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Rate of Weak Convergence of Random Walk with a Generalized Reflecting Barrier

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Highlights

- A random walk with a generalised reflecting barrier is examined in this study.
- The ergodicity of the process and the weak convergence of the ergodic distribution is discussed.
- An inequality is obtained for evaluating the rate of the weak convergence.
- The rate depends only on the probability characteristics of the ladder height of the random walk.

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Abstract

In this study, a random walk process with generalized reflecting barrier is considered and an inequality for rate of weak convergence of the stationary distribution of the process of interest is propounded. Though the rate of convergence is not thoroughly examined, the literature does provide a weak convergence theorem under certain conditions for the stationary distribution of the process under consideration. Nonetheless, one of the most crucial issues in probability theory is the convergence rate in limit theorems, as it affects the precision and effectiveness of using these theorems in practice. Therefore, for the rate of convergence for the examined process, comparatively simple inequality is represented. The obtained inequality demonstrates that the rate of convergence is correlated with the tail of the distribution of ladder heights of the random walk.

1. INTRODUCTION

Since the introduction of randomness into science, it has become obvious that the theoretical foundations of random walk processes play a crucial role, particularly in the fields of quantum physics and mathematical biology, reliability, stochastic finance, computer science, inventory, and queuing theories, among others. In some real-life problems, the theoretical basis of the random walk with reflecting barriers was especially required. The motion of a high-energy particle in a diluted environment, for instance, is described by the considered process in quantum physics [1]. Some valuable studies on reflecting barriers can be listed as follows: [2-6] etc.

Weak convergence, a critical probabilistic feature of stochastic processes, has been studied in literature [1-2,7-13]. Because theoretical results are usually complicated [1], some asymptotic approaches are taken into account ([2,4]), thus the obtained results are able to be applied in practice.

The rate of weak convergence is also important because it influences the precision and efficacy with which weak convergence theorems can be applied in practice [7,14]. Even though the weak convergence for stationary distribution of the random walk with reflecting barrier is obtained by asymptotic methods in the study [2], the rate of the weak convergence is not investigated. Consider the following real-world model before giving the mathematical design of the process.

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The Model. In this study, a company which its capital level begins from $\lambda z > 0$ $(z, \lambda > 0)$ as an initial level, is considered. The capital level increases with the premiums and the arrivals of the new customers at $T_n = \sum_{i=1}^{n} \xi_i$, $n = 1,2, \ldots$ random times. Moreover, the company loses money due to accidents. Let us consider the variation $(\{\eta_n\}, n \geq 1)$ of the capital level over time $(\{T_n\}, n \geq 1)$ with a stochastic process. Here, both the sequences $\{\eta_n\}$ and $\{T_n\}$ are random. The process moves up and down over time until the capital level drops below zero. A capital level falling below zero indicates an unmet cost $(-\zeta_1)$ which means the company encounters with a crisis. If an unmet cost occurs, the company takes out a loan immediately at λ times bigger of the size of the unmet cost. Thus, as soon as the company faces with a crisis, the process begins from a new initial level $(\lambda \zeta_1)$ and proceeds similarly until it encounters with a new crisis. Every time the company meets with an unmet cost $(\{-\zeta_n\}, n = 1,2, \ldots)$, it continues by carrying out the same procedure. Thus, the process moves by repeating the same kind of cycles. Hence, a random walk with a generalized reflecting barrier expresses such a mechanism.

To investigate the system expressed above, random walk with a generalized reflecting barrier should be examined asymptotically. The weak convergence presented in the study [2] provides an asymptotic estimate, however the convergence's speed depends on more delicate properties of the underlying distribution [1]. The striking feature of the obtained inequality in this study is that it depends only on some basic numerical and probabilistic characteristics of the ladder heights of random walk. The study's primary finding provides a more quantitative outcome by indicating the rate at which this convergence occurs and providing a bound on the maximum error of the convergence found in [2].

The rate of weak convergence for an ergodic distribution of a particular stochastic process has been examined in this work. The rate of convergence in limit theorems is a crucial piece of information. Usually, figuring out the rate is a quite challenging task. Even though there aren't many studies in this field in the literature, the ones that are have a significant impact on both science and practice such as the Berry-Esseen inequality for central limit theorems [1]. The current study's results are helpful in that regard. The fact that the rate can only be determined by taking into account the probability characteristics of the random walk's first ladder height is particularly intriguing.

The rest of the paper is followed by the mathematical construction of the process of interest. Then, after examining the ergodicity of the process in the third section, the preliminary results are given in the fourth section. Finally, the main purpose, which is an inequality for the rate of the weak convergence of the random walk with generalized reflecting barrier is presented. The study is concluded with the discussion on the obtained result.

2. MATHEMATICAL DESIGN OF PROCESS $X(t)$

The primary aim of the study is to attain an inequality for the rate of the weak convergence of the investigated process. To do so, primarily the process is required to be constructed. Therefore, this section establishes mathematically the random walk with generalized reflecting barrier. Correspondingly, denote the essential random variables and boundary functionals of interest.

Let $\{\xi_n\}$ and $\{\eta_n\}$, $n = 1,2, \ldots$ be two independent and identically distributed (iid) random sequences. Suppose that random variables $\xi_n, n = 1,2, \ldots$ take only positive values whereas random variables $\eta_n, n = 1,2, \ldots$ take on both positive and negative values. The following are the distribution functions for them:

$$\Phi(t) \equiv P\{\xi_1 \leq t\}; \quad F(x) \equiv P\{\eta_1 \leq x\}, t \geq 0, x \in R.$$

Give the following definitions for a renewal sequence $\{T_n\}$ and a random walk $\{S_n\}$ as follows:

$$T_0 \equiv S_0 \equiv 0; \ T_n = \sum_{i=1}^{n} \xi_i ; \quad S_n = \sum_{i=1}^{n} \eta_i , n = 1,2, \ldots \ .$$

Additionally, introduce the random variables that follows:

$$N_0 = 0;\ \zeta_0 = z \geq 0;\ \tau_0 \equiv 0;\ \ N_1 \equiv N_1(\lambda z) = \inf\{k \geq 1: \lambda z - S_k < 0\};$$
$$N_n \equiv N_n(\lambda\zeta_{n-1}) = \inf\{k \geq N_{n-1} + 1: \lambda\zeta_{n-1} - (S_k - S_{N_{n-1}}) < 0\};$$
$$\zeta_n \equiv \zeta_n(\lambda\zeta_{n-1}) = |\lambda\zeta_{n-1} - (S_{N_n} - S_{N_{n-1}})|;\ \tau_n = \sum_{i=1}^{N_n} \xi_i,\ \ n = 1,2,\ldots$$

Furthermore, set $\nu(t) = \max\{n \geq 0: T_n \leq t\},\ t > 0$. This allows us to construct the stochastic process as follows:

$$X(t) \equiv \sum_{n=0}^{\infty} \left(\lambda\zeta_n - (S_{\nu(t)} - S_{N_n}) \right) I_{[\tau_n\,;\,\tau_{n+1})}(t),$$

where $I_A(t)$ is an indicator function of the set A. Figure 1 displays a sample path of the process $X(t)$. The term *random walk with a generalized reflecting barrier* refers to the process $X(t)$.

