilon_1,\upsilon_1)=g(\upsilon_2,\upsilon_2)=g(\upsilon_3,\upsilon_3)=g(\upsilon_4,\upsilon_4)=g(\upsilon_5,\upsilon_5)=1,\ \ ,g(\upsilon_3,\upsilon_3)=-1.$$

Let the 1-form η is given by $\eta(\mathcal{F}_1)$=$g(\mathcal{F}_1,\upsilon_3)$, $\forall$ $\mathcal{F}_1\in\widetilde{\mathbb{V}}^5$ and the $(1,1)$-tensor field ϕ of $\widetilde{\mathbb{V}}^5$ as follows

$$\phi\upsilon_1=\upsilon_2,\ \phi\upsilon_2=\upsilon_1,\ \phi\upsilon_3=0,\ \phi\upsilon_4=\upsilon_5,\ \phi\upsilon_5=\upsilon_4.$$

Utilizing the linearity qualities of ϕ and g dictates how they interact.

$$\phi^2\upsilon_i=\upsilon_i+\eta(\upsilon_i)\zeta,\ \ \eta(\upsilon_3)=-1,$$

hold for i=$1,2,3,4,5$ and ζ=υ_3. Also, for ζ=υ_3, $\widetilde{\mathbb{V}}^5$ satisfies $g(\upsilon_i,\upsilon_3)$=$\eta(\upsilon_i)$, $g(\phi\upsilon_i,\upsilon_j)$=$g(\upsilon_i,\phi\upsilon_j)$ and $g(\phi\upsilon_i,\phi\upsilon_j)$=$g(\upsilon_i,\upsilon_j)$+$\eta(\upsilon_i)\eta(\upsilon_j)$, where $i,j=1,2,3,4,5$. Now, we can compute

$$[\upsilon_i,\upsilon_j]=\begin{cases}-e^u\upsilon_3, & \text{if } i=1,\ j=2,\\ -e^u\upsilon_1, & \text{if } i=1,\ j=4,\\ -e^u\upsilon_3, & \text{if } i=4,\ j=5,\\ 0, & \text{otherwise.}\end{cases}$$

We may use Koszul's formula for getting

$$\widetilde{\nabla}_{\upsilon_1}\upsilon_1=0,\ \ \widetilde{\nabla}_{\upsilon_1}\upsilon_2=\frac{e^u}{2}\upsilon_3,\ \ \widetilde{\nabla}_{\upsilon_1}\upsilon_3=-\frac{e^u}{2}\upsilon_2,\ \ \widetilde{\nabla}_{\upsilon_1}\upsilon_4=0,\ \ \widetilde{\nabla}_{\upsilon_1}\upsilon_5=0,$$

$$\widetilde{\nabla}_{\upsilon_2}\upsilon_1=-\frac{e^u}{2}\upsilon_3,\ \ \widetilde{\nabla}_{\upsilon_2}\upsilon_2=0,\ \ \widetilde{\nabla}_{\upsilon_2}\upsilon_3=-\frac{e^u}{2}\upsilon_1\ \ \widetilde{\nabla}_{\upsilon_2}\upsilon_4=0,\ \ \widetilde{\nabla}_{\upsilon_2}\upsilon_5=0,$$

$$\widetilde{\nabla}_{\upsilon_3}\upsilon_1=-\frac{e^u}{2}\upsilon_2,\ \ \widetilde{\nabla}_{\upsilon_3}\upsilon_2=-\frac{e^u}{2}\upsilon_1,\ \ \widetilde{\nabla}_{\upsilon_3}\upsilon_3=0,\ \ \widetilde{\nabla}_{\upsilon_3}\upsilon_4=-\frac{e^u}{2}\upsilon_5,\ \ \widetilde{\nabla}_{\upsilon_3}\upsilon_5=-\frac{e^u}{2}\upsilon_4,$$

