



## ON A CLASS OF STRONGLY $L_p$ -SUMMING SUBLINEAR OPERATORS AND THEIR PIETSCH DOMINATION THEOREM

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ABSTRACT. In this paper, we study a class of non commutative strongly  $l_p$ -summing sublinear operators and characterize this class of operators by given the extension of the Pietsch domination theorem. Some new properties are shown.

### 1. Introduction

The concept of strongly  $p$ -summing linear operators was introduced by Cohen [5] as a characterization of the conjugates of absolutely  $p^*$ -summing linear operators. A linear operator  $u$  between two Banach spaces  $X, Y$  is strongly  $p$ -summing for  $(1 < p \leq \infty)$  if there is a positive constant  $C$  such that for all  $n \in \mathbb{N}$ ,  $x_1, \dots, x_n \in X$  and  $y_1^*, \dots, y_n^* \in Y^*$ , we have

$$(1.1) \quad \left\| \left( \langle u(x_i), y_i^* \rangle \right)_{1 \leq i \leq n} \right\|_{l_1^n} \leq C \| (x_i) \|_{l_p^n(X)} \sup_{y \in B_Y} \| (y_i^*(y)) \|_{l_{p^*}^n}.$$

The smallest constant  $C$  which is noted by  $d_p(u)$ , such that the inequality (1.1) holds, is called the strongly  $p$ -summing norm on the space  $\mathcal{D}_p(X, Y)$  of all strongly  $p$ -summing linear operators from  $X$  into  $Y$  which is a Banach space. We have  $\mathcal{D}_1(X, Y) = \mathcal{B}(X, Y)$ , the vector space of all bounded linear operators from  $X$  into  $Y$ . In Achour et al. [2], the authors generalized this notion to the class of sublinear operators (which are positively homogenous and subadditive). Cohen deduced the domination theorem simply from the adjoint operator which is  $p^*$ -summing. That is not the case for sublinear operators because we have shown the Pietsch domination theorem by using Ky Fan's lemma.

Our main goal in this paper is to generalize the concept of strongly summing linear operator in the non commutative case. We characterize this type of operators by given with extension of the Pietsch domination theorem. The proof is different than that used in [2] because it is not adaptable to our case.

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The rest of this paper is organized as follows. Section 2 begins with a general formulation of our problem and gives the notations and assumptions used throughout the paper. In Section 3, we establish our main result.

## 2. Preliminaries

In this section, we recall some basic definitions and properties concerning the notion of sublinear operators and the theory of operator spaces (we consider that the reader is familiarized with this category).

If  $\mathcal{H}$  is a Hilbert space, we let  $B(\mathcal{H})$  denote the space of all bounded operators on  $\mathcal{H}$  and for every  $n$  in  $\mathbb{N}$  we let  $M_n$  denote the space of all  $n \times n$ -matrices of complex numbers, i.e.,  $M_n = B(l_2^n)$ . If  $X$  is a subspace of some  $B(\mathcal{H})$  and  $n \in \mathbb{N}$ , then  $M_n(X)$  denotes the space of all  $n \times n$ -matrices with  $X$ -valued entries which we, in the natural manner consider, as a subspace of  $B(l_2^n(X))$ . An *operator space* which

is a Banach lattice (resp. complete Banach lattice) is called a quantum Banach lattice (resp. quantum complete Banach lattice).

Let  $\mathcal{H}$  be a Hilbert space. We denote by  $S_p(\mathcal{H})$  ( $1 \leq p < \infty$ ) the Banach space of all compact operators  $u : \mathcal{H} \rightarrow \mathcal{H}$  such that  $Tr(|u|^p) < \infty$ , equipped with the norm

$$\|u\|_{S_p(\mathcal{H})} = (Tr(|u|^p))^{\frac{1}{p}}.$$

If  $\mathcal{H} = l_2$  (resp.  $l_2^n$ ), we denote simply  $S_p(l_2)$  by  $S_p$  (resp.  $S_p(l_2^n)$  by  $S_p^n$ ). We denote also by  $S_\infty(\mathcal{H})$  (resp.  $S_\infty$ ) the Banach space of all compact operators equipped with the norm induced by  $B(\mathcal{H})$  (resp.  $B(l_2)$ ) ( $S_\infty^n = B(l_2^n)$ ). Recall that if  $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$  ( $1 \leq p, q, r < \infty$ ), then

$$u \in B_{S_p(\mathcal{H})} \text{ iff there are } u_1 \in B_{S_q(\mathcal{H})}, u_2 \in B_{S_r(\mathcal{H})} \text{ such that } u = u_1 u_2,$$

where  $B_{S_p(\mathcal{H})}$  is the closed unit ball of  $S_p(\mathcal{H})$ . We also denote by  $S_p^+(\mathcal{H}) = \{a \in S_p(\mathcal{H}) : a \geq 0\}$ .

