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New Weighted Inequalities for Functions Whose Higher-Order Partial Derivatives Are Co-Ordinated Convex

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Abstract

The purpose of this study is to establish recent inequalities based on double integrals of mappings whose higher-order partial derivatives in absolute value are convex on the co-ordinates on rectangle from the plane. Also, some special cases of results improved in this study are examined.

1. Introduction

In the past century, Many scholars have been interested in Hermite-Hadamard inequalities Hermite-Hadamard inequalities have attracted the interest of a good many researchers because of wide application fields in numerical analysis and in the theory of some special means. A large number of researchers have worked on new results related to Hermite-Hadamard inequalities for various function classes. One of them is co-ordinated convex functions, and we examine generalizations of these types results for co-ordinated convex functions in this work.

We define a bidimensional interval $\Delta =: [a_1,a_2]\times[b_1,b_2]$ in $\mathbb{R}^2$ with $a_1<a_2$ and $b_1<b_2$. If the inequality

$$\varphi\left(t\varkappa+(1-t)z, t\tau+(1-t)w\right)\leq t\varphi\left(\varkappa,\tau\right)+(1-t)\varphi\left(z,w\right)$$

holds, $\varphi:\Delta\to\mathbb{R}$ is said to be convex on $\Delta,$ for all $(\varkappa,\tau),(z,w)\in\Delta$ and $t\in[0,1]$. If the partial functions $\varphi_\tau:[a_1,a_2]\to\mathbb{R}$, $\varphi_\tau(u)=\varphi(u,\tau)$ and $\varphi_\varkappa:[b_1,b_2]\to\mathbb{R}$, $\varphi_\varkappa(v)=\varphi(\varkappa,v)$ are convex for all $\varkappa\in[a_1,a_2]$ and $\tau\in[b_1,b_2]$, then $\varphi:\Delta\to\mathbb{R}$ is said to be convex on the co-ordinates on Δ (see, [1]).

In this case, the definition of co-ordinated convex function can be given as follows.

Definition 1.1. *Let $t,s\in[0,1]$ and $(\varkappa,u),(\tau,v)\in\Delta =: [a_1,a_2]\times[b_1,b_2]$. If the inequality*

$$\varphi(t\varkappa+(1-t)\tau, su+(1-s)v)\leq ts\varphi(\varkappa,u)+s(1-t)\varphi(\tau,u)+t(1-s)\varphi(\varkappa,v)+(1-t)(1-s)\varphi(\tau,v)$$

holds, then $\varphi:\Delta\to\mathbb{R}$ will be called co-ordinated convex on Δ.

It is clearly seen that every convex mapping is co-ordinated convex. Also, A coordinated convex function that is not convex does exist (see, [1]).

Furthermore, in [1], Hermite-Hadamard type inequalities for co-ordinated convex mapping on a rectangle from the plane $\mathbb{R}^2$ were established by Dragomir.

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Theorem 1.2. *Let $\varphi:\Delta\to\mathbb{R}$ be a co-ordinated convex mapping on Δ. Then we possess the inequalities:*

$$\begin{aligned}\varphi\left(\frac{a_1+a_2}{2},\frac{b_1+b_2}{2}\right) &\leq \frac{1}{2}\left[\frac{1}{a_2-a_1}\int_{a_1}^{a_2}\varphi\left(\varkappa,\frac{b_1+b_2}{2}\right)d\varkappa+\frac{1}{b_2-b_1}\int_{b_1}^{b_2}\varphi\left(\frac{a_1+a_2}{2},\tau\right)d\tau\right] \\ &\leq \frac{1}{(a_2-a_1)(b_2-b_1)}\int_{a_1}^{a_2}\int_{b_1}^{b_2}\varphi(\varkappa,\tau)\,d\tau d\varkappa \\ &\leq \frac{1}{4}\left[\frac{1}{a_2-a_1}\int_{a_1}^{a_2}\varphi(\varkappa,b_1)\,d\varkappa+\frac{1}{a_2-a_1}\int_{a_1}^{a_2}\varphi(\varkappa,b_2)\,d\varkappa\right. \\ &+ \left.\frac{1}{b_2-b_1}\int_{b_1}^{b_2}\varphi(a_1,\tau)\,d\tau+\frac{1}{b_2-b_1}\int_{b_1}^{b_2}\varphi(a_2,\tau)\,d\tau\right] \\ &\leq \frac{\varphi(a_1,b_1)+\varphi(a_1,b_2)+\varphi(a_2,b_1)+\varphi(a_2,b_2)}{4}.\end{aligned} \tag{1.1}$$

The above inequalities are sharp.

During the past several years, some mathematicians have worked on double integral inequalities for co-ordinated convex functions. For illustrate, Hadamard's type inequalities including Riemann-Liouville fractional integrals for convex and s-convex functions on the co-ordinates by some authors in [2] and [3]. Latif and Dragomir provided recent double integral inequalities based on the left side of Hermite- Hadamard type inequality by using co-ordinated convex functions in two variables in [4]. Novel weighted integral inequalities for functions whose partial derivatives in absolute value are convex on the co- ordinates on a rectangle from the plane are attained by Erden and Sarıkaya in [5] and [6]. some researchers derived Hermite-Hadamard type results based on the deference between the middle and the rightmost terms in (1.1) by using the derivatives of co-ordinated convex functions in [7]. Also, some mathematicians found out recent inequalities for co-ordinated convex functions in [8], [9], [10], and [11]. In [12], [13], and [14], some Hermite-Hadamard type results for different classes of co-ordinated convex mappings are developed.

On the other side, a large number of researchers have focused on inequalities involving higher-order differentiable functions. To illustrate, some integral inequalities for n-times differentiable functions are established in [15], [16] and [17]. In addition, Erden et al. gave weighted inequalities for $n-$times differentiable functions in [18]. Some mathematicians also focused on double integral inequalities including higher-order partial derivatives for two-dimensional functions in [19], [20] and [21].

In this work, we first establish a novel double integral equality based on higher-order partial derivatives. After that, recent inequalities for convex functions on the co-ordinates on the rectangle from the plane are provided. What is more, we observe relations between results in this work and inequalities presented in the earlier studies.

