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# $k-$FIBONACCI AND $k-$LUCAS GENERALIZED QUATERNIONS

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Abstract. We investigate the properties of $k-$Fibonacci and $k-$Lucas quaternions over the generalized quaternion algebra. After presenting generating functions and Binet's formulas for these types of quaternions, we calculate several well-known identities such as Catalan's, Cassini's and d'Ocagne's identities for $k-$Fibonacci and $k-$Lucas generalized quaternions.

## 1. Introduction

The famous integer sequence, Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$, is defined by the numbers which satisfy the second order recurrence relation $F_n = F_{n-1} + F_{n-2}$ with the initial conditions $F_0 = 0$ and $F_1 = 1$. Fibonacci numbers have many interesting properties and applications in various research areas. The Lucas sequence $\{L_n\}_{n=0}^{\infty}$ is defined with the Lucas numbers which are defined with the recurrence relation $L_n = L_{n-1} + L_{n-2}$ with the initial conditions $L_0 = 2$ and $L_1 = 1$. Sometimes, Lucas numbers are defined with the well-known identity $L_n = F_{n-1}+F_{n+1}$ between Fibonacci and Lucas numbers.

Pell sequence $\{P_n\}_{n=0}^{\infty}$ and Pell-Lucas sequence $\{PL_n\}_{n=0}^{\infty}$ are other well-known sequences. They are defined by the recurrence relations $P_n = 2P_{n-1} + P_{n-2}$ and $PL_n = 2PL_{n-1}+PL_{n-2}$ where the initial conditions are $P_0 = 0$ and $P_1 = 1$,$PL_0 = 2$ and $PL_1 = 2$, respectively.

Generating functions for the sequences $\{F_n\}_{n=0}^{\infty}$, $\{L_n\}_{n=0}^{\infty}$, $\{P_n\}_{n=0}^{\infty}$ and $\{PL_n\}_{n=0}^{\infty}$ are shown below:

$$\sum_{n=0}^{\infty} F_n x^n = \frac{x}{1-x-x^2},$$
$$\sum_{n=0}^{\infty} L_n x^n = \frac{2-x}{1-x-x^2},$$
$$\sum_{n=0}^{\infty} P_n x^n = \frac{x}{1-2x-x^2}$$

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and

$$\sum_{n=0}^{\infty} PL_n x^n = \frac{2-2x}{1-2x-x^2}$$

respectively. Binet's formulas for the Fibonacci, Lucas, Pell and Pell-Lucas numbers are

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$
$$L_n = \alpha^n + \beta^n$$
$$P_n = \frac{\gamma^n - \delta^n}{\gamma - \delta}$$

and

$$PL_n = \gamma^n + \delta^n$$

respectively, where $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$ are the roots of the characteristic equation $x^2 - x - 1 = 0$, and $\gamma = 1 + \sqrt{2}$, $\delta = 1 - \sqrt{2}$ are the roots of the characteristic equation $x^2 - 2x - 1 = 0$. The positive roots $\alpha$ and $\gamma$ are known as the golden ratio and the silver ratio, respectively (see [12, 13] for details).

Many authors generalized these sequences by changing the initial conditions or changing the recurrence relation slightly. One of the generalizations of the Fibonacci sequence is $k-$Fibonacci sequence introduced by Falcon and Plaza [6]. For any positive real number $k$, $k-$Fibonacci numbers are defined by the recurrence relation

$$F_{k,n} = kF_{k,n-1} + F_{k,n-2}$$

with the initial conditions $F_{k,0} = 0$ and $F_{k,1} = 1$. Falcon [7] defined $k-$Lucas numbers by the recurrence relation

$$L_{k,n} = kL_{k,n-1} + L_{k,n-2}$$

with the initial conditions $L_{k,0} = 2, L_{k,1} = k$.

For $k = 1$, the $k-$Fibonacci and $k-$Lucas sequences reduce to the classical Fibonacci and Lucas sequences. Similarly, for $k = 2$, the $k-$Fibonacci and $k-$Lucas sequences reduce to the Pell and Pell-Lucas sequences.

Generating functions for the $k-$Fibonacci and $k-$Lucas sequences are respectively

$$f_k(x) = \frac{x}{1-kx-x^2}$$

and

$$l_k(x) = \frac{2-kx}{1-kx-x^2}.$$

Binet's formulas for the $k-$Fibonacci and $k-$Lucas numbers are respectively

$$F_{k,n} = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

and

$$L_{k,n} = \alpha^n + \beta^n$$

where

$$\alpha = \frac{k+\sqrt{k^2+4}}{2} \text{ and } \beta = \frac{k-\sqrt{k^2+4}}{2} \tag{1.1}$$

are roots of the characteristic equation $x^2 - kx - 1 = 0$.