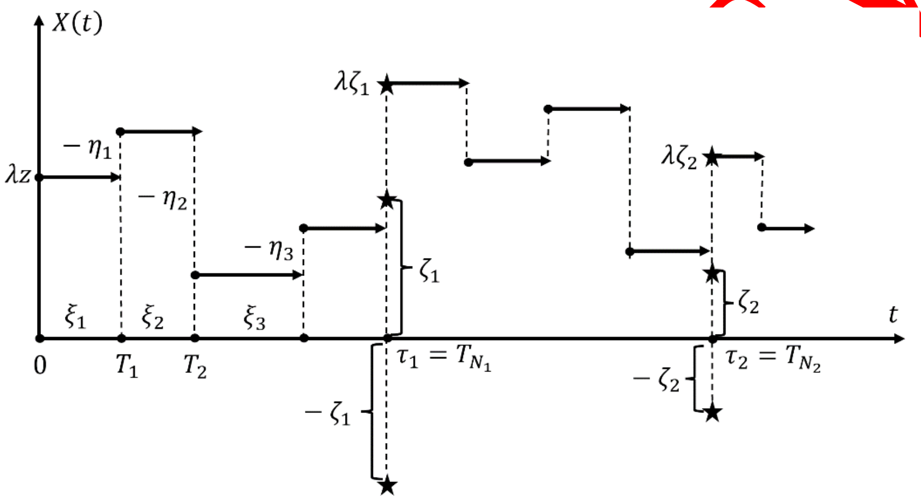

Figure 1. *Illustration of a trajectory of the process* $X(t)$

The primary aim is to obtain an inequality for the rate of the weak convergence for the defined process $X(t)$. To reach the goal, the ergodicity and the weak convergence for the stationary distribution of the process are investigated in the next section.

3. ERGODICITY AND WEAK CONVERGENCE OF PROCESS $X(t)$

The ergodicity of the random walk with generalized reflecting barrier, $X(t)$, is investigated in this section. In the study of [2], the ergodicity of the process $X(t)$ is proved under the following conditions on the initial sequences of random variables $\{\xi_n\}$ and $\{\eta_n\}, n = 1,2,\ldots$:

i) $E(\xi_1) < \infty$; **ii)** $E(\eta_1) > 0$; **iii)** $E(\eta_1^2) < \infty$; **iv)** η_1 has non-arithmetic distribution.

Put $\pi_\lambda(z) = \lim_{n\to\infty} P\{\zeta_n(\lambda\zeta_{n-1}) \leq z\}$. Now, denote by $Y_\lambda(t)$ the standardized stochastic process, i.e., $Y_\lambda(t) = X(t)/\lambda$, the random variables $Y_\lambda(0) = z; Y_\lambda(\tau_n) = \zeta_n; n = 1,2,\ldots$ forms an stationary Markov chain with stationary distribution $\pi_\lambda(x)$. Consequently, it is easily derived that the standardized stochastic process $Y_\lambda(t)$ is also stationary [2].

Furthermore, in [2], the weak convergence theorem for stationary distribution of the standardized process $Y_\lambda(t)$ is presented. Denote $Q_Y(x) \equiv \lim_{t\to\infty} P\{Y_\lambda(t) \leq x\}$.

Proposition 3.1 [2]. Under the above conditions on $\{\xi_n\}$ and $\{\eta_n\}, n = 1,2,...$, as follows, the stationary distribution $Q_Y(x)$ converges weakly to the distribution function $R(x)$:

$$\lim_{\lambda \to \infty} Q_Y(x) = R(x) \equiv \frac{2}{\mu_2} \int_0^x \left\{ \int_v^\infty (1 - F_+(u)) du \right\} dv,$$

where $\mu_2 \equiv E(\chi_1^{+2})$ and $F_+(x) \equiv P\{\chi_1^+ \leq x\}$. Here χ_1^+ is the first ladder height of random walk $\{S_n\}$.

This study aims to evaluate the difference $Q_Y(x) - R(x)$ for sufficiently large values of λ. To achieve this, some propositions and lemmas are presented for the preliminary research in the next section.

4. PRELIMINARY RESULTS FOR INEQUALITY OF WEAK CONVERGENCE RATE

In terms of having the main result, let us include primarily two essential independent random sequences, which are called ladder variables $(\chi_m^+, \nu_m^+), m = 1,2, ...$ Let us introduce the first ladder epoch ν_1^+ and the first ladder height χ_1^+ of the random walk $\{S_n\}, n \geq 0$ as follows: $\nu_1^+ = \min\{n \geq 1: S_n > 0\}$; $\chi_1^+ = S_{\nu_1^+} = \sum_{i=1}^{\nu_1^+} \eta_i$.

The random pairs (χ_n^+, ν_n^+), $n = 2,3, ...$, are mutually independent and identically distributed (iid) with the random pairs (χ_1^+, ν_1^+), respectively (see [1]).

By using ladder heights $\{\chi_n^+, n = 1,2, ...\}$, a renewal process $H(t)$ is defined as $H(t) \equiv \min\{n \geq 1: \sum_{i=1}^n \chi_i^+ > t\}, t \geq 0$. Then, the process $W(t) \equiv \sum_{i=1}^{H(t)} \chi_i^+ - t$ is called as *residual waiting time generated by ladder heights* or *residual waiting time* for shortness. Moreover, the cumulative distribution function (cdf) of $W(t)$ is denoted by $H(t,x)$, i.e., $H(t,x) \equiv P\{W(t) \leq x\}$.

In order to achieve the fundamental advancement of the study, we need to get asymptotic results on cdf and characteristic function of residual waiting time as an auxiliary goal. Therefore, let us include the following propositions.

Proposition 4.1. Suppose that $\mu_3 \equiv E(\chi_1^{+3}) < \infty$. Then, for all $x \geq 0$, $\bar{a}_n(x) \equiv \int_x^\infty v^{n-1}(1 - F_+(v))dv < \infty$; $n = 1,2,3$. Here, $F_+(x) \equiv P\{\chi_1^+ \leq x\}$.

Proof. Recall that the n[th] order of the moment for the positive-valued random variables $\{\chi_n^+, n \geq 1\}$, $\mu_n \equiv E(\chi_1^{+n}) = n \int_0^\infty v^{n-1}(1 - F_+(v))dv$ [4]. For $n = 1,2,3$, the relation that follows is hold:

$$\frac{\mu_n}{n} = \int_0^\infty v^{n-1}(1 - F_+(v))dv \geq \int_x^\infty v^{n-1}(1 - F_+(v))dv = \bar{a}_n(x).$$

Since $\mu_n < \infty, n = 1,2,3$, for all $x \geq 0$, $\bar{a}_n(x) < \infty, n = 1,2,3$ is hold.

The following corollary can be easily attained from Proposition 4.1.

Corollary 4.1. Regarding Proposition 4.1's requirements, the following inequalities are hold:

$$x\bar{a}_1(x) \leq \frac{\mu_2}{2};\ x^2\bar{a}_1(x) \leq \frac{\mu_3}{3} \text{ and } x\bar{a}_2(x) \leq \frac{\mu_3}{3}.$$

Proof. From Proposition 4.1, it is easy to see that the relations that follows are hold:

$$\frac{\mu_2}{2} \geq \int_x^\infty v\,\bar{F}_+(v)dv \geq \int_x^\infty x\,\bar{F}_+(v)dv = x\bar{a}_1(x),$$
$$\frac{\mu_3}{3} \geq \int_x^\infty v^2\,\bar{F}_+(v)dv \geq \int_x^\infty x^2\,\bar{F}_+(v)dv = x^2\bar{a}_1(x),$$
$$\frac{\mu_3}{3} \geq \int_x^\infty v^2\bar{F}_+(v)dv \geq \int_x^\infty xv\,\bar{F}_+(v)dv = x\bar{a}_2(x).$$

Here, $\bar{F}_+(x) \equiv 1 - F_+(x)$. Therefore, since $\mu_3 < \infty$ for all $x \geq 0$, the desired results are obtained.
By including $a_n(x) \equiv \int_0^x v^{n-1}\bar{F}_+(v)dv$ and $\bar{F}_+(x) = 1 - F_+(x)$, let us give the following proposition.

Proposition 4.2. Suppose that $\mu_3 \equiv E(\chi_1^{+3}) < \infty$. Then, $\lim_{x\to\infty} a_n(x) = \frac{\mu_n}{n}$ is hold.

Proof. By using the alternative definition of the n[th] order of moment of a positive-valued random variables and Proposition 4.1, the corresponding result is obtained.