$$\widetilde{\nabla}_{\upsilon_4}\upsilon_1=0,\ \ \widetilde{\nabla}_{\upsilon_4}\upsilon_2=0,\ \ \widetilde{\nabla}_{\upsilon_4}\upsilon_3=-\frac{e^u}{2}\upsilon_5,\ \ \widetilde{\nabla}_{\upsilon_4}\upsilon_4=0,\ \ \widetilde{\nabla}_{\upsilon_4}\upsilon_5=-\frac{e^u}{2}\upsilon_3,$$

$$\widetilde{\nabla}_{\upsilon_5}\upsilon_1=0,\ \ \widetilde{\nabla}_{\upsilon_5}\upsilon_2=0,\ \ \widetilde{\nabla}_{\upsilon_5}\upsilon_3=-\frac{e^u}{2}\upsilon_4,\ \ \widetilde{\nabla}_{\upsilon_5}\upsilon_4=-\frac{e^u}{2}\upsilon_3,\ \ \widetilde{\nabla}_{\upsilon_5}\upsilon_5=0.$$

Thus for υ_3=ζ and α=-$\frac{e^u}{2}$ we verified that $\widetilde{\nabla}_{\mathcal{F}_1}\zeta$=$\alpha\phi\mathcal{F}_1$ for all $\mathcal{F}_1\in\mathcal{T}\widetilde{\mathbb{V}}^5$, where $\mathcal{F}_1$=$\mathcal{F}_1\upsilon_1+\mathcal{F}_2\upsilon_2+\mathcal{F}_3\upsilon_3+\mathcal{F}_4\upsilon_4+\mathcal{F}_5\upsilon_5$. So, the manifold $\widetilde{\mathbb{V}}^5$ equipped with the structure (ϕ,ζ,η,g) is an $(\mathcal{LCS})_5$-manifold with α=-$\frac{e^u}{2}$ and $\varrho^\star$=-$\mathcal{F}_4\alpha$.

Let $\widetilde{\pi}:\mathbb{N}\to\widetilde{\mathbb{V}}$ and given by $\widetilde{\pi}(r,s,t)$=$(r,s,u,0,0)$. Then we define $\mathbb{N}$=$\{(r,s,u)\in\Re^3:u\neq0\}$, where (r,s,u) are the standard coordinates in $\Re^3$. Let $\{\upsilon_1,\upsilon_2,\upsilon_3\}$ on $\mathbb{N}$ given by

$$\upsilon_1=e^u\frac{\partial}{\partial r}+e^us\frac{\partial}{\partial u},\ \upsilon_2=\frac{\partial}{\partial s},\ \upsilon_3=\frac{\partial}{\partial u}.$$
$$g(\upsilon_1,\upsilon_1)=g(\upsilon_2,\upsilon_2)=1,\ \ g(\upsilon_3,\upsilon_3)=-1.$$

Also, the $(1,1)$-tensor field ϕ of $\mathbb{N}^3$ is given by

$$\phi\upsilon_1=\upsilon_2,\ \phi\upsilon_2=\upsilon_1,\ \phi\upsilon_3=0.$$

Utilizing the linearity qualities of ϕ and g dictates how they interact

$$\phi^2\upsilon_i = \upsilon_i + \eta(\upsilon_i)\zeta, \ \eta(\zeta) = -1,$$

for i=1, 2, 3 and ζ=υ_3. Again, for ζ=υ_3, $\mathbb{N}^3$ satisfies

$$g(\phi\upsilon_i, \phi\upsilon_j) = g(\upsilon_i, \upsilon_j) + \eta(\upsilon_i)\eta(\upsilon_j),$$

where i, j=1, 2, 3. Next, one can easily obtain

$$[\upsilon_1, \upsilon_2] = -e^u\upsilon_3, \quad [e_1, \upsilon_3] = -e^u\upsilon_1, \quad [\upsilon_2, \upsilon_3] = 0.$$