Let $\lambda, \mu \in \mathbb{R}$ and $\mathbb{H}(\lambda, \mu)$ be the generalized quaternion algebra with the basis $\{1, e_1, e_2, e_3\}$. The multiplication table of this algebra is

$$\begin{array}{c|cccc} \cdot & 1 & e_1 & e_2 & e_3 \\ \hline 1 & 1 & e_1 & e_2 & e_3 \\ e_1 & e_1 & -\lambda & e_3 & -\lambda e_2 \\ e_2 & e_2 & -e_3 & -\mu & \mu e_1 \\ e_3 & e_3 & \lambda e_2 & -\mu e_1 & -\lambda\mu. \end{array}$$

The algebra $\mathbb{H}(1,1)$ is the quaternion division algebra and $\mathbb{H}(1,-1)$ is the split-quaternion algebra. A generalized quaternion, $q$, in the algebra $\mathbb{H}(\lambda,\mu)$ is formulated as $q = a_0 + a_1e_1 + a_2e_2 + a_3e_3$ where $a_0, a_1, a_2$ and $a_3$ are real numbers. The conjugate of $q$ is given by $q^* = a_0 - a_1e_1 - a_2e_2 - a_3e_3$ and the norm of $q$ is $n(q) = qq^* = a_0^2 + a_1^2\lambda + a_2^2\mu + a_3^2\lambda\mu$.

Horadam [10] defined Fibonacci and Lucas quaternions as follows:

$$Q_n = F_n + F_{n+1}e_1 + F_{n+2}e_2 + F_{n+3}e_3$$

and

$$K_n = L_n + L_{n+1}e_1 + L_{n+2}e_2 + L_{n+3}e_3$$

respectively, where $F_n$ is the $n$th Fibonacci number and $L_n$ is the $n$th Lucas number. He defined a generalization of the Fibonacci quaternions with the relation

$$P_n = H_n + H_{n+1}e_1 + H_{n+2}e_2 + H_{n+3}e_3$$

where $H_1 = p$, $H_2 = p+q$ and $H_n = H_{n-1} + H_{n-2}$.

A considerable amount of literature has focused on Fibonacci and Lucas quaternions. Iyer [11] investigated a number of relations of Fibonacci and Lucas quaternions. Halici [8] studied the Fibonacci and Lucas quaternions and introduced several properties including the Binet's formulas as follows:

$$Q_n = \frac{1}{\sqrt{5}}\left(\widehat{\alpha}\alpha - \widehat{\beta}\beta\right)$$

and

$$K_n = \widehat{\alpha}\alpha + \widehat{\beta}\beta$$

where $\widehat{\alpha} = 1+\alpha e_1+\alpha^2 e_2+\alpha^3 e_3$ and $\widehat{\beta} = 1+\beta e_1+\beta^2 e_2+\beta^3 e_3$ where $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$.

Akyigit et al. [1] studied split Fibonacci and split Lucas quaternions on the algebra $\mathbb{H}(1,-1)$. They obtained Binet's formulas and focused on summation identities for these quaternions. Akyigit et al. [2] generalized the quaternions $Q_n$ and $K_n$ on the algebra $\mathbb{H}(\lambda,\mu)$.

Szynal-Liana & Wloch [18] and Cimen & Ipek [3] worked on Pell and Pell-Lucas quaternions which are defined as follows:

$$R_n = P_n + P_{n+1}e_1 + P_{n+2}e_2 + P_{n+3}e_3$$

and

$$S_n = PL_n + PL_{n+1}e_1 + PL_{n+2}e_2 + PL_{n+3}e_3$$

where $P_n$ and $PL_n$ are the $n$th Pell and Pell-Lucas numbers. Many properties of these quaternions can be found in studies [3, 18].

Ramirez [16] introduced the $k-$Fibonacci and $k-$Lucas quaternions on the algebra $\mathbb{H}(1,1)$ as follows:

$$D_{k,n} = F_{k,n} + F_{k,n+1}e_1 + F_{k,n+2}e_2 + F_{k,n+3}e_3$$

and

$$P_{k,n} = L_{k,n} + L_{k,n+1}e_1 + L_{k,n+2}e_2 + L_{k,n+3}e_3$$

respectively, where $F_{k,n}$ and $L_{k,n}$ are the $n$th $k-$Fibonacci and $k-$Lucas numbers. He obtained generating functions, Binet's formulas, Cassini's identity and some other identities. He also explained Catalan's identity as a conjecture. Furthermore, this conjecture was proved by Polatli and Kesim [15].