2. Integral identity

Before we can prove our primary findings, we establish the following equality involving mappings whose partial derivatives are continuous.

Lemma 2.1. *Assuming that $\varphi:[a_1,a_2]\times[b_1,b_2]=:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ is a continuous function such that the partial derivatives $\frac{\partial^{k+l}\varphi(t,s)}{\partial t^k\partial s^l}$, $k=0,1,2,...,n-1$, $l=0,1,2,...,m-1$ exists and are continuous on Δ, and suppose that the functions $g:[a_1,a_2]\to[0,\infty)$ and $h:[b_1,b_2]\to[0,\infty)$ are integrable. Additionally, $P_{n-1}(\varkappa,t)$ and $Q_{m-1}(\tau,s)$ are defined by*

$$P_{n-1}(\varkappa,t):=\begin{cases}\frac{1}{(n-1)!}\int\limits_{a_1}^{t}(u-t)^{n-1}g(u)\,du, & a_1\leq t<\varkappa \\ \\ \frac{1}{(n-1)!}\int\limits_{a_2}^{t}(u-t)^{n-1}g(u)\,du, & \varkappa\leq t\leq a_2\end{cases}$$

and

$$Q_{m-1}(\tau,s):=\begin{cases}\frac{1}{(m-1)!}\int\limits_{b_1}^{s}(u-s)^{m-1}h(u)\,dv, & b_1\leq s<\tau \\ \\ \frac{1}{(m-1)!}\int\limits_{b_2}^{s}(u-s)^{m-1}h(u)\,dv, & \tau\leq s\leq b_2\end{cases}$$

where $n,m \in \mathbb{N}\setminus\{0\}$. Then, for all $(\varkappa,\tau) \in [a_1,a_2]\times[b_1,b_2]$, we have the identity

$$\int_{a_1}^{a_2}\int_{b_1}^{b_2} P_{n-1}(\varkappa,t)\,Q_{m-1}(\tau,s)\,\frac{\partial^{n+m}\varphi(t,s)}{\partial t^n\partial s^m}dsdt = \sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{M_k(\varkappa)}{k!}\frac{M_l(\tau)}{l!}\frac{\partial^{k+l}\varphi(\varkappa,\tau)}{\partial\varkappa^k\partial\tau^l} - \sum_{l=0}^{m-1}\frac{M_l(\tau)}{l!}\int_{a_1}^{a_2} g(t)\frac{\partial^l\varphi(t,\tau)}{\partial\tau^l}dt$$
$$- \sum_{k=0}^{n-1}\frac{M_k(\varkappa)}{k!}\int_{b_1}^{b_2} h(s)\frac{\partial^k\varphi(\varkappa,s)}{\partial\varkappa^k}ds + \int_{a_1}^{a_2}\int_{b_1}^{b_2} h(s)g(t)\varphi(t,s)dsdt \tag{2.1}$$

where $M_k(\varkappa)$ and $M_l(\tau)$ are defined by

$$M_k(\varkappa) = \int_{a_1}^{a_2}(u-\varkappa)^k g(u)\,du, \quad k = 0,1,2,...$$

$$M_l(\tau) = \int_{b_1}^{b_2}(u-\tau)^l h(u)\,du, \quad l = 0,1,2,...$$

Proof. Applying integration by parts for partial derivatives given in the lemma, via fundamental analysis operations, the desired identity (2.1) can be obtained. □

3. Some inequalities for co-ordinated convex mappings

For convenience, we give the following notations used to simplify the details of some results given in this section;

$$A_n(\varkappa) = (a_2-a_1)\frac{(\varkappa-a_1)^{n+1}}{n+1} + \frac{(a_2-\varkappa)^{n+2}-(\varkappa-a_1)^{n+2}}{n+2},$$

$$B_n(\varkappa) = (a_2-a_1)\frac{(a_2-\varkappa)^{n+1}}{n+1} + \frac{(\varkappa-a_1)^{n+2}-(a_2-\varkappa)^{n+2}}{n+2},$$

$$C_m(\tau) = (b_2-b_1)\frac{(\tau-b_1)^{m+1}}{m+1} + \frac{(b_2-\tau)^{m+2}-(\tau-b_1)^{m+2}}{m+2}$$

and

$$D_m(\tau) = (b_2-b_1)\frac{(b_2-y)^{m+1}}{m+1} + \frac{(\tau-b_1)^{m+2}-(b_2-\tau)^{m+2}}{m+2}.$$

We start with the following result.

Theorem 3.1. *Suppose that all the assumptions of Lemma 2.1 hold. If $\left|\frac{\partial^{n+m}\varphi}{\partial t^n\partial s^m}\right|$ is a convex function on the co-ordinates on Δ, then the following inequality holds:*

$$\left|\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{M_k(\varkappa)}{k!}\frac{M_l(\tau)}{l!}\frac{\partial^{k+l}\varphi(\varkappa,\tau)}{\partial\varkappa^k\partial\tau^l} - \sum_{l=0}^{m-1}\frac{M_l(\tau)}{l!}\int_{a_1}^{a_2} g(t)\frac{\partial^l\varphi(t,\tau)}{\partial\tau^l}dt\right. \tag{3.1}$$
$$\left. - \sum_{k=0}^{n-1}\frac{M_k(\varkappa)}{k!}\int_{b_1}^{b_2} h(s)\frac{\partial^k\varphi(\varkappa,s)}{\partial\varkappa^k}ds + \int_{a_1}^{a_2}\int_{b_1}^{b_2} h(s)g(t)\varphi(t,s)dsdt\right|$$
$$\leq \frac{\|g\|_{[a_1,a_2],\infty}}{(a_2-a_1)\,n!}\frac{\|h\|_{[b_1,b_2]\infty}}{(b_2-b_1)\,m!}\left\{\left|\frac{\partial^{n+m}\varphi(a_1,b_1)}{\partial t^n\partial s^m}\right|A_n(\varkappa)C_m(\tau) + \left|\frac{\partial^{n+m}\varphi(a_1,b_2)}{\partial t^n\partial s^m}\right|A_n(\varkappa)D_m(\tau)\right.$$
$$\left. + \left|\frac{\partial^{n+m}\varphi(a_2,b_1)}{\partial t^n\partial s^m}\right|B_n(\varkappa)C_m(\tau) + \left|\frac{\partial^{n+m}\varphi(a_2,b_2)}{\partial t^n\partial s^m}\right|B_n(\varkappa)D_m(\tau)\right\}$$

for all $(\varkappa,\tau) \in [a_1,a_2]\times[b_1,b_2]$, where $\|g\|_{[a_1,a_2],\infty} = \sup_{u\in[a_1,a_2]}|g(u)|$ and $\|h\|_{[b_1,b_2],\infty} = \sup_{u\in[b_1,b_2]}|h(u)|$.