We acquire assuming Koszul's formula

$$\nabla_{\upsilon_1}\upsilon_1 = 0, \quad \nabla_{\upsilon_1}\upsilon_2 = \frac{e^u}{2}\upsilon_3, \quad \nabla_{\upsilon_1}\upsilon_3 = -\frac{e^u}{2}\upsilon_2, \quad \nabla_{\upsilon_2}e_1 = -\frac{e^u}{2}\upsilon_3, \quad \nabla_{\upsilon_2}\upsilon_2 = 0,$$

$$\nabla_{\upsilon_2}\upsilon_3 = -\frac{e^u}{2}\upsilon_1, \quad \nabla_{\upsilon_3}\upsilon_1 = -\frac{e^u}{2}\upsilon_2, \quad \nabla_{\upsilon_3}\upsilon_2 = -\frac{e^u}{2}\upsilon_1, \quad \nabla_{\upsilon_3}\upsilon_3 = 0.$$

Thus the data (ϕ, ζ, η, g) is an $(\mathcal{LCS})_3$-structure on $\mathbb{N}$. Consequently, if $\mathbb{N}^3$ equipped with the structure (ϕ, ζ, η, g) is $(\mathcal{LCS})_3$ manifold with α=-$\frac{e^u}{2}$ and $\varrho^\star$=-$\mathcal{F}_3\alpha$. We define the tangent space $\mathcal{T}\mathbb{N}$ of $\mathbb{N}^3$ as follows

$$\mathcal{T}\mathbb{N} = \mathcal{D} \oplus \mathcal{D}^\perp \oplus < \zeta >,$$

where $\mathcal{D}$=$< \upsilon_1 >$, $\mathcal{D}^\perp$=$< \upsilon_2 >$. Since $\phi\upsilon_1$=$\upsilon_2 \in \mathcal{D}^\perp$, for $\upsilon_1 \in \mathcal{D}$ and $\phi\upsilon_2$=$\upsilon_1 \in \mathcal{D}$, for $\upsilon_2 \in \mathcal{D}^\perp$. Then, $\mathbb{N}^3$ is an invariant submanifold of $\widetilde{\mathbb{V}}^5$. Also, from (5) we have $\hbar(\upsilon_i, \upsilon_j)$=$\widetilde{\nabla}_{\upsilon_i}\upsilon_j - \nabla_{\upsilon_i}\upsilon_j$. Using the values of $\widetilde{\nabla}_{\upsilon_i}\upsilon_j$ and $\nabla_{\upsilon_i}\upsilon_j$, we notice that $\hbar(\upsilon_i, \upsilon_j)$=0, $\forall\, i,\ j = 1, 2, 3$. i.e., $\mathbb{N}^3$ is totally geodesic. So, Theorem 1 is verified. Now, using (16) we get the QSMC $\bar{\bar{\nabla}}$ on $\mathbb{N}$ as follows

$$\bar{\bar{\nabla}}_{\upsilon_1}\upsilon_3 = -\left\{\frac{e^u+2}{2}\right\}\upsilon_2, \quad \bar{\bar{\nabla}}_{\upsilon_1}\upsilon_1 = 0, \quad \bar{\bar{\nabla}}_{\upsilon_1}\upsilon_2 = \left\{\frac{e^u-2}{2}\right\}\upsilon_3,$$
$$\bar{\bar{\nabla}}_{\upsilon_2}\upsilon_3 = -\left\{\frac{e^u+2}{2}\right\}\upsilon_1, \quad \bar{\bar{\nabla}}_{\upsilon_3}\upsilon_2 = -\frac{e^u}{2}\upsilon_1, \quad \bar{\bar{\nabla}}_{\upsilon_2}\upsilon_1 = -\left\{\frac{e^u+2}{2}\right\}\upsilon_3,$$
$$\bar{\bar{\nabla}}_{\upsilon_3}\upsilon_3 = 0, \quad \bar{\bar{\nabla}}_{\upsilon_2}\upsilon_2 = 0, \quad \bar{\bar{\nabla}}_{\upsilon_3}\upsilon_1 = 0.$$