Polatli et.al. [14] idefined and studied split $k-$Fibonacci and $k-$Lucas Quaternions on the algebra $\mathbb{H}(1,-1)$

$$M_{k,r} = F_{k,r} + F_{k,r+1}e_1 + F_{k,r+2}e_2 + F_{k,r+3}e_3$$

and

$$N_{k,r} = L_{k,r} + L_{k,r+1}e_1 + L_{k,r+2}e_2 + L_{k,r+3}e_3$$

respectively, where $F_{k,n}$ and $L_{k,n}$ are the $n$th $k-$Fibonacci and $k-$Lucas numbers and gave many properties for these quaternions such as Binet's formulas, generating functions, Catalan's and d'Ocagne's identities, and several summation and binomial formulas.

Catarino [4] introduced the Modified Pell and Modified $k-$Pell quaternions as follows:

$$MP_n = \sum_{s=0}^{3} q_{n+s}e_s$$

and

$$MP_{k,n} = \sum_{s=0}^{3} q_{k,n+s}e_s$$

where $q_n$ is the $n$th Modified Pell number and $q_{k,n}$ is the $n$th Modified $k-$Pell number. She also presented a number of properties of these quaternions.

Catarino and Vasco [5] studied on dual $k-$Pell and dual $k-$Pell–Lucas quaternions which defined by the following relations:

$$\widehat{R_{k,n}} = R_{k,n} + \epsilon R_{k,n+1}$$

and

$$\widehat{S_{k,n}} = S_{k,n} + \epsilon S_{k,n+1}$$

where $R_{k,n}$ and $S_{k,n}$ are the $k-$Pell and $k-$Pell–Lucas quaternions respectively, and $\epsilon$ is the dual unit which satisfies $\epsilon^2 = 0$. They gave several properties of these quaternions including generating functions, Binet's formulas and some identities.

In this paper, we present a generalization of all the studies mentioned above on the algebra $\mathbb{H}(\lambda,\mu)$. Firstly, we introduce the definitions of the $k-$Fibonacci and $k-$Lucas generalized quaternions.

**Definition 1.1.** For any integer $r$, the $k-$Fibonacci and $k-$Lucas generalized quaternions are

$$G_{k,r} = F_{k,r} + F_{k,r+1}e_1 + F_{k,r+2}e_2 + F_{k,r+3}e_3 \tag{1.2}$$

and

$$H_{k,r} = L_{k,r} + L_{k,r+1}e_1 + L_{k,r+2}e_2 + L_{k,r+3}e_3 \tag{1.3}$$

respectively, where $F_{k,n}$ and $L_{k,n}$ are the $n$th $k-$Fibonacci and $k-$Lucas numbers.

We abbreviate the $k-$Fibonacci and $k-$Lucas generalized quaternions to $k$FGQ and $k$LGQ respectively. The following table presents how $G_{k,r}$ and $H_{k,r}$ generalize the quaternions mentioned above.

| $(\lambda, \mu)$ | $(1,1)$ | $(1,-1)$ |
|---|---|---|
| $k=1$ | Fibonacci quaternions [10] | Split Fibonacci quaternions [1] |
| $k=2$ | Pell quaternions [3, 18] | Split Pell quaternions [17] |
| $k$ | $k-$Fibonacci quaternions [16] | Split $k-$Fibonacci quaternions [14] |

Furthermore, the $k-$Fibonacci and $k-$Lucas Quaternions have the following recurrence relations.

**Corollary 1.1.** *For any integer $r$, we have*

$$G_{k,r} = kG_{k,r-1} + G_{k,r-2} \tag{1.4}$$

*and*

$$H_{k,r} = kH_{k,r-1} + H_{k,r-2}. \tag{1.5}$$

We give generalizations of some well-known identities in next sections.

## 2. Generating Functions and Binet's Formulas for the $k-$Fibonacci and $k-$Lucas Generalized Quaternions

The following theorem states generating functions for $k$FGQ and $k$LGQ.