Proof. If we take absolute value of both sides of the equality (2.1), we find that

$$\left|\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{M_k(\varkappa)}{k!}\frac{M_l(\tau)}{l!}\frac{\partial^{k+l}\varphi(\varkappa,\tau)}{\partial\varkappa^k\partial\tau^l}-\sum_{l=0}^{m-1}\frac{M_l(\tau)}{l!}\int_{a_1}^{a_2}g(t)\frac{\partial^l\varphi(t,\tau)}{\partial\tau^l}dt\right.$$
$$\left.-\sum_{k=0}^{n-1}\frac{M_k(\varkappa)}{k!}\int_{b_1}^{b_2}h(s)\frac{\partial^k\varphi(\varkappa,s)}{\partial\varkappa^k}ds+\int_{a_1}^{a_2}\int_{b_1}^{b_2}h(s)g(t)\varphi(t,s)dsdt\right|\leq\int_{a_1}^{a_2}\int_{b_1}^{b_2}|P_{n-1}(\varkappa,t)||Q_{m-1}(\tau,s)|\left|\frac{\partial^{n+m}\varphi(t,s)}{\partial t^n\partial s^m}\right|dsdt.$$

Since $\left|\frac{\partial^{n+m}\varphi(t,s)}{\partial t^n\partial s^m}\right|$ is a convex function on the co-ordinates on Δ, we have

$$\begin{aligned}\left|\frac{\partial^{n+m}}{\partial t^n\partial s^m}\varphi\left(\frac{a_2-t}{a_2-a_1}a_1+\frac{t-a_1}{a_2-a_1}a_2,\frac{b_2-s}{b_2-b_1}b_1+\frac{s-b_1}{b_2-b_1}b_2\right)\right| &\leq \frac{a_2-t}{a_2-a_1}\frac{b_2-s}{b_2-b_1}\left|\frac{\partial^{n+m}\varphi(a_1,b_1)}{\partial t^n\partial s^m}\right| \\ &+ \frac{a_2-t}{a_2-a_1}\frac{s-b_1}{b_2-b_1}\left|\frac{\partial^{n+m}\varphi(a_1,b_2)}{\partial t^n\partial s^m}\right| \\ &+ \frac{t-a_1}{a_2-a_1}\frac{b_2-s}{b_2-b_1}\left|\frac{\partial^{n+m}\varphi(a_2,b_1)}{\partial t^n\partial s^m}\right| \\ &+ \frac{t-a_1}{a_2-a_1}\frac{s-b_1}{b_2-b_1}\left|\frac{\partial^{n+m}\varphi(a_2,b_2)}{\partial t^n\partial s^m}\right|.\end{aligned} \tag{3.2}$$

Utilizing the inequality (3.2), we can write

$$\begin{aligned}\int_{a_1}^{a_2}\int_{b_1}^{b_2}|P_{n-1}(\varkappa,t)||Q_{m-1}(\tau,s)|\left|\frac{\partial^{n+m}\varphi(t,s)}{\partial t^n\partial s^m}\right|dsdt &\leq \frac{1}{(a_2-a_1)(b_2-b_1)} \\ &\times \left\{\left|\frac{\partial^{n+m}\varphi(a_1,b_1)}{\partial t^n\partial s^m}\right|\int_{a_1}^{a_2}\int_{b_1}^{b_2}(a_2-t)|P_{n-1}(\varkappa,t)|(b_2-s)|Q_{m-1}(\tau,s)|dsdt\right. \\ &+ \left|\frac{\partial^{n+m}\varphi(a_1,b_2)}{\partial t^n\partial s^m}\right|\int_{a_1}^{a_2}\int_{b_1}^{b_2}(a_2-t)|P_{n-1}(\varkappa,t)|(s-b_1)|Q_{m-1}(\tau,s)|dsdt \\ &+ \left|\frac{\partial^{n+m}\varphi(a_2,b_1)}{\partial t^n\partial s^m}\right|\int_{a_1}^{a_2}\int_{b_1}^{b_2}(t-a_1)|P_{n-1}(\varkappa,t)|(b_2-s)|Q_{m-1}(\tau,s)|dsdt \\ &+ \left.\left|\frac{\partial^{n+m}\varphi(a_2,b_2)}{\partial t^n\partial s^m}\right|\int_{a_1}^{a_2}\int_{b_1}^{b_2}(t-a_1)|P_{n-1}(\varkappa,t)|(s-b_1)|Q_{m-1}(\tau,s)|dsdt\right\}.\end{aligned}$$

If we calculate the above four double inetgrals and also substitute the results in (3.3), because of $\|g\|_{[a_1,\varkappa],\infty}, \|g\|_{[\varkappa,a_2],\infty}\leq\|g\|_{[a_1,a_2],\infty}$ and $\|h\|_{[b_1,\tau]\infty}, \|h\|_{[\tau,b_2]\infty}\leq\|h\|_{[b_1,b_2]\infty}$, we obtain required inequality (3.1) which completes the proof. □