Now, let us examine the limit behavior for the function $\bar{a}_n(x)$ given in Proposition 4.1:

Proposition 4.3. Suppose that $\mu_3 \equiv E(\chi_1^{+3}) < \infty$.Then, $\lim_{x\to\infty} \bar{a}_n(x) = 0$. Here, $\bar{a}_n(x) \equiv \int_x^\infty v^{n-1}\bar{F}_+(v)dv$.
By using Proposition 4.3, we can obtain the following results to reach auxiliary purpose.

Proposition 4.4. Suppose that $\mu_3 \equiv E(\chi_1^{+3}) < \infty$. Then, the following results are hold:

a) $\lim_{x\to\infty} x^2\bar{a}_1(x) = 0$; **b)** $\lim_{x\to\infty} x\bar{a}_1(x) = 0$; **c)** $\lim_{x\to\infty} x\bar{a}_2(x) = 0$.

By using the above results, we can give the two-term asymptotic expansion for the cdf $H(t;x)$ of the residual waiting time $W(t)$ as in part of the secondary purpose of the study.

Lemma 4.1. Let $W(t)$ represent the residual waiting time, produced by ladder heights $\{\chi_n^+\}$,, where $n = 1,2,\ldots$ and $\mu_3 \equiv E(\chi_1^{+3}) < \infty$. Then, the following asymptotic expansion for the cdf $H(t;x)$ can be given, when $t \to \infty$:

$$H(t;x) \equiv P\{W(t) \leq x\} = \pi_+(x) + o\left(\frac{1}{t}\right).$$

Here, $\pi_+(x) = \frac{1}{\mu_1}\int_0^x \left(1 - F_+(t)\right)dt$; $\mu_1 \equiv E(\chi_1^+)$ and $F_+(t) \equiv P\{\chi_1^+ \leq t\}$.

Proof. The corresponding proof is placed in Appendix 1.

Define that a random variable ζ which has distribution $\pi_\lambda(z)$, i.e., $P\{\zeta \leq z\} \equiv \pi_\lambda(z) = \lim_{n\to\infty} P\{\zeta_n(\lambda\zeta_{n-1}) \leq z\}$. By means of Lemma 4.1 and the rest of the propositions for reaching the auxiliary aim, the asymptotic relation of the stationary limit distribution $\pi_\lambda(x)$ can be given in the following lemma.

Lemma 4.2. Suppose that the conditions of Proposition 3.1 are satisfied. Then, the asymptotic relation for $\pi_\lambda(x)$ can be written as follows, when $\lambda \to \infty$:

$$\pi_\lambda(x) \equiv \lim_{n\to\infty} P\{\zeta_n(\lambda\zeta_{n-1}) \leq x\} = \pi_+(x) + \frac{1}{\lambda}g(\lambda;x).$$

Here, $\pi_+(x) = \frac{1}{\mu_1}\int_0^x \big(1 - F_+(t)\big)dt\,;\ \mu_1 = E(\chi_1^+);\ \ F_+(t) = P\{\chi_1^+ \le t\}$ and $g(\lambda; x)$ is a measurable and bounded function with $\lim_{\lambda\to\infty} g(\lambda; x) = 0$.

Proof. The random variable ζ_1 is a residual waiting time generated by the ladder heights $\{\chi_n^+\}, n = 1,2, \dots$. Then, by using the Lemma 4.1, the asymptotic expansion for the distribution function $\pi_{1\lambda}(z)$ of random variable ζ_1 can be written as follows, when $\lambda \to \infty$:

$$H(\lambda z; x) \equiv P\{\zeta_1(\lambda z) \le x\} \equiv \pi_{1\lambda}(\lambda z; x) = \pi_+(x) + \frac{1}{\lambda} g_1(\lambda z; x).$$

Here, $\lim_{\lambda\to\infty} g_1(\lambda z; x) = 0$ for the measurable and bounded function $g_1(\lambda z; x)$. Then, the distribution of the random variable ζ_2 can be written as follows:

$$\pi_{2\lambda}(\lambda z; x) \equiv P\{\zeta_2(\lambda\zeta_1) \le x\} = \int_{v=+0}^{\infty} H(\lambda v; x) d_v H(\lambda z; v). \tag{1}$$

Here, $H(\lambda v; x) = \pi_+(x) + \frac{1}{\lambda} g_1(\lambda v; x)$, when $\lambda \to \infty$. By substituting $H(\lambda v; x)$ in (1):

$$\pi_{2\lambda}(\lambda z; x) = \int_{v=+0}^{\infty} \left\{\pi_+(x) + \frac{1}{\lambda} g_1(\lambda v; x)\right\} d_v H(\lambda z; v) = \pi_+(x) + \frac{1}{\lambda} g_2(\lambda z; x)$$

is acquired. Here, $g_2(\lambda z; x) \equiv \int_{v=+0}^{\infty} g_1(\lambda v; x) d_v H(\lambda z; v)$ is a measurable and bounded function and $\lim_{\lambda\to\infty} g_2(\lambda z; x) = 0$ is hold (see, [2]). Now, examine the distribution of ζ_3 as follows:

$$\pi_{3\lambda}(\lambda z; x) \equiv P\{\zeta_3(\lambda\zeta_2) \le x\} = \pi_+(x) + \frac{1}{\lambda} g_3(\lambda z; x). \tag{2}$$

Here, $g_3(\lambda v; x) = \int_{v=+0}^{\infty} g_1(\lambda v; x)\pi_{2\lambda}(\lambda z; dv)$ is a bounded and measurable function and $\lim_{\lambda\to\infty} g_3(\lambda z; x) = 0$ is hold (see, [2]). Similarly, by induction, the asymptotic expansion that follows can be expressed, when $\lambda \to \infty$:

$$\pi_{n\lambda}(\lambda z; x) = P\{\zeta_n(\lambda\zeta_{n-1}) \le x\} = \pi_+(x) + \tfrac{1}{\lambda} g_n(\lambda z; x);\ \ n = 1,2,3, \dots\ .$$

Here $\pi_+(x) = \frac{1}{\mu_1}\int_0^x \big(1 - F_+(v)\big)dv$ and the function $g_n(\lambda z; x)$ is a measurable function and $\lim_{\lambda\to\infty} g_n(\lambda z; x) = 0$ is hold for all $n = 1,2, \dots$ (see, [2]). By using Equation (2), the following relation can be obtained:

$$\pi_\lambda(x) \equiv \lim_{n\to\infty} \pi_{n\lambda}(\lambda z; x) = \lim_{n\to\infty} P\{\zeta_n(\lambda\zeta_{n-1}) \le x\} = \pi_+(x) + \frac{1}{\lambda}\lim_{n\to\infty} g_n(\lambda z; x).$$

For convenience, include $g(\lambda; x) \equiv \lim_{n\to\infty} g_n(\lambda z; x)$. Recall that $g(\lambda; x)$ is a measurable and bounded function and $\lim_{\lambda\to\infty} g(\lambda; x) = 0$. Then,

$$\pi_\lambda(x) = \pi_+(x) + \frac{1}{\lambda} g(\lambda; x).$$

Thus, this proves Lemma 4.2.

Now that we get the asymptotic relation for the stationary distribution $\pi_\lambda(x)$, let us focus on obtaining the asymptotic expansion for the characteristic function of the residual waiting time. To do so, give the following proposition.

Proposition 4.5. Suppose that the measurable function $g: R^+ \to R$ is a bounded function and, $g(0) = 0$ and $\lim_{x\to\infty} g(x) = 0$ are satisfied. It is then possible to write the following relation:

$$\lim_{T\to\infty} \int_{x=0}^{T} e^{i\alpha x} dg(x) = 0, \qquad \alpha > 0.$$

Proof. Since $\lim_{x\to\infty} g(x) = 0$, for sufficiently large T, $|g(T)| \to 0$. Therefore,

$$\left|\int_0^T e^{i\alpha x} dg(x)\right| \le \left|\int_0^T dg(x)\right| = |g(T) - g(0)| = |g(T)|.$$

For all $\varepsilon > 0$, it is possible to find such T_0 that $|g(T)| < \varepsilon$ is satisfied for all $T \geq T_0$. Then,

$$\left|\int_0^T e^{i\alpha x} dg(x)\right| < \varepsilon$$

is obtained. Therefore, $\lim_{T\to\infty} \int_0^T e^{i\alpha x} d_x g(x) = 0$ is hold.