**Theorem 2.1.** *The generating functions for $k$FGQ and $k$LGQ are*

$$G(x) = \sum_{n=0}^{\infty} G_{k,n}x^n = \frac{x + e_1 + (k+x)e_2 + (k^2+1+kx)e_3}{1-kx-x^2}$$

*and*

$$\begin{aligned} H(x) &= \sum_{r=0}^{\infty} H_{k,r}x^r \\ &= \frac{2-kx+(k+2x)e_1+(k^2+2+kx)e_2+[k^3+3k+(k^2+2)x]e_3}{1-kx-x^2} \end{aligned}$$

*respectively.*

The proof can be completed following similar steps as in [16]. Binet's formulas for $k$FGQ and $k$LGQ are given in the following theorem.

**Theorem 2.2.** *For any integer $r$, the $r$th $k$FGQ and $k$LGQ are, respectively,*

$$G_{k,r} = \frac{\widehat{\alpha}\alpha^r - \widehat{\beta}\beta^r}{\alpha - \beta} \tag{2.1}$$

*and*

$$H_{k,r} = \widehat{\alpha}\alpha^r + \widehat{\beta}\beta^r \tag{2.2}$$

*where $\widehat{\alpha} = 1 + \alpha e_1 + \alpha^2 e_2 + \alpha^3 e_3$ and $\widehat{\beta} = 1 + \beta e_1 + \beta^2 e_2 + \beta^3 e_3$.*

*Proof.* For the first equation, we have by Eq.(1.2)

$$\begin{aligned}
&\alpha G_{k,r} + G_{k,r-1} = \\
&\quad \alpha F_{k,r} + F_{k,r-1} + (\alpha F_{k,r+1} + F_{k,r})e_1 + (\alpha F_{k,r+2} + F_{k,r+1})e_2 \\
&\qquad\qquad\qquad\qquad\qquad\qquad + (\alpha F_{k,r+3} + F_{k,r+2})e_3.
\end{aligned}$$

Using the identity $\alpha^r = \alpha F_{k,r} + F_{k,r-1}$, we obtain

$$\alpha G_{k,r} + G_{k,r-1} = \widehat{\alpha}\alpha^r. \tag{2.3}$$

Similarly, we have

$$\beta G_{k,r} + G_{k,r-1} = \widehat{\beta}\beta^r. \tag{2.4}$$

After substraction Eq. (2.4) from Eq. (2.3), we get the Eq. (2.1). Moreover, Eq. (2.2) can be obtained similarly. □

Using the Binet's formulas for $k$FGQ and $k$LGQ, we investigate the properties of these quaternions.

**Lemma 2.1.** *We have*

$$\widehat{\alpha}\widehat{\beta} = u_1 + H_{k,0} + u_2\sqrt{k^2+4}, \tag{2.5}$$

$$\widehat{\beta}\widehat{\alpha} = u_1 + H_{k,0} - u_2\sqrt{k^2+4}, \tag{2.6}$$

$$(\widehat{\alpha})^2 = v_1 + H_{k,0} + \sqrt{k^2+4}(v_2 + G_{k,0}), \tag{2.7}$$

$$(\widehat{\beta})^2 = v_1 + H_{k,0} - \sqrt{k^2+4}(v_2 + G_{k,0}) \tag{2.8}$$

*where*

$$\begin{aligned}
u_1 &= \lambda\mu + \lambda - \mu - 1, \\
u_2 &= -\mu e_1 - \lambda k e_2 + e_3, \\
v_1 &= -\frac{\lambda\mu}{2}k^6 - \left(3\lambda\mu + \frac{\mu}{2}\right)k^4 \\
&\qquad - \left(\frac{9}{2}\lambda\mu + 2\mu + \frac{\lambda}{2}\right)k^2 - \lambda\mu - \lambda - \mu - 1,
\end{aligned}$$

and

$$v_2 = -\frac{\lambda\mu}{2}k^5 - \left(2\lambda\mu + \frac{\mu}{2}\right)k^3 - \left(\frac{3}{2}\lambda\mu + \frac{\lambda}{2} + \mu\right)k.$$

*Proof.* From the definition of $\widehat{\alpha}$ and $\widehat{\beta}$, we have

$$\begin{aligned}
\widehat{\alpha}\widehat{\beta} &= (1 + \alpha e_1 + \alpha^2 e_2 + \alpha^3 e_3)(1 + \beta e_1 + \beta^2 e_2 + \beta^3 e_3) \\
&= \lambda\mu + \lambda - \mu + 1 + ke_1 + (k^2+1)e_2 + (k^3+3k)e_3 \\
&\qquad\qquad + \sqrt{k^2+4}(-\mu e_1 - k\lambda e_2 + e_3) \\
&= u_1 + H_{k,0} + u_2\sqrt{k^2+4}.
\end{aligned}$$