Remark 3.2. *Under the same assumptions of Theorem 3.1 with $n=m=1$, then the following inequality holds:*

$$\begin{aligned}&\left|M_0(\varkappa)M_0(\tau)\varphi(\varkappa,\tau)-M_0(\tau)\int_{a_1}^{a_2}g(t)\varphi(t,\tau)dt-M_0(\varkappa)\int_{b_1}^{b_2}h(s)\varphi(\varkappa,s)ds+\int_{a_1}^{a_2}\int_{b_1}^{b_2}g(t)h(s)\varphi(t,s)dsdt\right| \\ \leq\ &\frac{\|g\|_{[a_1,a_2],\infty}}{(a_2-a_1)}\frac{\|h\|_{[b_1,b_2]\infty}}{(b_2-b_1)}\times\left\{\left|\frac{\partial^2\varphi(a_1,b_1)}{\partial t\partial s}\right|A_1(\varkappa)C_1(\tau)+\left|\frac{\partial^2\varphi(a_1,b_2)}{\partial t\partial s}\right|A_1(\varkappa)D_1(\tau)\right. \\ +\ &\left.\left|\frac{\partial^2\varphi(a_2,b_1)}{\partial t\partial s}\right|B_1(\varkappa)C_1(\tau)+\left|\frac{\partial^2\varphi(a_2,b_2)}{\partial t\partial s}\right|B_1(\varkappa)D_1(\tau)\right\}\end{aligned} \tag{3.3}$$

which was given by Erden and Sarikaya in [22] (in case of $\lambda=0$).

Remark 3.3. *If we take $g(u)=h(u)=1$ in (3.3), then we get*

$$\begin{aligned}&\left|(a_2-a_1)(b_2-b_1)\varphi(\varkappa,\tau)-(b_2-b_1)\int_{a_1}^{a_2}\varphi(t,\tau)dt-(a_2-a_1)\int_{b_1}^{b_2}\varphi(\varkappa,s)ds+\int_{a_1}^{a_2}\int_{b_1}^{b_2}\varphi(t,s)dsdt\right| \quad (3.4)\\ \leq\ & \frac{1}{(a_2-a_1)(b_2-b_1)}\left\{\left|\frac{\partial^2\varphi(a_1,b_1)}{\partial t\partial s}\right|A_1(\varkappa)C_1(\tau)+\left|\frac{\partial^2\varphi(a_1,b_2)}{\partial t\partial s}\right|A_1(\varkappa)D_1(\tau)\right.\\ +\ & \left.\left|\frac{\partial^2\varphi(a_2,b_1)}{\partial t\partial s}\right|B_1(\varkappa)C_1(\tau)+\left|\frac{\partial^2\varphi(a_2,b_2)}{\partial t\partial s}\right|B_1(\varkappa)D_1(\tau)\right\}\end{aligned}$$

which was given by Erden and Sarıkaya in [6].

Remark 3.4. *Taking $\varkappa=\dfrac{a_1+a_2}{2}$ and $\tau=\dfrac{b_1+b_2}{2}$ in (3.4), it is found that*

$$\begin{aligned}&\left|(a_2-a_1)(b_2-b_1)\varphi\left(\frac{a_1+a_2}{2},\frac{b_1+b_2}{2}\right)-(b_2-b_1)\int_{a_1}^{a_2}\varphi\left(t,\frac{b_1+b_2}{2}\right)dt\right.\\ -\ & \left.(a_2-a_1)\int_{b_1}^{b_2}\varphi\left(\frac{a_1+a_2}{2},s\right)ds+\int_{a_1}^{a_2}\int_{b_1}^{b_2}\varphi(t,s)dsdt\right|\\ \leq\ & \frac{(a_2-a_1)^2(b_2-b_1)^2}{16}\left\{\frac{\left|\frac{\partial^2\varphi(a_1,b_1)}{\partial t\partial s}\right|+\left|\frac{\partial^2\varphi(a_1,b_2)}{\partial t\partial s}\right|+\left|\frac{\partial^2\varphi(a_2,b_1)}{\partial t\partial s}\right|+\left|\frac{\partial^2\varphi(a_2,b_2)}{\partial t\partial s}\right|}{4}\right\}\end{aligned}$$

which was given by Latif and Dragomir in [4].

Corollary 3.5. *Under the same assumptions of Theorem 3.1 with $g(u)=h(u)=1$, then we have the inequality*

$$\begin{aligned}&\left|\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{X_k(\varkappa)}{k!}\frac{Y_l(\tau)}{l!}\frac{\partial^{k+l}\varphi(\varkappa,\tau)}{\partial\varkappa^k\partial\tau^l}-\sum_{l=0}^{m-1}\frac{Y_l(\tau)}{l!}\int_{a_1}^{a_2}\frac{\partial^l\varphi(t,\tau)}{\partial\tau^l}dt-\sum_{k=0}^{n-1}\frac{X_k(\varkappa)}{k!}\int_{b_1}^{b_2}\frac{\partial^k\varphi(\varkappa,s)}{\partial\varkappa^k}ds+\int_{a_1}^{a_2}\int_{b_1}^{b_2}\varphi(t,s)dsdt\right| \quad (3.5)\\ \leq\ & \frac{1}{n!(a_2-a_1)}\frac{1}{m!(b_2-b_1)}\left\{\left|\frac{\partial^{n+m}\varphi(a_1,b_1)}{\partial t^n\partial s^m}\right|A_n(\varkappa)C_m(\tau)+\left|\frac{\partial^{n+m}\varphi(a_1,b_2)}{\partial t^n\partial s^m}\right|A_n(\varkappa)D_m(\tau)+\left|\frac{\partial^{n+m}\varphi(a_2,b_1)}{\partial t^n\partial s^m}\right|B_n(\varkappa)C_m(\tau)\right.\\ +\ & \left.\left|\frac{\partial^{n+m}\varphi(a_2,b_2)}{\partial t^n\partial s^m}\right|B_n(\varkappa)D_m(\tau)\right\}\end{aligned}$$

where $X_k(\varkappa)$ and $Y_l(\tau)$ are defined by

$$X_k(x)=\frac{(a_2-\varkappa)^{k+1}+(-1)^k(\varkappa-a_1)^{k+1}}{(k+1)} \quad (3.6)$$

and

$$Y_l(y)=\frac{(b_2-\tau)^{l+1}+(-1)^l(\tau-b_1)^{l+1}}{(l+1)}, \quad (3.7)$$

respectively. This result is a Ostrowski type inequality for mappings whose absolute value of heigher degree partial derivatives are co-ordinated convex.