To complete the secondary purpose, the two-term asymptotic expansion for the characteristic function $\left(\varphi_\zeta(\alpha)\right)$ of the random variable ζ can be given in the following lemma by means of Lemma 4.2 and Proposition 4.5.

Lemma 4.3. Assume that Proposition 3.1's requirements are met. Then, the asymptotic expansion for the characteristic function of the positive-valued random variable ζ which has the distribution $\pi_\lambda(x)$, can be expressed as follows:

$$\varphi_\zeta(\alpha) \equiv E\left(e^{i\alpha\zeta}\right) = \hat{\varphi}_+(\alpha) + o\left(\frac{1}{\lambda}\right). \tag{3}$$

Here, $\hat{\varphi}_+(\alpha) = \frac{\varphi_+(\alpha)-1}{i\alpha\mu_1}$; $\varphi_+(\alpha) \equiv E\left(e^{i\alpha\chi_1^+}\right)$ and $\mu_1 = E(\chi_1^+)$.

Proof. The characteristic function $\varphi_\zeta(\alpha)$ of the random variable ζ is $\varphi_\zeta(\alpha) \equiv E\left(e^{i\alpha\zeta}\right) = \int_{x=0}^{\infty} e^{i\alpha x} d\pi_\lambda(x)$ by definition. By Lemma 4.2, the following asymptotic expansion for the distribution function $\pi_\lambda(x)$ can be given as follows, when $\lambda \to \infty$:

$$\pi_\lambda(x) = \pi_+(x) + o\left(\frac{1}{\lambda}\right). \tag{4}$$

Here, $\pi_+(x) = \frac{1}{\mu_1}\int_0^x \left(1 - F_+(u)\right) du$; $F_+(x) \equiv P\{\chi_1^+ \le x\}$ and $\mu_1 \equiv E(\chi_1^+)$. By considering the asymptotic expansion (4) in (3) the following equality is hold, when $\lambda \to \infty$:

$$\varphi_\zeta(\alpha) = \int_{x=0}^{\infty} e^{i\alpha x} d\left[\pi_+(x) + \frac{1}{\lambda} g(\lambda z; x)\right] = \int_{x=0}^{\infty} e^{i\alpha x} d\pi_+(x) + \frac{1}{\lambda}\int_{x=0}^{\infty} e^{i\alpha x} d_x g(\lambda z; x).$$

Proposition 4.5 allows us to write the asymptotic expansion as follows when $\lambda \to \infty$:

$$\varphi_\zeta(\alpha) \equiv \int_{x=0}^{\infty} e^{i\alpha x} d\pi_+(x) + o\left(\frac{1}{\lambda}\right) = \frac{1}{\mu_1}\int_{x=0}^{\infty} e^{i\alpha x}\big(1 - F_+(x)\big)dx + o\left(\frac{1}{\lambda}\right).$$

Denoting $u \equiv 1 - F_+(x)$ and $v \equiv e^{i\alpha x}$, we get $\varphi_\zeta(\alpha) = \frac{\varphi_+(\alpha)-1}{i\alpha\mu_1} + o\left(\frac{1}{\lambda}\right)$. Here, $\varphi_+(\alpha) \equiv E\left(e^{i\alpha\chi_1^+}\right)$. The characteristic function $\left(\hat{\varphi}_+(\alpha)\right)$ of the residual waiting time of the renewal process generated by the ladder heights (χ_n^+) is $\hat{\varphi}_+(\alpha) \equiv \frac{\varphi_+(\alpha)-1}{i\alpha\mu_1}$ [15-16]. Hence, $\varphi_\zeta(\alpha) = \hat{\varphi}_+(\alpha) + o\left(\frac{1}{\lambda}\right)$ is obtained.

As intended, the required preliminary results are obtained to have the inequality that we are aiming for the rate of the weak convergence of the random walk with generalized reflecting barrier $X(t)$. Therefore, we can give the main result in the next section.

5. INEQUALITY FOR RATE OF WEAK CONVERGENCE OF STATIONARY DISTRIBUTION FOR PROCESS $Y_\lambda(t)$

One of the most important topics in probability theory is the convergence rate of limit theorems, as it influences the accuracy and practical utility of these theorems. Subsequently, in this section, the primary purpose of the study, which is attaining an inequality for the rate of weak convergence for the random walk with generalized reflecting barrier, is presented. Before moving on to the main purpose, let us include the following alternative definition of $N_1(\lambda z)$ and $S_{N_1(\lambda z)}$ according to Dynkin principle by using ladder variables, $N_1(\lambda z) = \sum_{i=1}^{H(\lambda z)} \nu_i^+$; $S_{N_1(\lambda z)} = \sum_{i=1}^{H(\lambda z)} \chi_i^+$ where $H(x) = \min\{n \geq 1: \sum_{i=1}^{n} \chi_i^+ > x\}$ (see [16]). With the help of the results acquired in Section 4, the main purpose can be given by the following theorem.

Theorem 5.1. Suppose that the conditions of Proposition 3.1 are satisfied and $\mu_3 \equiv E(\chi_1^{+3}) < \infty$. Then, the inequality that follows can be expressed, when $\lambda \to \infty$:

$$|Q_Y(x) - R(x)| \leq \frac{2m_1\mu_2\big(1 - R(x)\big) + 2m_2\mu_1\big(1 - \pi_+(x)\big)}{\lambda m_1 \mu_2}.$$

Here, $Q_Y(x) \equiv \lim_{t\to\infty} P\{Y_\lambda(t) \leq x\}$, $R(x) \equiv \frac{2}{\mu_2}\int_0^x \{\int_v^\infty \big(1 - F_+(u)\big)du\}dv$, $\pi_+(x) = \frac{1}{\mu_1}\int_0^x \big(1 - F_+(t)\big)\,dt$; $F_+(x) \equiv P\{\chi_1^+ \leq x\}$, $\mu_k = E(\chi_1^{+k})$; $m_k = E(\eta_1^k)$, $k = 1,2$.

Proof. The characteristic function of the process $Y_\lambda(t) = \frac{X(t)}{\lambda}, \lambda > 0$ by definition, can be written by means of the characteristic function of the process $X(t)$ as $\varphi_Y(\alpha) = \varphi_X\left(\frac{\alpha}{\lambda}\right)$. According to [2],

$$\varphi_X(\alpha) = \frac{1}{E\{N_1(\lambda\zeta_1)\}}\int_0^\infty e^{i\alpha\lambda z}\frac{\varphi_{S_{N_1(\lambda z)}}(-\alpha) - 1}{\varphi_\eta(-\alpha) - 1}d\pi_\lambda(z), \alpha \neq 0.$$

Therefore, the equality $\varphi_Y(\alpha) = \frac{1}{I_2(\lambda)}\int_0^\infty I_1(\lambda, z)d\pi_\lambda(z)$ can be written. Here (see [2]),

$$I_1(\lambda, z) \equiv e^{i\alpha z}\left[\varphi_{S_{N_1(\lambda z)}}\left(-\frac{\alpha}{\lambda}\right) - 1\right]; \quad I_2(\lambda) \equiv E\big(N_1(\lambda\zeta)\big)\left[\varphi_\eta\left(-\frac{\alpha}{\lambda}\right) - 1\right]. \tag{5}$$