The others can be proved similarly. □

This lemma gives us the following useful properties:

$$\widehat{\alpha}\widehat{\beta}+\widehat{\beta}\widehat{\alpha} = 2(u_1+H_{k,0}), \tag{2.9}$$

$$\widehat{\alpha}\widehat{\beta}-\widehat{\beta}\widehat{\alpha} = 2u_2\sqrt{k^2+4}, \tag{2.10}$$

$$(\widehat{\alpha})^2+(\widehat{\beta})^2 = 2(v_1+H_{k,0}), \tag{2.11}$$

$$(\widehat{\alpha})^2-(\widehat{\beta})^2 = 2\sqrt{k^2+4}(v_2+G_{k,0}). \tag{2.12}$$

We can give negative indices for $k$FGQ and $k$LGQ. Using the identities $F_{-r} = (-1)^{r+1}F_r$ and $L_{-r} = (-1)^r L_r$, we have

$$G_{k,-r} = (-1)^r\Big[-F_r+F_{r+1}e_1-F_{r+2}e_2+F_{r+3}e_3\Big]$$

and

$$H_{k,-r} = (-1)^r\Big[L_r-L_{r+1}e_1+L_{r+2}e_2-L_{r+3}e_3\Big].$$

## 3. Cassini's, Catalan's and d'Ocagne's Identities

In this section, we introduce Catalan's, Cassini's and d'Ocagne's identities for $k$FGQ and $k$LGQ. The following theorem provides the Catalan's identities.

**Theorem 3.1.** *For any integers $r$ and $s$, we have*

$$G_{k,r+s}G_{k,r-s}-G_{k,r}^2 = (-1)^{r+s+1}\left[(H_{k,0}+u_1)F_{k,s}^2+u_2F_{k,2s}\right] \tag{3.1}$$

*and*

$$H_{k,r+s}H_{k,r-s}-H_{k,r}^2 = (-1)^{r+s}(k^2+4)\left[(H_{k,0}+u_1)F_{k,s}^2+u_2F_{k,2s}\right]. \tag{3.2}$$

*Proof.* By using the Binet formula for $k$FGQ, we have

$$\begin{aligned}
G_{k,r+s}G_{k,r-s}-G_{k,r}^2 &= \frac{1}{k^2+4}\Big[\Big(\widehat{\alpha}\alpha^{r+s}-\widehat{\beta}\beta^{r+s}\Big)\Big(\widehat{\alpha}\alpha^{r-s}-\widehat{\beta}\beta^{r-s}\Big) \\
&\qquad -\Big(\widehat{\alpha}\alpha^{r}-\widehat{\beta}\beta^{r}\Big)^2\Big] \\
&= \frac{1}{k^2+4}\Big[-\widehat{\alpha}\widehat{\beta}\alpha^{r+s}\beta^{r-s}-\widehat{\beta}\widehat{\alpha}\beta^{r+s}\alpha^{r-s} \\
&\qquad +\widehat{\beta}\widehat{\alpha}\alpha^r\beta^r+\widehat{\alpha}\widehat{\beta}\alpha^r\beta^r\Big] \\
&= \frac{1}{k^2+4}\Big[-\alpha^{r-s}\beta^{r-s}\Big(\widehat{\alpha}\widehat{\beta}\alpha^{2s}+\widehat{\beta}\widehat{\alpha}\beta^{2s}\Big) \\
&\qquad +(-1)^r\,2\,(H_{k,0}+u_1)\Big].
\end{aligned}$$

Substituting Eqs. (2.5) and (2.6) into the last equation and making some elementary operations, we obtain

$$\begin{aligned}
G_{k,r+s}G_{k,r-s}-G_{k,r}^2 &= (-1)^{r+s+1}u_2F_{k,2s} \\
&\quad +\frac{1}{k^2+4}(-1)^r(H_{k,0}+u_1)[2-(-1)^sL_{k,2s}].
\end{aligned}$$

The identity $(k^2+4)F_{k,2s}^2 = L_{k,2s}-2(-1)^s$ gives the Eq. (3.1). Accordingly, Eq. (3.2) can be proved similarly. □

If we take $s=1$, we obtain Cassini's identities for $k$FGQ and $k$LGQ as in the following result.

**Corollary 3.1.** *For any integer $r$, we have*

$$G_{k,r+1}G_{k,r-1} - G_{k,r}^2 = (-1)^r \left[H_{k,0} + u_1 + ku_2\right] \tag{3.3}$$

*and*

$$H_{k,r+1}H_{k,r-1} - H_{k,r}^2 = (-1)^{r+1}(k^2+4)\left[H_{k,0} + u_1 + ku_2\right]. \tag{3.4}$$

In the next theorem, d'Ocagne's identities for the $k$FGQ and $k$LGQ are given.