Corollary 3.6. *Under the same assumptions of Theorem 3.1 with $\varkappa=\frac{a_1+a_2}{2}$ and $\tau=\frac{b_1+b_2}{2}$, then we have the inequality*

$$\begin{aligned}&\left|\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{M_k\left(\frac{a_1+a_2}{2}\right)}{k!}\frac{M_l\left(\frac{b_1+b_2}{2}\right)}{l!}\frac{\partial^{k+l}\varphi\left(\frac{a_1+a_2}{2},\frac{b_1+b_2}{2}\right)}{\partial\varkappa^k\partial\tau^l}-\sum_{l=0}^{m-1}\frac{M_l\left(\frac{b_1+b_2}{2}\right)}{l!}\int_{a_1}^{a_2}g(t)\frac{\partial^l\varphi\left(t,\frac{b_1+b_2}{2}\right)}{\partial\tau^l}dt\right.\\ -\ & \left.\sum_{k=0}^{n-1}\frac{M_k\left(\frac{a_1+a_2}{2}\right)}{k!}\int_{b_1}^{b_2}h(s)\frac{\partial^k\varphi\left(\frac{a_1+a_2}{2},s\right)}{\partial\varkappa^k}ds+\int_{a_1}^{a_2}\int_{b_1}^{b_2}h(s)g(t)\varphi(t,s)dsdt\right|\\ \leq\ & \frac{\|g\|_{[a_1,a_2],\infty}}{(n+1)!}\frac{\|h\|_{[b_1,b_2]\infty}}{(m+1)!}\frac{((a_2-a_1)^{n+1}}{2^{n+1}}\frac{(b_2-b_1)^{m+1}}{2^{m+1}}\\ \times\ & \left\{\left|\frac{\partial^{n+m}\varphi(a_1,b_1)}{\partial t^n\partial s^m}\right|+\left|\frac{\partial^{n+m}\varphi(a_1,b_2)}{\partial t^n\partial s^m}\right|+\left|\frac{\partial^{n+m}\varphi(a_2,b_1)}{\partial t^n\partial s^m}\right|+\left|\frac{\partial^{n+m}\varphi(a_2,b_2)}{\partial t^n\partial s^m}\right|\right\}\end{aligned}$$

*which is "**weighted mid-point**" inequality for functions whoose absolute value of heigher degree partial derivatives are co-ordinated convex.*

We establish some weighted integral inequalities by using convexity of $\left|\frac{\partial^{n+m}\varphi}{\partial t^n \partial s^m}\right|^q$.

Theorem 3.7. *Suppose that all the assumptions of Lemma 2.1 hold. If $\left|\frac{\partial^{n+m}\varphi}{\partial t^n \partial s^m}\right|^q$ is a convex function on the co-ordinates on Δ, $\frac{1}{p}+\frac{1}{q}=1$ and $q>1$, then the following inequality holds:*

$$\begin{aligned}
&\left|\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{M_k(\varkappa)}{k!}\frac{M_l(\tau)}{l!}\frac{\partial^{k+l}\varphi(\varkappa,\tau)}{\partial\varkappa^k\partial\tau^l}-\sum_{l=0}^{m-1}\frac{M_l(\tau)}{l!}\int_{a_1}^{a_2}g(t)\frac{\partial^l\varphi(t,\tau)}{\partial\tau^l}dt-\sum_{k=0}^{n-1}\frac{M_k(\varkappa)}{k!}\int_{b_1}^{b_2}h(s)\frac{\partial^k\varphi(\varkappa,s)}{\partial\varkappa^k}ds\right. \\
&+\left.\int_{a_1}^{a_2}\int_{b_1}^{b_2}h(s)g(t)\varphi(t,s)dsdt\right| \\
\leq\; & \frac{\|g\|_{[a_1,a_2],\infty}}{n!\,(np+1)^{\frac{1}{p}}}\frac{\|h\|_{[b_1,b_2],\infty}}{m!\,(mp+1)^{\frac{1}{p}}}(a_2-a_1)^{\frac{1}{q}}(b_2-b_1)^{\frac{1}{q}}\times\left[(\varkappa-a_1)^{np+1}+(a_2-\varkappa)^{np+1}\right]^{\frac{1}{p}}\left[(\tau-b_1)^{mp+1}+(b_2-\tau)^{mp+1}\right]^{\frac{1}{p}} \\
\times\; & \left[\frac{\left|\frac{\partial^{n+m}\varphi(a_1,b_1)}{\partial t^n\partial s^m}\right|^q+\left|\frac{\partial^{n+m}\varphi(a_1,b_2)}{\partial t^n\partial s^m}\right|^q+\left|\frac{\partial^{n+m}\varphi(a_2,b_1)}{\partial t^n\partial s^m}\right|^q+\left|\frac{\partial^{n+m}\varphi(a_2,b_2)}{\partial t^n\partial s^m}\right|^q}{4}\right]^{\frac{1}{q}}
\end{aligned} \tag{3.8}$$

for all $(\varkappa,\tau)\in[a_1,a_2]\times[b_1,b_2]$, where $\|g\|_{[a_1,a_2],\infty}=\sup_{u\in[a_1,a_2]}|g(u)|$ and $\|h\|_{[b_1,b_2],\infty}=\sup_{u\in[b_1,b_2]}|h(u)|$.

