Konuralp Journal of Mathematics
Volume 5 No. 2 pp. 114–118 (2017) ©KJM

TWO EXAMPLES ON UNIONS OF ABELIAN p-GROUPS

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Abstract. We give two constructive examples of groups that are special unions of abelian p-groups. This sheds some light on certain problematic aspects in the present subject.

1. Introduction and Definitions

All groups considered herein are *abelian p-torsion* for some arbitrary but fixed prime p. The classical terminology and notation are in agreement with [8]. For instance, if G and A are arbitrary groups, then $G\nabla A$ denotes the torsion product between them. Likewise, $p^\omega G = \cap_{n<\omega} p^n G$ is the subgroup of G consisting of all elements of infinite height, and G is called *separable* provided $p^\omega G = \{0\}$. The other specific notions and notations follow mainly [7] and some of them are listed below.

P. Hill showed in [9] that the p-group G is a direct sum of cyclic groups if, and only if, it is the countable set-theoretic union of pure subgroups which are direct sums of cyclic groups. Such unions are not necessarily ascending.

On the other vein, it was stated in [2] and [3] (for more detailed information see [4], [5] and [6], respectively, as well) the notion of *nice basis* for a group G thus: A p-group G is said to have *a nice basis* provided it is the countable group-theoretic (i.e., an ascending) union of nice subgroups which are direct sums of cyclic groups. It was established in [3] that both separable p-groups and simply presented p-groups possess nice bases.

In view of these two settings above, it is quite natural to formulate the following concept:

Definition 1.1. A p-group G has an *almost nice basis* if G is the countable set-theoretic union $\cup_{n<\omega} G_n$ of subgroups G_n which are nice in G and are direct sums of cyclic groups.

Date: March 11, 2017 and, in revised form, May 31, 2017.
2010 *Mathematics Subject Classification.* 20K10.
Key words and phrases. Abelian Groups, Balanced Subgroups, Nice Subgroups, Unions.

In what follows we shall show that if G is a group of length strictly greater than $\omega \cdot 2$, then it does not contain in general an almost nice basis, thus expanding a result from [3] (compare with Example 2.1 below).

The above can be generalized like this:

Definition 1.2. A group G is said to have a λ-*nice basis* if $G = \cup_{\alpha<\lambda} G_\alpha$ is a smooth union, where $G_\alpha \subseteq G_{\alpha+1} \leq G$ are nice in G direct sums of cyclic groups. If, in addition, all G_α are pure in G, it is said to be a group with a λ-*nice pure basis.*

When $\lambda = \omega$, we will obtain the ordinary nice basis for a group. Apparently, if $\lambda \geq \omega_1$, then any group G with a λ-nice basis is, itself, separable, so that it has an ω-nice basis (i.e., a nice basis). Here we will be mainly concentrated on $\lambda = \omega_1$.

In that way, utilizing results from [1], it follows that if G and A are groups with an ω_1-nice pure basis, whose cardinalities are not really big, then $G \nabla A$ is a direct sum of cyclic groups. In fact, [1, Corollary 7] works to get the pursued assertion (see also the proof of [1, Theorem 33]).

The next concept is a folklore (see, e.g., [10] and [11]): A subgroup H of a p-group G is called *separable* if, for each $g \in G$, there is a corresponding countable subgroup K of H such that the following equality holds:

$$sup\{\text{height}_G(g+k) \ : \ k \in K\} = sup\{\text{height}_G(g+h) \ : \ h \in H\}.$$

Clearly, nice subgroups are themselves separable.

Definition 1.3. A p-group G is said to have a *weak nice basis* if $G = \cup_{n<\omega} G_n$, where $G_n \subseteq G_{n+1} \leq G$ and all G_n are separable in G direct sums of cyclic groups.

Thus, it is self-evident that, every group with a nice basis also possesses a weak nice basis.

Let us now recollect that a subgroup S of a p-group G is said to be *closed* (in the p-adic topology), provided G/S is a separable group (i.e., without elements of infinite height), that is, $p^\omega(G/S) = \{0\}$. Routinely, closed subgroups are themselves nice.

Definition 1.4. A p-group G is said to have a *strong nice basis* if $G = \cup_{n<\omega} G_n$, where $G_n \subseteq G_{n+1} \leq G$ and all G_n are closed in G direct sums of cyclic groups.

So, any group with a strong nice basis will possess a nice basis as well. However, these two concepts do coincide when the former group is separable, because for subgroups the properties "closed" and "nice" are then equivalent.

In regard to the uncountable unions stated above in Definition 1.2, we shall illustrate in the sequel that even balanced subgroups in such unions will not be enough to obtain a direct sum of cyclic groups (compare with Example 2.2 below).

2. The Examples

We distribute our two construction into the following two subsections as follows:

2.1. **Almost Nice Basis.** We start here with

Example 2.1. There is a p-group G of length beyond $\omega \cdot 2$ such that $p^\omega G$ is countable and such that G does not have an almost nice basis.

Proof. Let $B = \bigoplus_{n<\omega} \mathbb{Z}(p^n)$ be the standard direct sum of cyclic groups and let G be any group such that

(1) $G/p^\omega G = \overline{B}$ is torsion-complete;

(2) $p^\omega G$ is a reduced countable group which is not a direct sum of cyclic groups.