Let  $\mathcal{H}_1, \mathcal{H}_2$  be Hilbert spaces. Let  $X \subset B(\mathcal{H}_1)$  and  $Y \subset B(\mathcal{H}_2)$  be operator spaces. A linear operator  $u : X \rightarrow Y$  is called completely bounded (in short *c.b.*) if the operators

$$\begin{aligned} u_n & : M_n(X) & \longrightarrow & M_n(Y), \\ & (x_{ij})_{1 \leq i, j \leq n} & \longmapsto & (u(x_{ij}))_{1 \leq i, j \leq n}, \end{aligned}$$

are uniformly bounded when  $n \rightarrow \infty$ , i.e.,  $\sup \{\|u_n\|, n \geq 1\} < \infty$ .

In this case we put  $\|u\|_{cb} = \sup \{\|u_n\|, n \geq 1\}$  and we denote by  $cb(X, Y)$  the Banach space of all *c.b.* maps from  $X$  into  $Y$  which is also an operator space ( $M_n(cb(X, Y)) = cb(X, M_n(Y))$ ) (see Blecher & Paulsen [4] and Effros & Ruan [6]). We denote also by  $X \otimes_{\min} Y$  a subspace of  $B(\mathcal{H} \otimes_2 \mathcal{K})$  with induced norm.

Consider  $Y \subset A$  (a commutative  $C^*$ -algebra)  $\subset B(\mathcal{H})$  and let  $X$  be an arbitrary operator space. Then,

$$\mathcal{B}(X, Y) = cb(X, Y),$$

and

$$(2.1) \quad \|u\| = \|u\|_{cb}.$$

We have  $M_n \otimes_{\min} Y \equiv M_n \otimes_{\epsilon} Y$  is isometrically ( $M_n \otimes_{\epsilon} Y$  is the injective tensor product of  $M_n$  by  $Y$  in the commutative case).(see Blecher & Paulsen [4] and Effros & Ruan [6]).

Let  $\mathcal{OH}$  be the Hilbert operator space introduced by Pisier in [10, Proposition 1.5, p. 18]. We recall that  $\mathcal{OH}$  is homogeneous, namely, every bounded linear operator  $u : \mathcal{OH} \rightarrow \mathcal{OH}$  is automatically *c.b.* and

$$(2.2) \quad \|u\| = \|u\|_{cb}.$$

Before continuing our notation we announce the following property. It will be needed in the sequel.

Let  $X \subset B(\mathcal{H})$  be an operator space. For all  $n$  in  $\mathbb{N}$  and  $1 \leq p < \infty$ , we have

$$(2.3) \quad \|v\|_{cb} = \sup_{a,b \in B_{S_{2p}^+(\mathcal{H})}^+} \left( \sum_1^n \|ax_i b\|_{S_p(\mathcal{H})}^p \right)^{\frac{1}{p}} = \left\| \sum_1^n e_j \otimes x_j \right\|_{l_p^n \otimes_{\min} X},$$

If  $p = \infty$ , we have

$$(2.4) \quad \|v\|_{cb} = \left\| \sum_1^n e_j \otimes x_j \right\|_{l_{\infty}^n \otimes_{\min} X} = \left\| \sum_1^n e_j \otimes x_j \right\|_{l_{\infty}^n \otimes_{\epsilon} X} = \|v\|,$$

and  $v : l_{p^*}^n \rightarrow X$  such that  $v(e_i) = x_i$  ( $p^*$  is the conjugate of  $p$ , i.e.,  $\frac{1}{p} + \frac{1}{p^*} = 1$ ).(see Blecher & Paulsen [4] , Effros & Ruan [6] and Mazrag [8, 8]).

Now, let  $X$  be an operator space. As usual we denote by  $l_p(X)$  (resp.  $l_p^n(X)$ ) for  $1 \leq p < \infty$  the space of sequences  $(x_1, \dots, x_n, \dots)$  (resp. finite sequences  $(x_1, \dots, x_n)$ ) in  $X$  equipped with the norm  $(\sum_{n=1}^{\infty} \|x_n\|^p)^{\frac{1}{p}} < \infty$  (resp.  $(\sum_{i=1}^n \|x_i\|^p)^{\frac{1}{p}}$ ) with which becomes an operator space. For more informations on this the reader can consult Pisier [9].

### 3. Main result

We will extend to sublinear operators the class of strongly  $p$ -summing operators defined in 1973 by Cohen [5]. We prove directly the principal result of this work, which is a Pietsch domination-type theorem. In [2], the authors used Ky Fan's lemma which is not adjustable in the non commutative case. For the linear case, Cohen deduce it obviously by duality because the adjoint of an operator strongly  $p$ -summing is absolutely  $p^*$ -summing. That is not the case for sublinear operators.