Proof. Taking absolute value of (2.1), from Hölder's inequality, it follows that

$$\begin{aligned}
&\left|\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{M_k(\varkappa)}{k!}\frac{M_l(\tau)}{l!}\frac{\partial^{k+l}\varphi(\varkappa,\tau)}{\partial\varkappa^k\partial\tau^l}-\sum_{l=0}^{m-1}\frac{M_l(\tau)}{l!}\int_{a_1}^{a_2}g(t)\frac{\partial^l\varphi(t,\tau)}{\partial\tau^l}dt\right. \\
-\;&\left.\sum_{k=0}^{n-1}\frac{M_k(\varkappa)}{k!}\int_{b_1}^{b_2}h(s)\frac{\partial^k\varphi(\varkappa,s)}{\partial\varkappa^k}ds+\int_{a_1}^{a_2}\int_{b_1}^{b_2}h(s)g(t)\varphi(t,s)dsdt\right| \\
\leq\;&\left(\int_{a_1}^{a_2}\int_{b_1}^{b_2}|P_{n-1}(\varkappa,t)|^p\,|Q_{m-1}(\tau,s)|^p\,dsdt\right)^{\frac{1}{p}}\left(\int_{a_1}^{a_2}\int_{b_1}^{b_2}\left|\frac{\partial^{n+m}\varphi(t,s)}{\partial t^n\partial s^m}\right|^q dsdt\right)^{\frac{1}{q}}.
\end{aligned} \tag{3.9}$$

By utulizing the definition of $P_{n-1}(\varkappa,t)$ and $Q_{m-1}(\tau,s)$, we find that

$$\begin{aligned}
\left(\int_a^b\int_c^d|P_{n-1}(\varkappa,t)|^p\,|Q_{m-1}(\tau,s)|^p\,dsdt\right)^{\frac{1}{p}} \leq\;& \frac{\|g\|_{[a_1,a_2],\infty}}{n!\,(np+1)^{\frac{1}{p}}}\frac{\|h\|_{[b_1,b_2],\infty}}{m!\,(mp+1)^{\frac{1}{p}}} \\
\times\;&\left[(\varkappa-a_1)^{np+1}+(a_2-\varkappa)^{np+1}\right]^{\frac{1}{p}}\left[(\tau-b_1)^{mp+1}+(b_2-\tau)^{mp+1}\right]^{\frac{1}{p}}.
\end{aligned} \tag{3.10}$$

Since $\left|\frac{\partial^{n+m}\varphi(t,s)}{\partial t^n\partial s^m}\right|^q$ is a convex function on the co-ordinates on Δ, we also have

$$\begin{aligned}
\left|\frac{\partial^{n+m}}{\partial t^n\partial s^m}\varphi\left(\frac{a_2-t}{a_2-a_1}a_1+\frac{t-a_1}{a_2-a_1}a_2,\frac{b_2-s}{b_2-b_1}b_1+\frac{s-b_1}{b_2-b_1}b_2\right)\right|^q \leq\;& \frac{a_2-t}{a_2-a_1}\frac{b_2-s}{b_2-b_1}\left|\frac{\partial^{n+m}\varphi(a_1,b_1)}{\partial t^n\partial s^m}\right|^q \\
+\;&+\frac{a_2-t}{a_2-a_1}\frac{s-b_1}{b_2-b_1}\left|\frac{\partial^{n+m}\varphi(a_1,b_2)}{\partial t^n\partial s^m}\right|^q \\
+\;&\frac{t-a_1}{a_2-a_1}\frac{b_2-s}{b_2-b_1}\left|\frac{\partial^{n+m}\varphi(a_2,b_1)}{\partial t^n\partial s^m}\right|^q \\
+\;&+\frac{t-a_1}{a_2-a_1}\frac{s-b_1}{b_2-b_1}\left|\frac{\partial^{n+m}\varphi(a_2,b_2)}{\partial t^n\partial s^m}\right|^q.
\end{aligned} \tag{3.11}$$

Using the inequality (3.11), it follows that

$$\left(\int_{a_1}^{a_2}\int_{b_1}^{b_2}\left|\frac{\partial^{n+m}\varphi(t,s)}{\partial t^n\partial s^m}\right|^q dsdt\right)^{\frac{1}{q}} \leq (a_2-a_1)^{\frac{1}{q}}(b_2-b_1)^{\frac{1}{q}} \times \left[\frac{\left|\frac{\partial^{n+m}\varphi(a_1,b_1)}{\partial t^n\partial s^m}\right|^q+\left|\frac{\partial^{n+m}\varphi(a_1,b_2)}{\partial t^n\partial s^m}\right|^q+\left|\frac{\partial^{n+m}\varphi(a_2,b_1)}{\partial t^n\partial s^m}\right|^q+\left|\frac{\partial^{n+m}\varphi(a_2,b_2)}{\partial t^n\partial s^m}\right|^q}{4}\right]^{\frac{1}{q}}. \tag{3.12}$$

Substituting the inequalities (3.10) and (3.12) in (3.9), we deduce the inequality (3.8). Hence, the proof is completed. □

Remark 3.8. *Under the same assumptions of Theorem 3.7 with $n=m=1$, then the following inequality holds:*

$$\begin{aligned}&\left|M_0(\varkappa)M_0(\tau)\varphi(\varkappa,\tau)-M_0(\tau)\int_{a_1}^{a_2}g(t)\varphi(t,\tau)dt-M_0(\varkappa)\int_{b_1}^{b_2}h(s)\varphi(\varkappa,s)ds+\int_{a_1}^{a_2}\int_{b_1}^{b_2}g(t)h(s)\varphi(t,s)dsdt\right| \qquad (3.13)\\ &\leq \|g\|_{[a,b],\infty}\|h\|_{[c,d]\infty}(a_2-a_1)^{\frac{1}{q}}(b_2-b_1)^{\frac{1}{q}}\times\left[\frac{(\varkappa-a_1)^{p+1}+(a_2-\varkappa)^{p+1}}{p+1}\right]^{\frac{1}{p}}\left[\frac{(\tau-b_1)^{p+1}+(b_2-\tau)^{p+1}}{p+1}\right]^{\frac{1}{p}}\\ &\times\left[\frac{\left|\frac{\partial^2\varphi(a_1,b_1)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2\varphi(a_1,b_2)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2\varphi(a_2,b_1)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2\varphi(a_2,b_2)}{\partial t\partial s}\right|^q}{4}\right]^{\frac{1}{q}}\end{aligned}$$

which was given by Erden and Sarikaya in [22] (in case of $\lambda=0$).