By using the preceding relations, one can get $\bar{\mathcal{R}}$.

$$\bar{\mathcal{R}}(\upsilon_1, \upsilon_2)\upsilon_1 = \frac{(e^u+2)^2}{4}\upsilon_2, \ \bar{\mathcal{R}}(\upsilon_1, \upsilon_2)\upsilon_2 = -\frac{(3e^{2u}-4)}{4}\upsilon_1, \ \bar{\mathcal{R}}(\upsilon_2, \upsilon_3)\upsilon_2 = \frac{e^u(e^u+2)}{4}\upsilon_3.$$

Also, the $\bar{\mathcal{R}}ic$ and $\bar{\tau}$ have the value

$$\bar{\mathcal{R}}ic(\upsilon_1, \upsilon_1) = -\frac{(3e^{2u}-4)}{4}, \ \bar{\mathcal{R}}ic(\upsilon_2, \upsilon_2) = 0, \ \bar{\mathcal{R}}ic(\upsilon_3, \upsilon_3) = \frac{e^u(e^u+2)}{4},$$

$$\bar{\tau} = -[(e^{2u}-1) + \frac{e^u}{2}].$$

Since, $\mathbb{N}$ in invariant on $\widetilde{\mathbb{V}}$. Therefore, from the equations (1) and (17) we obtain

$$2\kappa\bar{\mathcal{R}}ic(\upsilon_i,\upsilon_i)+[2(\alpha-1)+2\mu-l\bar{\tau}-\frac{1}{n}(pn+2)]g(\upsilon_i,\upsilon_i) \quad (44)$$
$$+2[\alpha-1+\nu]\eta(\upsilon_i)\eta(\upsilon_i)=0,$$

for all $i\in\{1,2,3\}$. From the equation (44), we can easily calculate

$$\mu = \frac{1}{6}[(3p+2)-(3l-2\kappa)\bar{\tau}+2\nu-4(\alpha-1)]. \quad (45)$$

$$\nu = -\frac{1}{6}(3p+2)-\frac{\kappa e^u(e^u+2)}{4}+\mu-\frac{l\bar{\tau}}{2}. \quad (46)$$

With help of equations (45), (46) and the value of $\bar{\tau}$, we obtain

$$\mu = \frac{(3p+2)}{6}-\frac{l(2e^{2u}-2+e^u)}{4}+\frac{\kappa(3e^{2u}-4)}{8}-\alpha+1.$$

Thus the data $(g,\mathcal{F}_1,\mu,\nu,\kappa,l)$ is a CERYS of type (κ,l) with respect to QSMC $\bar{\tilde{\nabla}}$ on $(\mathbb{N}^3,g)$. Now, we conclude that:

Case(a):
For $\kappa=1$ and $l=0$, $(\mathbb{N}^3,g)$ also admits the CERS, which is
(i) expanding if $p>-\frac{3}{4}e^{2u}+2\alpha-\frac{5}{3}$,
(ii) steady if $p=-\frac{3}{4}e^{2u}+2\alpha-\frac{5}{3}$,
(iii) shrinking if $p<-\frac{3}{4}e^{2u}+2\alpha-\frac{5}{3}$.

Case(b):
For $\kappa=0$ and $l=1$, then $(\mathbb{N}^3,g)$ admits the CEYS, which is
(i) expanding if $p>e^u(e^u+\frac{1}{2})+2\alpha-\frac{11}{3}$,
(ii) steady if $p=e^u(e^u+\frac{1}{2})+2\alpha-\frac{11}{3}$,
(iii) shrinking if $p<e^u(e^u+\frac{1}{2})+2\alpha-\frac{11}{3}$.

Case(c):
For $\kappa=1$ and $l=-1$, $(\mathbb{N}^3,g)$ admits the CEES, which is
(i) expanding if $p>-\frac{e^u}{4}(7e^u+2)-\frac{2}{3}+2\alpha$,
(ii)steady if $p=-\frac{e^u}{4}(7e^u+2)-\frac{2}{3}+2\alpha$,
(iii) shrinking if $p<-\frac{e^u}{4}(7e^u+2)-\frac{2}{3}+2\alpha$.