**Theorem 3.2.** *For any integers $r$ and $s$, we have*

$$G_{k,r}G_{k,s+1} - G_{k,r+1}G_{k,s} = (-1)^s[(H_{k,0}+u_1)F_{k,r-s} + u_2 L_{k,r-s}] \tag{3.5}$$

*and*

$$H_{k,r}H_{k,s+1} - H_{k,r+1}H_{k,s} = (-1)^{s+1}(k^2+4)[(H_{k,0}+u_1)F_{k,r-s} + u_2 L_{k,r-s}]. \tag{3.6}$$

*Proof.* The Binet's formula for $k$FGQ gives

$$\begin{aligned} G_{k,r}G_{k,s+1} - G_{k,r+1}G_{k,s} \quad &= \\ &\frac{1}{k^2+4}\left[\left(\widehat{\alpha}\alpha^r - \widehat{\beta}\beta^r\right)\left(\widehat{\alpha}\alpha^{s+1} - \widehat{\beta}\beta^{s+1}\right)\right. \\ &\qquad \left. - \left(\widehat{\alpha}\alpha^{r+1} - \widehat{\beta}\beta^{r+1}\right)\left(\widehat{\alpha}\alpha^s - \widehat{\beta}\beta^s\right)\right] \\ &= \frac{\sqrt{k^2+4}}{k^2+4}(-1)^s\left[\widehat{\alpha}\widehat{\beta}\alpha^{r-s} - \widehat{\beta}\widehat{\alpha}\beta^{r-s}\right]. \end{aligned}$$

Substituting the Eqs. (2.5) and (2.6) into the last equation and making some operations, we get Eq. (3.5). The Eq. (3.6) can be handled in the same way. □

## 4. Some Identities for the $k-$Fibonacci and $k-$Lucas Generalized Quaternions

In this section, we present adaptations of some well-known identities between Fibonacci and Lucas numbers for $k$FGQ and $k$LGQ. The following identity explains the summation and subtraction of squares of $k$FGQ and $k$LGQ.

**Theorem 4.1.** *For any integer $r$, we have*

$$\begin{aligned} H_{k,r}^2 + G_{k,r}^2 \quad &= \quad \frac{k^2+5}{k^2+4}\Big[(v_1+H_{k,0})L_{k,2r} + (k^2+4)(v_2+G_{k,0})F_{k,2r}\Big] \\ &\qquad + \frac{2(-1)^r(k^2+3)}{k^2+4}(u_1+H_{k,0}) \end{aligned}$$

*and*

$$\begin{aligned} H_{k,r}^2 - G_{k,r}^2 \quad &= \quad \frac{k^2+3}{k^2+4}\Big[(v_1+H_{k,0})L_{k,2r} + (k^2+4)(v_2+G_{k,0})F_{k,2r}\Big] \\ &\qquad + \frac{2(-1)^r(k^2+5)}{k^2+4}(u_1+H_{k,0}). \end{aligned}$$

*Proof.* By using the Binet formulas for $k$FGQ and $k$LGQ, we have

$$\begin{aligned} H_{k,r}^2 + G_{k,r}^2 \quad &= \quad \frac{1}{k^2+4}\left(\widehat{\alpha}\alpha^r - \widehat{\beta}\beta^r\right)^2 + \left(\widehat{\alpha}\alpha^r + \widehat{\beta}\beta^r\right)^2 \\ &= \quad \frac{1}{k^2+4}\left[(\widehat{\alpha})^2\alpha^{2r} + (\widehat{\beta})^2\beta^{2r} - \widehat{\alpha}\widehat{\beta}\alpha^r\beta^r - \widehat{\beta}\widehat{\alpha}\alpha^r\beta^r\right] \\ &\qquad + \left[(\widehat{\alpha})^2\alpha^{2r} + (\widehat{\beta})^2\beta^{2r} + \widehat{\alpha}\widehat{\beta}\alpha^r\beta^r + \widehat{\beta}\widehat{\alpha}\alpha^r\beta^r\right]. \end{aligned}$$

By substituting $(\widehat{\alpha})^2, (\widehat{\beta})^2$ from Lemma 2.1 and $\widehat{\alpha}\widehat{\beta}+\widehat{\beta}\widehat{\alpha}$ from Eq. (2.9) into the last equation and simplifying the result with some elementary operations, we obtain the first identity in theorem. The second identity can be proved in a similar way. □

Now, we provide several interesting identities in the following theorems.