Include $\hat{S}_{N_1(\lambda z)} \equiv S_{N_1(\lambda z)} - \lambda z$ for the shortness. Then,

$$\varphi_{S_{N_1(\lambda z)}}\left(-\frac{\alpha}{\lambda}\right) \equiv E\left(\exp\left(-i\frac{\alpha}{\lambda}\, S_{N_1(\lambda z)}\right)\right) = e^{-i\alpha z}E\left(\exp\left(-i\frac{\alpha}{\lambda}\, \hat{S}_{N_1(\lambda z)}\right)\right) \tag{6}$$

is obtained. Therefore, when $\lambda \to \infty$:

$$E\left(\exp\left(-i\frac{\alpha}{\lambda}\hat{S}_{N_1(\lambda z)}\right)\right) = 1 - \frac{i\alpha}{\lambda}E\left(\hat{S}_{N_1(\lambda z)}\right) + o\left(\frac{1}{\lambda}\right) \tag{7}$$

is hold. The asymptotic expansion for $E\left(S_{N_1(\lambda z)}\right)$ that follows can be expressed:

$$E\left(S_{N_1(\lambda z)}\right) = E\big(H(\lambda z)\big)\mu_1 = \lambda z + \hat{\mu}_1 + o(1)$$

where $\hat{\mu}_1 = \frac{\mu_2}{2\mu_1}$. Note that $\hat{\mu}_1 \equiv E(\hat{\chi}_1^+)$ is the expected value of the residual waiting time $(\hat{\chi}_1^+)$ generated by the ladder heights $(\chi_i^+, i = 1,2,\dots)$. Hence,

$$E\left(\hat{S}_{N_1(\lambda z)}\right) = \hat{\mu}_1 + o(1), \hat{\mu}_1 = \mu_2/(2\mu_1). \tag{8}$$

By substituting Equation (8) into Equation (7), the following asymptotic expansion is derived:

$$E\left\{\exp\left(-i\frac{\alpha}{\lambda}\hat{S}_{N_1(\lambda z)}\right)\right\} = 1 - \frac{i\alpha}{\lambda}\hat{\mu}_1 + o\left(\frac{1}{\lambda}\right). \tag{9}$$

The following asymptotic expansion is obtained for $\varphi_{S_{N_1(\lambda z)}}\left(-\frac{\alpha}{\lambda}\right)$ by substituting Equation (9) into Equation (6):

$$\varphi_{S_{N_1(\lambda z)}}\left(-\frac{\alpha}{\lambda}\right) - 1 = e^{-i\alpha z} - 1 - e^{-i\alpha z}\frac{i\alpha}{\lambda}\hat{\mu}_1 + o\left(\frac{1}{\lambda}\right) \tag{10}$$

is obtained. By substituting Equation (10) in the definition of $I_1(\lambda, z)$ in Equation (5),

$$I_1(\lambda, z) = e^{i\alpha z}\left\{\varphi_{S_{N_1(\lambda z)}}\left(-\frac{\alpha}{\lambda}\right) - 1\right\} = 1 - e^{i\alpha z} - \frac{i\alpha}{\lambda}\hat{\mu}_1 + o\left(\frac{1}{\lambda}\right) \tag{11}$$

$$I_1(\lambda, z) = e^{i\alpha z}\left\{\varphi_{S_{N_1(\lambda z)}}\left(-\frac{\alpha}{\lambda}\right) - 1\right\} = 1 - e^{i\alpha z} - \frac{i\alpha}{\lambda}\hat{\mu}_1 + o\left(\frac{1}{\lambda}\right) \tag{11}$$

can be written. Now, take the integral of the following expansion (11) with respect to $\pi_\lambda(z)$ in the interval $[0, \infty]$:

$$\int_0^\infty I_1(\lambda, z)d\pi_\lambda(z) = 1 - \varphi_\zeta(\alpha) - \frac{i\alpha}{\lambda}\hat{\mu}_1 + \frac{1}{\lambda}\int_0^\infty g(\lambda z)d\pi_\lambda(z).$$

Here, $\int_0^\infty g(\lambda z)d\pi_\lambda(z) \to 0$ is hold as $\lambda \to \infty$ (see [2], Prop 3.1., p.254), then,

$$\int_0^\infty I_1(\lambda, z)d\pi_\lambda(z) = 1 - \varphi_\zeta(\alpha) - \frac{i\alpha}{\lambda}\hat{\mu}_1 + o\left(\frac{1}{\lambda}\right), \tag{12}$$

where $\varphi_\zeta(\alpha) \equiv E\left(e^{i\alpha\zeta}\right)$; the random variable ζ has the distribution function $P\{\zeta \le x\} = \pi_\lambda(x)$ and $\hat{\mu}_1 = \frac{\mu_2}{2\mu_1}$. According to Lemma 4.3, Equation (12) can be given as follows:

$$\int_0^\infty I_1(\lambda, z)d\pi_\lambda(z) = 1 - \hat{\varphi}_+(\alpha) - \frac{i\alpha}{\lambda}\hat{\mu}_1 + o\left(\frac{1}{\lambda}\right). \tag{13}$$

Here $\hat{\varphi}_+(\alpha) \equiv \varphi_1^+(\alpha) = \frac{\varphi_+(\alpha)-1}{i\alpha\mu_1}$; $\varphi_+(\alpha) \equiv E\left(e^{i\alpha\chi_1^+}\right)$ and $\hat{\mu}_1 \equiv E(\hat{\chi}_1^+)$. Now, obtain two-term asymptotic expansion for $I_2(\lambda)$. By definition, $I_2(\lambda)$ is as follows:

$$I_2(\lambda) \equiv E\big(N_1(\lambda\zeta)\big)\left[\varphi_\eta\left(-\frac{\alpha}{\lambda}\right) - 1\right].$$

The expression $\varphi_\eta\left(-\frac{\alpha}{\lambda}\right)$ can be written as follows:

$$\varphi_\eta\left(-\frac{\alpha}{\lambda}\right) - 1 = -\frac{i\alpha}{\lambda} m_1\left\{1 - \frac{i\alpha}{\lambda}\frac{m_2}{2m_1} + o\left(\frac{1}{\lambda}\right)\right\}. \tag{14}$$

Here $m_1 \equiv E(\eta_1), m_2 \equiv E(\eta_1^2)$. According to Wald identity, the following equality can be written for $E\big(N_1(\lambda\zeta)\big)$:

$$E\big(N_1(\lambda\zeta)\big) = E\big(H(\lambda\zeta)\big)E(\nu_1^+). \tag{15}$$

Here, $E\big(H(\lambda\zeta)\big)$ is a renewal function which is generated by the ladder heights $(\{\chi_n^+\}, n \geq 1)$ and the refined renewal theorem allows it to be expressed as the expansion below:

$$E\big(H(\lambda\zeta)\big) = \frac{\lambda\beta_1}{\mu_1} + \frac{\mu_2}{2\mu_1^2} + o(1).$$

Here, $\beta_1 \equiv E(\zeta) = \int_0^\infty z d\pi_\lambda(z)$ and $\lim_{\lambda\to\infty}\beta_1 = \hat{\mu}_1 = \frac{\mu_2}{2\mu_1}$ holds (see [2]). Thus, it is possible to write the expansion that follows:

$$E\big(H(\lambda\zeta)\big) = \frac{\lambda\frac{\mu_2}{2\mu_1}}{\mu_1} + \frac{\mu_2}{2\mu_1^2} + o(1) = \lambda\frac{\mu_2}{2\mu_1^2}\left\{1 + \frac{1}{\lambda} + o\left(\frac{1}{\lambda}\right)\right\}.$$