10. Conclusion

The investigation of a CERYS on Riemannian (or pseudo-Riemannian) manifolds is crucial in differential geometry, relativity theory and physics. RY flow is the most visible representative of modern physics. In addition to differential geometry, the CERYS is a new idea that works with geometric and physical applications. We characterized the submanifolds of a $(\mathcal{LCS})_n$-manifold that admits the CERYS with a QSMC in our study.

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References

[1] Atceken, M., Hui, S. K., Slant and pseudo-slant submanifolds in $(\mathcal{LCS})_n$-manifolds, *Czechoslovak Math. J.*, 63 (2013), 177-190. http://eudml.org/doc/252505

[2] Basu, N., Bhattacharyya, A., Conformal Ricci soliton in Kenmotsu manifold. *Global Journal of Advanced Research on Classical and Modern Geometries*, 4 (2015), 15-21.

[3] Baishya, K. K., Eyasmin, S., Generalized weakly Ricci-symmetric $(\mathcal{LCS})_n$-Spacetimes, *J. of Geom. and Physics*, 132 (2018), 415-422. https://doi.org/10.1016/j.geomphys.2018.05.029

[4] De, U. C., Sardar, A., De, K., Ricci-Yamabe solitons and 3-dimensional Riemannian manifolds, *Turk J. of Math.*, 6(3) (2022), 1078-1088. https://doi.org/10.55730/1300-0098.3143

[5] De, U. C., Haseeb, A., On generalized Sasakian space forms with $\mathcal{M}$-projective curvature tensor, *Adv. Pure Appl. Math.*, 9 (2018), 67-73. https://doi.org/10.1515/apam-2017-0041

[6] Fischer, A. E., An introduction to conformal Ricci flow, *Classical and Quantum Gravity*, 21 (2004), 171-218. https://doi.org/10.1088/0264-9381/21/3/011

[7] Güler, S., Crasmareanu, M., Ricci-Yamabe maps for Riemannian flows and their volume variation and volume entropy, *Turk J. Math.*, 43 (2019), 2631-2641. https://doi.org/10.3906/mat-1902-38

[8] Golab, S., On semi-symmetric and quarter symmetric linear connections, *Tensor* (*N.S.*), 29 (1975), 249-254.

[9] Prasad, R., Haseeb, A., On a Lorentzian para-Sasakian manifold with respect to the quarter symmetric-metric connection, *Novi Sad J. Math.*, 46 (2016), 103-116. https://doi.org/10.30755/NSJOM.04279

[10] Hui, S. K. Pal, T., Totally real submanifolds of $(\mathcal{LCS})_n$-manifolds, *Facta Universitatis (NIS) Ser. Math. Inform.*, 33 (2018.), 141-152. https://doi.org/10.22190/FUMI1802141H

[11] Hui, S. K., Prasad, R., Pal, T., Ricci solitons on submanifolds of $(\mathcal{LCS})_n$-manifolds, *Ganita*, 68 (2018), 53-63. https://doi.org/10.48550/arXiv.1707.06815

[12] Ahmad, M., Jun, J. B., Haseeb, A., Hypersurfaces of an almost r-paracontact Riemannian manifold endowed with a quarter symmetric metric connection, *Bull. Korean Math. Soc.*, 46 (2009), 477-487. https://doi.org/10.4134/BKMS.2009.46.3.477

[13] Matsumoto, K., On Lorentzian almost paracontact manifolds, *Bull. of Yamagata Univ. Nat. Sci.*, 12(1989), 151-156.

[14] Mihai, I., Rosca, R., On Lorentzian para-Sasakian manifolds, Classical Analysis, World Scientific Publ. Singapore, 1992.