Corollary 3.9. *Substituting $(\varkappa,\tau)=(a_1,b_1)$, (a_1,b_2), (a_2,b_1) and (a_2,b_2) in (3.13). Subsequently, if we add the obtained results and use the triangle inequality for the modulus, we get the inequality*

$$\begin{aligned}&\left|M_0(\varkappa)M_0(\tau)\frac{\varphi(a_1,b_1)+\varphi(a_1,b_2)+\varphi(a_2,b_1)+\varphi(a_2,b_2)}{4}\right. \qquad (3.14)\\ &+\left.\int_{a_1}^{a_2}\int_{b_1}^{b_2}g(t)h(s)\varphi(t,s)dsdt-\frac{1}{2}M_0(\tau)\int_{a_1}^{a_2}g(t)\left[\varphi(t,b_1)+\varphi(t,b_2)\right]dt-\frac{1}{2}M_0(\varkappa)\int_{b_1}^{b_2}h(s)\left[\varphi(a_1,s)+\varphi(a_2,s)\right]ds\right|\\ &\leq \|g\|_{[a_1,a_2],\infty}\|h\|_{[b_1,b_2],\infty}\frac{(a_2-a_1)^2(b_2-b_1)^2}{4(p+1)^{\frac{1}{p}}}\times\left[\frac{\left|\frac{\partial^2\varphi(a_1,b_1)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2\varphi(a_1,b_2)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2\varphi(a_2,b_1)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2\varphi(a_2,b_2)}{\partial t\partial s}\right|^q}{4}\right]^{\frac{1}{q}}\end{aligned}$$

which is a weighted Hermite-Hadamard type inequality for double integrals.

Remark 3.10. *If we take $g(u)=h(u)=1$ in (3.14), then we have*

$$\begin{aligned}&\left|\frac{\varphi(a_1,b_1)+\varphi(a_1,b_2)+\varphi(a_2,b_1)+\varphi(a_2,b_2)}{4}\right.\\ &+\left.\frac{1}{(a_2-a_1)(b_2-b_1)}\int_{a_1}^{a_2}\int_{b_1}^{b_2}\varphi(t,s)dsdt-\frac{1}{2(a_2-a_1)}\int_{a_1}^{a_2}\left[\varphi(t,b_1)+\varphi(t,b_2)\right]dt-\frac{1}{2(b_2-b_1)}\int_{b_1}^{b_2}\left[\varphi(a_1,s)+\varphi(a_2,s)\right]ds\right|\\ &\leq \frac{(a_2-a_1)(b_2-b_1)}{4(p+1)^{\frac{1}{p}}}\left[\frac{\left|\frac{\partial^2\varphi(a_1,b_1)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2\varphi(a_1,b_2)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2\varphi(a_2,b_1)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2\varphi(a_2,b_2)}{\partial t\partial s}\right|^q}{4}\right]^{\frac{1}{q}}\end{aligned}$$

which was deduced by Sarikaya et al. in [7].

Remark 3.11. *If we take $g(u)=h(u)=1$ in (3.13), then we get*

$$\left|(a_2-a_1)(b_2-b_1)\varphi(\varkappa,\tau)-(b_2-b_1)\int_{a_1}^{a_2}\varphi(t,\tau)dt-(a_2-a_1)\int_{b_1}^{b_2}\varphi(\varkappa,s)ds+\int_{a_1}^{a_2}\int_{b_1}^{b_2}\varphi(t,s)dsdt\right| \tag{3.15}$$

$$\leq (a_2-a_1)^{\frac{1}{q}}(b_2-b_1)^{\frac{1}{q}}\times\left[\frac{(\varkappa-a_1)^{p+1}+(a_2-\varkappa)^{p+1}}{p+1}\right]^{\frac{1}{p}}\left[\frac{(\tau-b_1)^{p+1}+(b_2-\tau)^{p+1}}{p+1}\right]^{\frac{1}{p}}$$

$$\times\left[\frac{\left|\frac{\partial^2\varphi(a_1,b_1)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2\varphi(a_1,b_2)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2\varphi(a_2,b_1)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2\varphi(a_2,b_2)}{\partial t\partial s}\right|^q}{4}\right]^{\frac{1}{q}}$$

which was given by Erden and Sarıkaya in [6].

Remark 3.12. *Taking $\varkappa=\frac{a_1+a_2}{2}$ and $\tau=\frac{b_1+b_2}{2}$ in (3.15), we get*

$$\left|(a_2-a_1)(b_2-b_1)\varphi\left(\frac{a_1+a_2}{2},\frac{b_1+b_2}{2}\right)-(b_2-b_1)\int_{a_1}^{a_2}\varphi\left(t,\frac{b_1+b_2}{2}\right)dt\right.$$

$$\left.-\ (a_2-a_1)\int_{b_1}^{b_2}\varphi\left(\frac{a_1+a_2}{2},s\right)ds+\int_{a_1}^{a_2}\int_{b_1}^{b_2}\varphi(t,s)dsdt\right|$$

$$\leq \frac{(a_2-a_1)^2(b_2-b_1)^2}{4(p+1)^{\frac{2}{p}}}\times\left[\frac{\left|\frac{\partial^2\varphi(a_1,b_1)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2\varphi(a_1,b_2)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2\varphi(a_2,b_1)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2\varphi(a_2,b_2)}{\partial t\partial s}\right|^q}{4}\right]^{\frac{1}{q}}$$

which was given by Latif and Dragomir in [4].

Similarly, the other reults related to Theorem 3.7 can be obtained as in Corollary 3.5 and 3.6.

Theorem 3.13. *Suppose that all the assumptions of Lemma 2.1 hold. If $\left|\frac{\partial^{n+m}\varphi}{\partial t^n\partial s^m}\right|^q$ is a convex function on the co-ordinates on Δ, $\frac{1}{p}+\frac{1}{q}=1$ and $q\geq 1$, then the following inequality holds:*

$$\left|\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{M_k(\varkappa)}{k!}\frac{M_l(\tau)}{l!}\frac{\partial^{k+l}\varphi(\varkappa,\tau)}{\partial\varkappa^k\partial\tau^l}-\sum_{l=0}^{m-1}\frac{M_l(\tau)}{l!}\int_{a_1}^{a_2}g(t)\frac{\partial^l\varphi(t,\tau)}{\partial\tau^l}dt\right. \tag{3.16}$$

$$\left.-\ \sum_{k=0}^{n-1}\frac{M_k(\varkappa)}{k!}\int_{b_1}^{b_2}h(s)\frac{\partial^k\varphi(\varkappa,s)}{\partial\varkappa^k}ds+\int_{a_1}^{a_2}\int_{b_1}^{b_2}h(s)g(t)\varphi(t,s)dsdt\right|$$