On the other hand, when $m_1 \neq 0$, $E(\nu_1^+) = \mu_1/m_1$ is hold [1]. By substituting the expansion (14) and Equation (15) into Equation (13), the following asymptotic expansion

$$E\big(N_1(\lambda\zeta)\big) = E\big(H(\lambda\zeta)\big)E(\nu_1^+) = \lambda\frac{\mu_2}{2m_1\mu_1}\left\{1 + \frac{1}{\lambda} + o\left(\frac{1}{\lambda}\right)\right\} \tag{16}$$

can be written. By substituting the expansions Equation (14) and Equation (16) in the definition $I_2(\lambda)$, the asymptotic expansion that follows is attained, when $\lambda \to \infty$:

$$I_2(\lambda) = E\big(N_1(\lambda\zeta)\big)\left[\varphi_\eta\left(-\frac{\alpha}{\lambda}\right) - 1\right] = (-i\alpha\hat{\mu}_1)\left\{1 + \frac{1 - i\alpha\widehat{m}_1}{\lambda} + o\left(\frac{1}{\lambda}\right)\right\}.$$

Therefore, the following can be written:

$$\frac{1}{I_2(\lambda)} = -\frac{1}{i\alpha\hat{\mu}_1}\left\{1 - \frac{1 - i\alpha\widehat{m}_1}{\lambda} + o\left(\frac{1}{\lambda}\right)\right\}. \tag{17}$$

Here, $\hat{\mu}_1 = \frac{\mu_2}{2\mu_1}$; $\widehat{m}_1 = \frac{m_2}{2m_1}$. Substitute the expansions Equation (13) and Equation (17) in the definition of $\varphi_Y(\alpha)$, i.e., $\varphi_Y(\alpha) = \frac{1}{I_2(\lambda)}\int_0^\infty I_1(\lambda, z)d\pi_\lambda(z)$, then $\varphi_Y(\alpha) = \varphi_2^+(\alpha) + \varphi_2^+(\alpha)\frac{(i\alpha\widehat{m}_1 - 1)}{\lambda} + \frac{1}{\lambda} + o\left(\frac{1}{\lambda}\right)$. Then, the following calculations are hold:

$$\varphi_Y(\alpha) - \varphi_2^+(\alpha) = \frac{1}{\lambda}\left\{i\alpha\varphi_2^+(\alpha)\widehat{m}_1 - i\alpha\varphi_3^+(\alpha)\hat{\hat{\mu}}_1\right\} + o\left(\frac{1}{\lambda}\right). \tag{18}$$

Here, $\varphi_3^+(\alpha) \equiv \frac{\varphi_2^+(\alpha) - 1}{i\alpha\hat{\hat{\mu}}_1}$; $\varphi_2^+(\alpha) \equiv \frac{\varphi_1^+(\alpha) - 1}{i\alpha\hat{\mu}_1} =$; $\varphi_1^+(\alpha) \equiv \frac{\varphi_+(\alpha) - 1}{i\alpha\mu_1}$; $\hat{\hat{\mu}}_1 \equiv E\big(\hat{\chi}_1^+\big) = \frac{\mu_3}{3\mu_2}$; $\widehat{m}_1 = \frac{m_2}{2m_1}$; $\varphi_+(\alpha) \equiv E\big(e^{i\alpha\chi_1^+}\big)$; $\mu_k = E\big(\chi_1^{+k}\big)$; $m_k \equiv E\big(\eta_1^k\big)$; $k = 1,2,3$.

Hence, the following inequality is hold by using Equation (18):

$$Q_Y(x) - R(x) = \frac{\hat{\hat{\mu}}_1}{\lambda}\hat{R}'(x) - \frac{\widehat{m}_1}{\lambda}R'(x) + o\left(\frac{1}{\lambda}\right). \tag{19}$$

Here, $\hat{R}(x) \equiv \frac{1}{\hat{\hat{\mu}}_1}\int_0^x \left(1 - R(t)\right)dt = \frac{3\mu_2}{\mu_3}\int_0^x \left(1 - R(t)\right)dt\,;\ R(x) \equiv \frac{1}{\hat{\mu}_1}\int_0^x \left(1 - \pi_+(v)\right)dv = \frac{2}{\mu_2}\int_0^x \{\int_v^\infty (1 - F_+(u))du\}dv\,;\ \hat{\hat{\mu}}_1 \equiv E\left(\hat{\hat{\chi}}_1^+\right) = \frac{\mu_3}{3\mu_2};\ \widehat{m}_1 = \frac{m_2}{2m_1}$.

The expressions $R'(x)$ and $\hat{R}'(x)$ can be obtained as follows by utilizing the initial ladder height (χ_1^+):

$$R'(x) = \frac{1}{\hat{\mu}_1}\left(1 - \pi_+(x)\right) = \frac{2\mu_1}{\mu_2}\left(1 - \pi_+(x)\right), \tag{20}$$

$$\hat{R}'(x) = \left\{\frac{1}{\hat{\hat{\mu}}_1}\int_0^x \left(1 - R(t)\right)dt\right\}_x' = \frac{3\mu_2}{\mu_3}\left(1 - R(x)\right). \tag{21}$$

By substituting the expression (20) and Equation (21) into Equation (19), the following expansion is hold:

$$Q_Y(x) - R(x) = \frac{m_1\mu_2\left(1 - R(x)\right) - m_2\mu_1\left(1 - \pi_+(x)\right)}{\lambda m_1\mu_2} + o\left(\frac{1}{\lambda}\right). \tag{22}$$

Recall that, $R(x) = \frac{1}{\hat{\mu}_1}\int_0^x \left(1 - \pi_+(u)\right)du\,;\ \pi_+(x) = \frac{1}{\mu_1}\int_0^x \left(1 - F_+(u)\right)du$. Here $F_+(x) \equiv P\{\chi_1^+ \le x\}$. Hence, $\int_0^x \left(1 - F_+(u)\right)du = \int_0^\infty \left(1 - F_+(u)\right)du - \int_x^\infty \left(1 - F_+(u)\right)du = \mu_1 - \bar{a}_1(x)$ is obtained. Therefore, for sufficiently large values of λ,

$$|Q_Y(x) - R(x)| \le \frac{2m_1\mu_2\left(1 - R(x)\right) + 2m_2\mu_1\left(1 - \pi_+(x)\right)}{\lambda m_1\mu_2}.$$

Thus, the theorem is proved.

The obtained inequality gives an upper bound for the rate of the weak convergence of the considered process. This result is essentially proportional to the tail of the distribution of ladder heights and the tail of the distribution of the residual waiting time generated by ladder heights. Moreover, the first two moments of the first jump and the first ladder height are also used in the expression of the obtained inequality. Further discussion and interpretation are made in the next section.

6. CONCLUSION

The literature has examined weak convergence, a crucial probabilistic aspect of stochastic processes [1-2,7-13]. As theoretical results are frequently complex [1], asymptotic methods are considered [2,10], allowing for the practical application of the results. Because it affects the accuracy and practical usefulness of weak convergence theorems, the rate of weak convergence is also significant [1,14]. The study [2] uses asymptotic methods to obtain the weak convergence for stationary distribution of the random walk with reflecting barrier, but it does not investigate the rate of the weak convergence.

In this study, the asymptotic rate of the weak convergence for the stationary distribution of a random walk with generalized reflecting barrier is investigated and an inequality is obtained. The acquired inequality indicates that the rate of convergence is related with the tail of the cdf of residual waiting time of ladder heights $(\chi_n^+, n = 1,2,\dots)$ of a random walk. The presented inequality is, while approximate, practical for implementation due to its concise form as follows:

$$|Q_Y(x) - R(x)| \leq \frac{2m_1\mu_2\big(1 - R(x)\big) + 2m_2\mu_1\big(1 - \pi_+(x)\big)}{\lambda m_1\mu_2}.$$

Moreover, this inequality gives a bound on maximal error of the weak convergence. Similar asymptotic results can be investigated for future studies by considering random walk with two reflected barriers which is frequently encountered in physics. The results under any distribution assumption allow for the evaluation of various distributions in a variety of dimensions, including inter-arrival times and demands; additionally, they allow for the evaluation of the ladder height and residual waiting time that arise from demands. When the results are considered in relation to the specific problem, they become very intriguing. As an upcoming research project, the obtained result can be applied to a number of real-world problems, from inventory to quantum physics.