For the convenience of the reader, we recall the definition of sublinear operators. For more details see Achour & Mezrag [1] and Tiaiba [11].

**Definition 1.** *An operator  $T$  from a Banach space  $X$  into a Banach lattice  $Y$  is said to be sublinear if for all  $x, y$  in  $X$  and  $\lambda$  in  $R_+$ , we have*

- (i)  $T(\lambda x) = \lambda T(x)$  (i.e., positively homogeneous),
- (ii)  $T(x + y) \leq T(x) + T(y)$  (i.e., subadditive).

Note that the sum of two sublinear operators is a sublinear operator and the multiplication by a positive number is also a sublinear operator.

Let us denote by

$$\mathcal{SB}(X, Y) = \{\text{bounded sublinear operators } T : X \longrightarrow Y\}.$$

We define the class of strongly  $l_p$ -summing sublinear operators as follows.

**Definition 2.** *Let  $X$  be a Banach space and  $Y$  be a quantum Banach lattice. A sublinear operator  $T : X \longrightarrow Y$  is strongly  $l_p$ -summing ( $1 < p < \infty$ ), if there is a positive constant  $C$  such that for any  $n \in \mathbb{N}$ ,  $x_1, \dots, x_n \in X$  and  $y_1^*, \dots, y_n^* \in Y^*$ , we have*

$$(3.1) \quad \|(\langle T(x_i), y_i^* \rangle)\|_{l_1^n} \leq C \|x_i\|_{l_p^n(X)} \sup_{a, b \in B_{S_{2p^*}}^+} \|(ay_i^*b)\|_{l_p^n(S_{p^*}(\mathcal{K}))}.$$

Where  $\mathcal{K}$  is a Hilbert space such that  $Y^* \subset B(\mathcal{K})$ , because by Blecher [3] is an operator space and consequently a quantum complete Banach lattice.

We denote by  $\mathcal{D}_p(X, Y)$  the class of all strongly  $l_p$ -summing sublinear operators from  $X$  into  $Y$  and by  $d_{l_p}(T)$  the smallest constant  $C$  such that the inequality (3.1) holds.

Consider  $T \in \mathcal{SB}(X, Y)$ . The operator  $T$  is strongly  $p$ -summing if and only if, for all  $n \in \mathbb{N}$  and all  $v \in \mathcal{B}(l_p^n, Y^*)$  ( $v(e_i) = y_i^*$  or  $v = \sum_{i=1}^n e_i \otimes y_i^*$ ), we have by (2.3)

$$(3.2) \quad \sum_{i=1}^n |\langle T(x_i), v(e_i) \rangle| \leq C \left( \sum_{i=1}^n \|x_i\|^p \right)^{\frac{1}{p}} \|v\|_{cb}.$$

For  $p = 1$ , we have  $\mathcal{D}_{l_1}(X, Y) = \mathcal{SB}(X, Y)$ .

The proof of the following proposition is easy by using (3.2).

**Proposition 3.** *Let  $E, X$  be Banach spaces and  $Y, F$  be quantum Banach lattices. Let  $T \in \mathcal{SB}(X, Y)$ ,  $R \in \mathcal{B}(Y, F)$  and  $S \in \mathcal{B}^+(E, X)$  (i.e.,  $S(x) \geq 0$ , for all  $x \geq 0$ ). We have that*

(i) *If  $T$  is strongly  $l_p$ -summing, then  $R \circ T$  is strongly  $l_p$ -summing and*

$$d_{l_p}(RT) \leq \|R\| d_{l_p}(T).$$

(ii) *If  $T$  is strongly  $l_p$ -summing, then  $T \circ S$  is strongly  $l_p$ -summing and*

$$d_{l_p}(T \circ S) \leq \|S\| d_{l_p}(T) \|S\|.$$

The main result of this paper is the following extension of the Pietsch domination theorem for sublinear operators. We note that a non-commutative multilinear version has been given by Mezrag in [7, 8].

**Theorem 4.** *Let  $X$  be a Banach space and  $Y$  be a quantum Banach lattice. An operator  $T$  in  $\mathcal{SB}(X, Y)$  is strongly  $l_p$ -summing ( $1 < p \leq \infty$ ), if there is a set  $I$ , a families  $a_\alpha, b_\alpha$  in  $B_{S_{2p^*}}^+$  and an ultrafilter  $\mathcal{U}$  on  $I$  such that for all  $x$  in  $X$  and  $y^*$*

in  $Y^*$ , we have

$$(3.3) \quad |\langle T(x), y^* \rangle| \leq d_{l_p}(T) \|x\| \lim_{\mathcal{U}} \|a_\alpha y^* b_\alpha\|_{S_{p^*}(\mathcal{K})}.$$

Conversely, if there is a positive constant  $C$ , a set  $I$ , a families  $a_\alpha, b_\alpha$  in  $B_{S_{2p^*}}^+$  and an ultrafilter  $\mathcal{U}$  on  $I$  such that for all  $x$  in  $X$  and  $y^*$  in  $Y^*$ , we have

$$(3.4) \quad |\langle T(x), y^* \rangle| \leq C \|x\| \lim_{\mathcal{U}} \|a_\alpha y^* b_\alpha\|_{S_{p^*}(\mathcal{K})},$$

then  $T \in \mathcal{D}_{l_p}(X, Y)$  and  $d_{l_p}(T) \leq C$ .