Suppose $\{N_n\}$ for $n < \omega$ is a collection of nice subgroups of G which are direct sums of cyclic groups and $\bigcup_{n<\omega} N_n = G$. For each $n < \omega$, let

$$N'_n = ([N_n + p^\omega G]/p^\omega G) \cap \overline{B}[p].$$

Therefore, $\overline{B}[p]$, which is a complete metric space, is the union of the N'_n.

Claim 1: For some $n < \omega$ and $m < \omega$ we assert that $(p^m\overline{B})[p] \subseteq N'_n$.

If for some fixed $n < \omega$ we have that $(p^m\overline{B})[p]$ is not contained in N'_n for all $m < \omega$, then N'_n will be nowhere dense in $\overline{B}[p]$. Since $\overline{B}[p]$ is a complete metric space, our Claim 1 follows from the classical well-known Baire Category Theorem.

Claim 2: $p^\omega G/N_n(\omega) = p^\omega G/(N_n \cap p^\omega G)$ is divisible.

Let $x \in p^\omega G$, and choose $y \in G$ such that $py = x$ and $\text{height}_G(y) \geq m$. So, $y + p^\omega G \in (p^m\overline{B})[p] \subseteq N'_n$. It follows that there is an element $z \in N_n$ such that $y + p^\omega G = z + p^\omega G$. Consequently, $y = z + w$, where $w \in p^\omega G$, so that $x = py = pz + pw \in N_n(\omega) + p(p^\omega G)$. Thereby, we have $p^\omega G = N_n(\omega) + p(p^\omega G)$, which establishes our Claim 2.

Furthermore, since G is reduced and $N_n(\omega)$ is nice in $p^\omega G$, it follows that $p^\omega G = N_n(\omega) \subseteq N_n$ and, in particular, $p^\omega G$ must be a direct sum of cyclic groups. However, this contradicts the construction of the initial group G. □

2.2. **Balanced Union.** Recall that if κ is a cardinal, then a p-group G is *κ-Σ-cyclic* if every subgroup K of G with $|K| < \kappa$ is a direct sum of cyclic groups.

We begin here with a demonstration that the aforementioned Hill's theorem is *not* longer true for uncountable unions.

Example 2.2. There is a separable p-group G which is *not* a direct sum of cyclic groups which is the smoothly ascending union of balanced subgroups $\{B_i\}_{i<\omega_1}$ such that each B_i is a direct sum of cyclic groups.

Proof. Using the notation from [1], let B be a separable p-group such that $\{\aleph_2\} \in \mathcal{K}_B$ (e.g., any torsion-complete p-group of cardinality at least $\aleph_2$). Let T be a p-group of cardinality $\aleph_1$ which is $\aleph_1$-Σ-cyclic but *not* a direct sum of cyclic groups, i.e., with $\{\aleph_1\} \in \mathcal{K}_T$. If $G = T\nabla B$, it follows that G is a separable group and $\{\aleph_1, \aleph_2\} \in \mathcal{K}_G$. In particular, G will not be a direct sum of cyclic groups, though it is an almost direct sum of cyclic groups (see [10]).

On the other hand, we can clearly express T as the smoothly ascending union of pure subgroups T_i for $i < \omega_1$. Letting $B_i := T_i\nabla B$, we then have that:

(a) G is the smoothly ascending union of the $\{B_i\}_{i<\omega_1}$, which follows readily from general properties of the torsion product.

(b) Every subgroup B_i is balanced in G – in fact, there exists a (pure) short-exact sequence

$$0 \to T_i \nabla B(= B_i) \to T \nabla B(= G) \to (T/T_i) \nabla B(= G/B_i) \to 0.$$

Therefore, since G is separable, any B_i must be isotype in G. In addition,

$$p^\omega(G/B_i) \cong p^\omega(T/T_i) \nabla p^\omega B = p^\omega(T/T_i) \nabla \{0\} = \{0\},$$

so that any B_i is closed (that is, nice) in G. $\square$

3. Left-open Problems

Here we shall pose a few problems of interest for the subject.

First, it is well known that a subgroup S of a p-group G is called *saturated* if, for each $g \in G$ and some $n \in \mathbb{N}$ depending on g with $ng \in S$, it must be that $g \in S$. Are saturated subgroups themselves nice?

Second, we begin with questions concerning uncountable smooth balanced unions of p-primary abelian groups. Suppose $A = \cup_{\alpha<\mu} A_\alpha$ is a smooth union, where $A_\alpha \subseteq A_{\alpha+1}$ are balanced subgroups of A and $|\mu| \geq \aleph_1$.

Problem 1. If A_α is an almost totally projective group (in particular, an almost direct sum of countable groups) for all α, is then A an almost totally projective group (in particular, an almost direct sum of countable groups)?

The eventual positive resolution of this query will considerably extend a result due to Balof-Keef from [1] established for almost direct sums of cyclic groups.

Problem 2. Describe groups with weak nice bases. Does it follow that they have similar properties to these of groups with (almost) nice bases?

Problem 3. Describe groups with strong nice bases. Is it true that they have analogous properties to these of groups with (almost) nice bases?

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