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

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APPENDIX 1

Proof (Lemma 4.1.). One-dimensional distribution of the residual waiting time $H(t;x) \equiv P\{W(t) \leq x\}$ can be written as $H(t;x) = \int_{0^-}^{t} [F_+(t+x-v) - F_+(t-v)] dU_+(v)$ (see [1], p.369]). Here, $U_+(t)$ respresents the renewal function which is generated by ladder heights $\{\chi_n^+\}, n = 1,2,\ldots$, i.e., $U_+(t) = \sum_{n=0}^{\infty} F_+^{*(n)}(t)$. Moreover $F_+^{*(n)}(t)$ is the n^th^ convolution that is $F^{*(n)}(t) = \int_0^t F_+^{*(n-1)}(t-v) dF_+(v)$. The aim is to prove that $\lim_{t\to\infty} t[H(t,x) - \pi_+(x)] = 0$. Note that $\tilde{N}(s) \equiv L_s(N(t))$ denotes Laplace transform and $N^*(s)$ represents Laplace – Stiltijes Transform of the function $N(t)$ in the rest of the paper. For shortness, denote $K(t;x) \equiv t[H(t;x) - \pi_+(x)]$. According to Tauber-Abel theorem, $\lim_{t\to\infty} K(t;x) = \lim_{s\to 0} s\tilde{K}(s;x)$ can be written. Let us examine the function $\tilde{K}(s;x)$. By using property of Laplace transformation, $\tilde{K}(s;x)$ can be written as follows:

$$\tilde{K}(s;x) = -\left[\frac{\partial}{\partial s}\tilde{H}(s;x) + \frac{1}{s^2}\pi_+(x)\right]. \tag{1}$$

Here $\tilde{H}(s;x) \equiv L_s(H(t;x))$. For shortness, denote that $G(t;x) \equiv \bar{F}_+(t) - \bar{F}_+(t+x)$ where $\bar{F}_+(t) = 1 - F_+(t)$ and $F_+(t) \equiv P\{\chi_1^+ \leq t\}$. $H(t;x) = G(t;x) * U_+(t)$ can be written, therefore, by applying Laplace transform to the obtained convolution,

$$\tilde{H}(s;x) = \tilde{G}(s;x) U_+^*(s) \tag{2}$$

is obtained, where $\tilde{G}(s;x) \equiv \int_0^{\infty} e^{-st} G(t) dt$ and $U_+^*(s) = \int_{-0}^{\infty} e^{-st} dU_+(t)$. By means of linearity property of Laplace transformation, $\tilde{G}(s;x) = \tilde{\bar{F}}_+(s) - L_s\{\bar{F}_+(t+x)\}$ is hold. By including $\varphi_+(s) \equiv E(e^{-s\chi_1^+}) = \int_0^{\infty} e^{-st} dF_+(t)$,

$$\tilde{\tilde{F}}_+(s) = \frac{1-\varphi_+(s)}{s} \tag{3}$$

is obtained. Similarly, $L_s\{\bar{F}_+(t+x)\}$ can be written as follows:

$$L_s\{\bar{F}_+(t+x)\} = e^{sx}\left[\tilde{\tilde{F}}_+(s) - M(s;x)\right] = e^{sx}\left[\frac{1-\varphi_+(s)}{s} - M(s;x)\right]. \tag{4}$$

Here, $M(s;x) \equiv \int_{v=0}^{x} e^{-sv}\bar{F}_+(v)dv$. By means of Equation (3) and Equation (4), the function $\tilde{G}(s;x)$ can be written as follows:

$$\tilde{G}(s;x) = \frac{[1-\varphi_+(s)][1-e^{sx}]}{s} + e^{sx}M(s;x). \tag{5}$$

Moreover, $U_+^*(s)$ is obtained as follows:

$$U_+^*(s) = s\tilde{U}_+(s) - U_+(-0) = s\tilde{U}_+(s) = \frac{1}{1-\varphi_+(s)}. \tag{6}$$

Therefore, considering Equation (5) and Equation (6) in Equation (2), $\tilde{H}(s;x)$ is obtained as follows:

$$\tilde{H}(s;x) = \tilde{G}(s;x)U_+^*(s) = \frac{1-e^{sx}}{s} + \frac{e^{sx}}{1-\varphi_+(s)}M(s;x). \tag{7}$$

Let us examine the derivative of Equation (7) with respect to s. For the convenience, include

$$J_1(s;x) = \frac{1-e^{sx}}{s}; \qquad J_2(s;x) = \frac{e^{sx}}{1-\varphi_+(s)}M(s;x)$$

where $\varphi_+(s) \equiv E\left(e^{-sX_1^+}\right)$. Then, $\tilde{H}(s;x)$ can be expressed as follows:

$$\tilde{H}(s;x) = J_1(s;x) + J_2(s;x). \tag{8}$$

By taking the first derivative of Equation (8),

$$\frac{\partial}{\partial s}\tilde{H}(s;x) = \frac{\partial J_1(s;x)}{\partial s} + \frac{\partial J_2(s;x)}{\partial s} \tag{9}$$

and they are stated as follows:

$$\frac{\partial J_1(s;x)}{\partial s} = \frac{e^{sx}(1-sx)-1}{s^2}; \tag{10}$$

$$\frac{\partial J_2(s;x)}{\partial s} = \frac{xe^{sx}M(s;x)}{1-\varphi_+(s)} + \frac{e^{sx}\varphi_+'(s)M(s;x)}{\left(1-\varphi_+(s)\right)^2} + \frac{e^{sx}M'(s;x)}{1-\varphi_+(s)}. \tag{11}$$

Here, $\varphi_+'(s) = \frac{d}{ds}\varphi_+(s)$ and $M'(s;x) = \frac{d}{ds}M(s;x)$. Primarily, examine $\frac{\partial J_1(s;x)}{\partial s}$. By using Taylor expansion, the following expansion is hold when $s \to 0$:

$$e^{sx}(1-sx) - 1 = -\frac{s^2x^2}{2}\left\{1 + \frac{2}{3}sx + o(s)\right\}. \tag{12}$$

Substituting the expansion in Equation (12) into Equation (10), the following expansion is obtained for $\frac{\partial J_1(s;x)}{\partial s}$ when $s \to 0$:

$$\frac{\partial J_1(s;x)}{\partial s}=\frac{1}{s^2}\left\{-\frac{s^2x^2}{2}\left\{1+\frac{2}{3}sx+o(s)\right\}\right\}=-\frac{x^2}{2}+o(1). \tag{13}$$

Now, let us examine the expression $\frac{\partial J_2(s;x)}{\partial s}$. To make the notation more readable, let us express $\frac{\partial J_2(s;x)}{\partial s}$ given in Equation (11) as follows alternatively:

$$\frac{\partial J_2(s;x)}{\partial s}=R_1(s;x)+R_2(s;x)+R_3(s;x). \tag{14}$$

Here, $R_1(s;x)=\frac{xe^{sx}M(s;x)}{1-\varphi_+(s)}$; $R_2(s,x)=\frac{e^{sx}\varphi_+'(s)M(s;x)}{(1-\varphi_+(s))^2}$; $R_3(s,x)=\frac{e^{sx}\frac{\partial M(s;x)}{\partial s}}{1-\varphi_+(s)}$.