**Theorem 4.2.** *For any integer $r$, we have*

$$\sum_{i=0}^{r}\binom{r}{i}k^iG_{k,i}=G_{k,2r}$$

*and*

$$\sum_{i=0}^{r}\binom{r}{i}k^iH_{k,i}=H_{k,2r}.$$

*Proof.* For $k$FGQ, we have

$$\begin{aligned}\sum_{i=0}^{r}\binom{r}{i}k^iG_{k,i} &= \sum_{i=0}^{r}\binom{r}{i}k^i\left[\frac{\widehat{\alpha}\alpha^r-\widehat{\beta}\beta^r}{\alpha-\beta}\right]\\ &= \frac{\widehat{\alpha}}{\alpha-\beta}\sum_{i=0}^{r}\binom{r}{i}k^i\alpha^i-\frac{\widehat{\beta}}{\alpha-\beta}\sum_{i=0}^{r}\binom{r}{i}k^i\beta^i\\ &= \frac{\widehat{\alpha}}{\alpha-\beta}(1+k\alpha)^r-\frac{\widehat{\beta}}{\alpha-\beta}(1+k\beta)^r.\end{aligned}$$

Since $\alpha$ and $\beta$ are roots of the equation $x^2-kx-1=0$, we have $1+k\alpha=\alpha^2$ and $1+k\beta=\beta^2$. Using these two equations, we obtain

$$\begin{aligned}\sum_{i=0}^{r}\binom{r}{i}k^iG_{k,i} &= \frac{\widehat{\alpha}\alpha^{2r}-\widehat{\beta}\beta^{2r}}{\alpha-\beta}\\ &= G_{k,2r}.\end{aligned}$$

In the same manner, we can see the second identity. □

**Theorem 4.3.** *For any integers $r$, $s$ and $t$, we have*

$$H_{k,r+s}G_{k,r+t}-H_{k,r+t}G_{k,r+s}=2(-1)^{r+s}(u_1+H_{k,0})F_{t-s}.$$

*Proof.* Using Binet's formulas for $k$FGQ and $k$LGQ, we have

$$\begin{aligned}&H_{k,r+s}G_{k,r+t}H_{k,r+t}G_{k,r+s}\\ &\quad=\frac{1}{\sqrt{k^2+4}}\Big[\left(\widehat{\alpha}\alpha^{r+s}+\widehat{\beta}\beta^{r+s}\right)\left(\widehat{\alpha}\alpha^{r+t}-\widehat{\beta}\beta^{r+t}\right)\\ &\qquad-\left(\widehat{\alpha}\alpha^{r+t}+\widehat{\beta}\beta^{r+t}\right)\left(\widehat{\alpha}\alpha^{r+s}-\widehat{\beta}\beta^{r+s}\right)\Big]\\ &\quad=\frac{1}{\sqrt{k^2+4}}\Big[-\widehat{\alpha}\widehat{\beta}\alpha^{r+s}\beta^{r+t}+\widehat{\beta}\widehat{\alpha}\beta^{r+s}\alpha^{r+t}\\ &\qquad+\widehat{\alpha}\widehat{\beta}\alpha^{r+t}\beta^{r+s}-\widehat{\beta}\widehat{\alpha}\alpha^{r+s}\beta^{r+t}\Big]\\ &\quad=\frac{(\widehat{\alpha}\widehat{\beta}+\widehat{\beta}\widehat{\alpha})}{\sqrt{k^2+4}}\Big[-\alpha^{r+s}\beta^{r+t}+\alpha^{r+t}\beta^{r+s}\Big].\end{aligned}$$

After we substitute Eq. (2.9) into the last equation, we obtain the result of the theorem. □