[15] Mantica, C. A., Suh, Y. J., Pseudo-$\mathcal{Q}$-symmetric Riemannian manifolds, *International Journal of Geometric Methods in Modern Physics*, 10 (2013), 25 pages. https://doi.org/10.1142/S0219887813500138

[16] Yadav, S. K., Yildiz, A., Q-curvature tensor on f-Kenmotsu 3-manifolds, *Universal Journal of Mathematics and Applications*, 5(3) (2022), 96-106. https://doi.org/10.32323/ujma.1055272

[17] Yadav, S. K., Dwivedi, P. K., Suthar, D. L., On $(\mathcal{LCS})_{2n+1}$-manifolds satisfying certain conditions on the concircular curvature tensor, *Thai J. Math.*, 9(3) (2011), 597-603.

[18] Mantica, C. A., Molinari, L. G., A note on concircular structure space-times, *Commun. Korean Math. Soc.*, 34(2) (2019), 633-635. https://doi.org/10.4134/CKMS.c180138

[19] Maksimovic, M. D., Zlatanović., M. L., Quarter-symmetric metric connection on a cosymplectic manifold, *Mathematics*, 11(9) (2023), 2209. https://doi.org/10.3390/math11092209

[20] Pokhariyal, G. P., Mishra, R. S., Curvature tensors and their relativistic significance II, *Yokohama Math. J.*, 19(2) (1971), 97-103. http://hdl.handle.net/11295/38452

[21] Prasad, B., A pseudo-projective curvature tensor on a Riemannian manifold, *Bull. Calcutta Math. Soc.*, 94(3) (2002), 163-166.

[22] Shaikh, A. A., On Lorentzian almost paracontact manifolds with a structure of the concircular type, *Kyungpook Math. J.*, 43 (2003), 305-314.

[23] Shaikh, A. A., Matsuyama, Y., Hui, S. K., On invariant submanifolds of $(\mathcal{LCS})_n$-manifolds, *Journal of the Egyptian Mathematical Society,* 24 (2016), 263-269. https://doi.org/10.1016/j.joems.2015.05.008

[24] Shaikh, A. A., Some results on $(\mathcal{LCS})_n$-manifolds, *J. Korean Math. Soc.*, 46 (2009), 449-461. https://doi.org/10.4134/JKMS.2009.46.3.449

[25] Yano, K., Concircular geometry I, Concircular transformations, *Proc. Imp.Acad. Tokyo,* 16 (1940), 195-200. https://doi.org/10.3792/PIA/1195579139

[26] Yano, K., Kon, M., Structures on manifolds, World Scientific publishing, Singapore, 1984. https://doi.org/10.1142/0067

[27] Yau, S. T., Harmonic functions on complete Riemannian manifolds, *Commu. Pure Appl. Math.*, 28 (1975), 201-228. https://doi.org/10.1002/cpa.3160280203

[28] Haseeb, A., Khan, M. A., Conformal η-Ricci-Yamabe solitons within the framework of ϵ-$\mathcal{LP}$-Sasakian 3-manifolds, *Advances in Mathematical Physics,* (2022), Article ID 3847889, 8 pages. https://doi.org/10.1155/2022/3847889

[29] Haseeb, A., Chaubey, S. K., Khan, M. A., Riemannian 3 manifolds and Ricci-Yamabe Solitons, *International Journal of Geometric Methods in Modern Physics*, 20(1) (2023): 2350015, 13 pages. https://doig/10.1142/S0219887823500159

[30] Zhang, P., Li, Y., Roy, S., Dey, S., Bhattacharyya, A., Geometrical structure in a perfect fluid spacetime with conformal Ricci-Yamabe soliton, *S*ymmetry, 14(3) (2022), 594. https://doi.org/10.3390/sym14030594

[31] Li, Y., Gezer, A., Karakas, E., Some notes on the tangent bundle with a Ricci quarter-symmetric metric connection, *AIMS Mathematics*, 8(8) (2023), 17335-17353. https://doi.org/10.3934/math.2023886