Now, let us focus on $R_i(s;x)$, $i=1,2,3$. To examine the asymptotic expansion of the function $R_1(s;x)$ when $s\to 0$, by using Taylor expansion, since $\mu_3<\infty$, the expansions that follows can be written:

$$e^{sx}=1+sx+\frac{s^2x^2}{2}+o(s^2); \tag{15}$$

$$\frac{1}{1-\varphi_+(s)}=\frac{1}{s\mu_1}\left\{1+s\hat{\mu}_1-s^2\left(\frac{\hat{\mu}_2}{2}-\hat{\mu}_1^2\right)+o(s^2)\right\}; \tag{16}$$

$$M(s;x)=a_1(x)\left\{1-sa_{21}(x)+\frac{s^2}{2}a_{31}(x)+o(s^2)\right\}. \tag{17}$$

Here $a_n(x)=\int_{v=0}^{x}v^{n-1}\bar{F}_+(v)dv$, $n=1,2$; $a_{n1}(x)=\frac{a_n(x)}{a_1(x)}$, $n=2,3$; $\hat{\mu}_k=\frac{\mu_{k+1}}{(k+1)\mu_1}$, $k=1,2$; $\mu_k\equiv E(\chi_1^{+k}), k=1,2$. By using expansions Equation (15) - Equation (17), $R_1(s;x)$ can be given as follows:

$$R_1(s;x)=\frac{xe^{sx}M(s;x)}{1-\varphi_+(s)}=\frac{x\pi_+(x)}{s}+C_{21}(x)+o(1). \tag{18}$$

Here, $C_{21}(x)=\frac{xa_1(x)}{\mu_1}[\hat{\mu}_1+x-a_{21}(x)]$. Similarly, since $\mu_3<\infty$ is hold, Taylor expansion can be written as follows, when $s\to 0$:

$$\varphi_+'(s)=-\mu_1\left\{1-2\hat{\mu}_1s+\frac{3\hat{\mu}_2}{2}s^2+o(s^2)\right\}. \tag{19}$$

By using Equation (19),

$$\frac{\varphi_+'(s)}{(1-\varphi_+(s))^2}=-\frac{1}{s^2\mu_1}\left\{1+s^2\left(\frac{\hat{\mu}_2}{2}-\hat{\mu}_1^2\right)+o(s^2)\right\} \tag{20}$$

is obtained. By the help of Equation (15), Equation (17) and Equation (20), $R_2(s;x)$ can be recorded as follows when $s\to 0$:

$$R_2(s;x)=-\frac{\pi_+(x)}{s^2}-\frac{\pi_+(x)}{s}[x-a_{21}(x)]+C_{22}(x)+o(1). \tag{21}$$

Here $C_{22}(x)=-\frac{a_1(x)}{\mu_1}\left[\frac{x^2}{2}+d_1-xa_{21}(x)+a_{31}(x)\right]$; $d_1=\frac{\hat{\mu}_2}{2}-\hat{\mu}_1^2$. Additionally, give the following Taylor expansion for $R_3(s,x)$ when $s\to 0$ by using Equation (17):

$$\frac{\partial M(s;x)}{\partial s}=-a_2(x)\{1-sa_{32}(x)+o(s)\}. \tag{22}$$

Then, $R_3(s;x)$ can be written as follows by the help of the expansions (15), (16) and (22):

$$R_3(s;x) = -\frac{\pi_+(x)}{s\mu_1}a_{21}(x) + C_{23}(x) + o(1). \tag{23}$$

Here, $C_{23}(x) = -\frac{a_2(x)}{\mu_1}[\hat{\mu}_1 + x - a_{32}(x)]$. Substituting the expansions (18), (21) and (23) into (13), the asymptotic expansion that follows can be attain for $J_2'(s;x)$ when $s \to 0$:

$$\frac{\partial J_2(s;x)}{\partial s} = R_1(s;x) + R_2(s;x) + R_3(s;x) = -\frac{\pi_+(x)}{s^2} + C_2(x) + o(1). \tag{24}$$

Here, $C_2(x) = C_{21}(x) + C_{22}(x) + C_{23}(x)$. By considering Equation (13) and Equation (24) into Equation (8), the following asymptotic expansion is hold, when $s \to 0$:

$$\frac{\partial}{\partial s}\tilde{H}(s;x) = J_1'(s;x) + J_2'(s;x) = -\frac{\pi_+(x)}{s^2} + C(x) + o(1). \tag{25}$$

Here, $C(x) = -\frac{x^2}{2} + C_2(x)$. By substituting Equation (25) in Equation (1),

$$\tilde{K}(s,x) = -\left[\frac{\partial}{\partial s}\tilde{H}(s,x) + \frac{1}{s^2}\pi_+(x)\right] = -C(x) + o(1).$$

Here, $C(x) = -\frac{x^2}{2} + C_2(x)$. According to Proposition 4.1, $\bar{a}_n(x) < \infty, n = 1,2,3$ is satisfied for all $x \geq 0$. Considering that $\pi_+(x) = \frac{1}{\mu_1}a_1(x)$ is hold, by substituting $a_n(x) = \frac{\mu_n}{n} - \bar{a}_n(x)$ in $C_2(x)$:

$$C_2(x) = -\frac{\mu_3}{6\mu_1} + \frac{x^2}{2} - \frac{x^2\bar{a}_1(x)}{2\mu_1} + \frac{x\bar{a}_2(x)}{\mu_1} + \bar{a}_2(x)C_F - x\bar{a}_1(x)C_F + \frac{\bar{a}_1(x)d_1}{\mu_1} \tag{26}$$

is obtained. Considering Equation (26) in definition of $C(x)$,

$$C(x) = -\frac{\mu_3}{6\mu_1} - \frac{x^2\bar{a}_1(x)}{2\mu_1} + \frac{x\bar{a}_2(x)}{\mu_1} + \bar{a}_2(x)C_F - x\bar{a}_1(x)C_F - \frac{\bar{a}_1(x)d_1}{\mu_1}$$

is hold. According to Corollary 4.1,

$$x^2\bar{a}_1(x) \leq \frac{\mu_3}{3};\;\; x\bar{a}_2(x) \leq \frac{\mu_3}{3};\;\; \bar{a}_2(x) \leq \frac{\mu_2}{2};\;\; x\bar{a}_1(x) \leq \frac{\mu_2}{2};\;\; \bar{a}_1(x) \leq \mu_1$$

is known. Then, the inequality that follows can be expressed for $|C(x)|$:

$$|C(x)| = \left|\frac{\mu_3}{6\mu_1}\right| + \left|\frac{x^2\bar{a}_1(x)}{2\mu_1}\right| + \left|\frac{x\bar{a}_2(x)}{\mu_1}\right| + |\bar{a}_2(x)C_F| + |x\bar{a}_1(x)C_F| + \left|\frac{\bar{a}_1(x)d_1}{\mu_1}\right| \leq \frac{5\mu_3}{6\mu_1} + \frac{3\mu_2^2}{4\mu_1^2}.$$

Since $\mu_3 < \infty$, $|C(x)| < \infty$ is hold. Proposition 4.4 suggests that

$$\lim_{x\to\infty} x^2\bar{a}_1(x) = 0;\;\; \lim_{x\to\infty} x\bar{a}_1(x) = 0 \text{ and } \lim_{x\to\infty} x\bar{a}_2(x) = 0.$$

Then for all $x \in R$,

$$\lim_{x\to\infty} C(x) = -\frac{\mu_3}{6\mu_1}$$

is obtained. $|C(x)| < \infty$ is satisfied for all $x \in R$. By using Equation (25), $\lim_{s\to 0} s\widetilde{K}(s,x) = \lim_{s\to 0} s\big(-C(x)\big) = 0$. Therefore, according to Tauber – Abel Theorem, $\lim_{s\to 0} s\widetilde{K}(s;x) = 0$ is satisfied. Then $\lim_{t\to\infty} K(t;x) = 0$ is hold. By using $K(t;x)$, $\lim_{t\to\infty} K(t;x) = \lim_{t\to\infty} t[H(t;x) - \pi_+(x)] = 0$. Thus, the asymptotic expansion that follows can be recorded as $t \to \infty$, $t[H(t;x) - \pi_+(x)] = o(1)$. Then the following expansion can be given: $H(t;x) = \pi_+(x) + o\left(\frac{1}{t}\right)$.

This completes the proof.