**Theorem 4.4.** *For any integers $r$ and $s$, we have*

$$G_{k,r+s} + (-1)^s G_{k,r-s} = G_{k,r} L_{k,s}.$$

*Proof.* Binet's formulas for $k$FGQ and $k$LGQ give

$$\begin{aligned} G_{k,r+s} + (-1)^s G_{k,r-s} &= \frac{1}{\sqrt{k^2+4}} \Big[ \widehat{\alpha}\alpha^{r+s} - \widehat{\beta}\beta^{r+s} \\ &\quad + (-1)^s \left( \widehat{\alpha}\alpha^{r-s} - \widehat{\beta}\beta^{r-s} \right) \Big] \\ &= \frac{1}{\sqrt{k^2+4}} \Big[ \widehat{\alpha}\alpha^{r+s} - \widehat{\beta}\beta^{r+s} \\ &\quad + \left( \widehat{\alpha}\alpha^{r}\beta^{s} - \widehat{\beta}\alpha^{s}\beta^{r} \right) \Big] \\ &= \frac{1}{\sqrt{k^2+4}} \Big[ \widehat{\alpha}\alpha^{r} \left( \alpha^s + \beta^s \right) - \widehat{\beta}\beta^{r} \left( \alpha^s + \beta^s \right) \Big] \\ &= G_{k,r} L_{k,s}. \end{aligned}$$

□

We know that $\mathbb{H}(\lambda, \mu)$ is non-commutative. It can be seen what happens when any two $k$FGQs or $k$LGQs are displaced.

**Theorem 4.5.** *For any integers $r$ and $s$, we have*

$$G_{k,r} G_{k,s} = G_{k,s} G_{k,r} + 2(-1)^{s+1} u_2 F_{k,r-s}$$

*and*

$$H_{k,r} H_{k,s} = H_{k,s} H_{k,r} + 2(-1)^{s} (k^2+4) u_2 F_{k,r-s}.$$

*Proof.* For the first equation in theorem, from the Binet formula for $k$FGQ, we have

$$\begin{aligned} G_{k,r} G_{k,s} - G_{k,s} G_{k,r} &= \frac{1}{k^2+4} \Big[ \left( \widehat{\alpha}\alpha^{r} - \widehat{\beta}\beta^{r} \right) \left( \widehat{\alpha}\alpha^{s} - \widehat{\beta}\beta^{s} \right) \\ &\quad - \left( \widehat{\alpha}\alpha^{s} - \widehat{\beta}\beta^{s} \right) \left( \widehat{\alpha}\alpha^{r} - \widehat{\beta}\beta^{r} \right) \Big] \\ &= \frac{1}{k^2+4} \Big[ - \widehat{\alpha}\widehat{\beta}\alpha^{r}\beta^{s} - \widehat{\beta}\widehat{\alpha}\alpha^{s}\beta^{r} + \widehat{\alpha}\widehat{\beta}\alpha^{s}\beta^{r} + \widehat{\beta}\widehat{\alpha}\alpha^{s}\beta^{r} \Big]. \end{aligned}$$

Substituting Eqs. (2.5) and (2.6) into the last equation and making some simplifications get the first equation in theorem. Taking similar steps, the second identity in theorem can be obtained. □

In the next theorem, we give some interesting identities for $k$FGQ and $k$LGQ without proof.

**Theorem 4.6.** *We have the followings*

$$
\begin{aligned}
&G_{k,r}H_{k,s} = G_{k,s}H_{k,r} + 2(-1)^{r+1}(u_1 + H_{k,0})F_{k,s-r},\\
&G_{k,r}H_{k,s} = H_{k,r}G_{k,s} + 2(-1)^{r+1}\Big[(u_1 + H_{k,0})F_{k,s-r} - u_2 L_{k,s-r}\Big],\\
&G_{k,r+s}F_{r+s} - G_{k,r-s}F_{r-s} = F_{k,2s}G_{k,2r},\\
&H_{k,r+s}L_{r+s} - H_{k,r-s}L_{r-s} = (k^2+4)F_{k,2s}G_{k,2r},\\
&G_{k,r} = kG_{k,r-1} + G_{k,r-2},\\
&H_{k,r} = kH_{k,r-1} + H_{k,r-2},\\
&H_{k,r} = G_{k,r-1} + G_{k,r+1},\\
&\sum_{n=0}^{\infty} G_{k,n+m}x^n = \frac{G_{k,m} + xG_{k,m-1}}{1 - kx - x^2},\\
&\sum_{n=0}^{\infty} H_{k,n+m}x^n = \frac{H_{k,m} + xH_{k,m-1}}{1 - kx - x^2},
\end{aligned}
$$

$$
\begin{aligned}
G_{k,r} - e_1G_{k,r+1} - e_2G_{k,r+2} - e_3G_{k,r+3}
&= \left[1 + \lambda + (k^2+1)\mu + (k^4 + 3k^2 + 1)\lambda\mu\right] F_{k,r}\\
&\quad + \left[k\lambda + (k^3 + 2k)\mu + (k^5 + 4k^3 + 3k)\lambda\mu\right] F_{k,r+1}.
\end{aligned}
$$

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