$$\leq \frac{1}{[(a_2-a_1)(b_2-b_1)]^{\frac{1}{q}}}\frac{\|g\|_{[a_1,a_2],\infty}}{n!(n+1)^{\frac{1}{p}}}\frac{\|h\|_{[b_1,b_2],\infty}}{m!(m+1)^{\frac{1}{p}}}\times\left[(\varkappa-a_1)^{n+1}+(a_2-\varkappa)^{n+1}\right]^{\frac{1}{p}}\left[(\tau-b_1)^{m+1}+(b_2-\tau)^{m+1}\right]^{\frac{1}{p}}$$

$$\times\left\{\left|\frac{\partial^{n+m}\varphi(a_1,b_1)}{\partial t^n\partial s^m}\right|^q A_n(\varkappa)C_m(\tau)+\left|\frac{\partial^{n+m}\varphi(a_1,b_2)}{\partial t^n\partial s^m}\right|^q A_n(\varkappa)D_m(\tau)\right.$$

$$\left.+\ \left|\frac{\partial^{n+m}\varphi(a_2,b_1)}{\partial t^n\partial s^m}\right|^q B_n(\varkappa)C_m(\tau)+\left|\frac{\partial^{n+m}\varphi(a_2,b_2)}{\partial t^n\partial s^m}\right|^q B_n(\varkappa)D_m(\tau)\right\}^{\frac{1}{q}}$$

for all $(\varkappa,\tau)\in[a_1,a_2]\times[b_1,b_2]$, where $\|g\|_{[a_1,a_2],\infty}=\sup_{u\in[a_1,a_2]}|g(u)|$ and $\|h\|_{[b_1,b_2],\infty}=\sup_{u\in[b_1,b_2]}|h(u)|$.

Proof. We take absolute value of (2.1). Because of $\frac{1}{p}+\frac{1}{q}=1$, $\frac{1}{p}+\frac{1}{q}$ can be written instead of 1. Using Hölder's inequality, we find that

$$\left|\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{M_k(\varkappa)}{k!}\frac{M_l(\tau)}{l!}\frac{\partial^{k+l}\varphi(\varkappa,\tau)}{\partial\varkappa^k\partial\tau^l}-\sum_{l=0}^{m-1}\frac{M_l(\tau)}{l!}\int_{a_1}^{a_2}g(t)\frac{\partial^l\varphi(t,\tau)}{\partial\tau^l}dt\right. \tag{3.17}$$

$$\left.-\sum_{k=0}^{n-1}\frac{M_k(\varkappa)}{k!}\int_{b_1}^{b_2}h(s)\frac{\partial^k\varphi(\varkappa,s)}{\partial\varkappa^k}ds+\int_{a_1}^{a_2}\int_{b_1}^{b_2}h(s)g(t)\varphi(t,s)dsdt\right|$$

$$\leq\left(\int_{a_1}^{a_2}\int_{b_1}^{b_2}|P_{n-1}(\varkappa,t)||Q_{m-1}(\tau,s)|dsdt\right)^{\frac{1}{p}}\times\left(\int_{a_1}^{a_2}\int_{b_1}^{b_2}|P_{n-1}(\varkappa,t)||Q_{m-1}(\tau,s)|\left|\frac{\partial^{n+m}\varphi(t,s)}{\partial t^n\partial s^m}\right|^q dsdt\right)^{\frac{1}{q}}.$$

By simple calculations, we can write

$$\int_{a_1}^{a_2}\int_{b_1}^{b_2}|P_{n-1}(\varkappa,t)||Q_{m-1}(\tau,s)|dsdt\leq\frac{\|g\|_{[a_1,a_2],\infty}}{(n+1)!}\frac{\|h\|_{[b_1,b_2],\infty}}{(m+1)!}\times\left[(\varkappa-a_1)^{n+1}+(a_2-\varkappa)^{n+1}\right]\left[(\tau-b_1)^{m+1}+(b_2-\tau)^{m+1}\right] \tag{3.18}$$

By similar methods in the proof of Theorem 3.1, from (3.11), we obtain

$$\begin{aligned}\int_{a_1}^{a_2}\int_{b_1}^{b_2}|P_{n-1}(\varkappa,t)||Q_{m-1}(\tau,s)|\left|\frac{\partial^{n+m}\varphi(t,s)}{\partial t^n\partial s^m}\right|^q dsdt &\leq \frac{\|g\|_{[a_1,a_2],\infty}}{(a_2-a_1)n!}\frac{\|h\|_{[b_1,b_2],\infty}}{(b_2-b_1)m!}\\ &\times\left\{\left|\frac{\partial^{n+m}\varphi(a_1,b_1)}{\partial t^n\partial s^m}\right|^q A_n(\varkappa)C_m(\tau)+\left|\frac{\partial^{n+m}\varphi(a_1,b_2)}{\partial t^n\partial s^m}\right|^q A_n(\varkappa)D_m(\tau)\right.\\ &+\left.\left|\frac{\partial^{n+m}\varphi(a_2,b_1)}{\partial t^n\partial s^m}\right|^q B_n(\varkappa)C_m(\tau)+\left|\frac{\partial^{n+m}\varphi(a_2,b_2)}{\partial t^n\partial s^m}\right|^q B_n(\varkappa)D_m(\tau)\right\}.\end{aligned} \tag{3.19}$$

Substituting the inequalities (3.18) and (3.19) in (3.17), we easily deduce the required inequality (3.16) which completes the proof. □

Remark 3.14. *In case $(p,q)=(\infty,1)$, if we take limit as $p\to\infty$ in Theorem 3.13, then the inequality (3.16) becomes the inequality (3.1). Thus, we obtain all of the results which are similar to Theorem 3.1.*

4. Conclusion

In this paper, Ostrowski type inequalities for co-ordinated convex functions are developed. It is also shown that the results provided in this paper are potential generalizations of the existing comparable results in the literature. Infuture directions, one may find similar results through different types of co-ordinated convexity.
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Declarations

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