*Proof.* We prove the first implication. Consider

$$S = \left\{ (a, b) \in B_{S_{2p^*}(\mathcal{K})} \times B_{S_{2p^*}(\mathcal{K})} : a, b \geq 0 \right\}$$

where  $\mathcal{K}$  is the Hilbert space such that  $Y^* \subset B(\mathcal{K})$ .

Let  $\mathcal{C}$  be the set of all functions on  $S$  with values in  $\mathbb{R}$  of the form

$$f_{((x_i), (y_i^*))}(a, b) = \sum_{i=1}^n (d_{l_p}(T) \left( \frac{1}{p} \|x\|^p + \frac{1}{p^*} \|ay_i^* b\|_{S_{p^*}(\mathcal{K})}^{p^*} \right) - |\langle T(x_i), y_i^* \rangle|).$$

where  $(x_i)_{1 \leq i \leq n} \subset X$  and  $(y_i^*)_{1 \leq i \leq n} \subset Y^*$ .

$\mathcal{C}$  is a convex cone. Indeed, let  $g, h$  be in  $\mathcal{C}$  and  $\lambda \geq 0$  such that

$$\begin{aligned} g_{((x'_i), (y'_i{}^*))}(a, b) = \\ \sum_{i=1}^k \left( \frac{d_{l_p}(T)}{p} \|x'_i\|^p + \frac{d_{l_p}(T)}{p^*} \|ay'_i{}^* b\|_{S_{p^*}(\mathcal{K})}^{p^*} - |\langle T(x'_i), y'_i{}^* \rangle| \right), \end{aligned}$$

and

$$\begin{aligned} h_{((x''_i), (y''_i{}^*))}(a, b) = \\ \sum_{i=1}^l \left( \frac{d_{l_p}(T)}{p} \|x''_i\|^p + \frac{d_{l_p}(T)}{p^*} \|ay''_i{}^* b\|_{S_{p^*}(\mathcal{K})}^{p^*} - |\langle T(x''_i), y''_i{}^* \rangle| \right). \end{aligned}$$

Since  $T$  is positively homogeneous, we get

$$\begin{aligned} & \lambda g_{((x'_i), (y'_i{}^*))}(a, b) \\ &= \sum_{i=1}^k \lambda \left( \frac{d_{l_p}(T)}{p} \|x'_i\|^p + \frac{d_{l_p}(T)}{p^*} \|ay'_i{}^* b\|_{S_{p^*}(\mathcal{K})}^{p^*} - |\langle T(x'_i), y'_i{}^* \rangle| \right) \\ &= \sum_{i=1}^k \left( d_{l_p}(T) \left( \frac{1}{p} \left\| \lambda^{\frac{1}{p}} x'_i \right\|^p + \frac{1}{p^*} \left\| a \lambda^{\frac{1}{p^*}} y'_i{}^* b \right\|_{S_{p^*}(\mathcal{K})}^{p^*} \right) - \left| \langle T(\lambda^{\frac{1}{p}} x'_i), \lambda^{\frac{1}{p^*}} y'_i{}^* \rangle \right| \right) \\ &= f_{\left( \left( \lambda^{\frac{1}{p}} x'_i \right), \left( \lambda^{\frac{1}{p^*}} y'_i{}^* \right) \right)}(a, b), \end{aligned}$$

and finally we have

$$g + h = \sum_{i=1}^n \left( \frac{d_{l_p}(T)}{p} \|x_i\|^p + \frac{d_{l_p}(T)}{p^*} \|ay_i^* b\|_{S_{p^*}(\mathcal{K})}^{p^*} - |\langle T(x_i), y_i^* \rangle| \right),$$

with  $n = k + l$ ,

$$x_i = \begin{cases} x'_i & \text{if } 1 \leq i \leq k, \\ x''_i & \text{if } k + 1 \leq i \leq n, \end{cases}$$

and

$$y_i^* = \begin{cases} y'_i{}^* & \text{if } 1 \leq i \leq k, \\ y''_i{}^* & \text{if } k + 1 \leq i \leq n. \end{cases}$$

Now, using the following elementary identity

$$(3.5) \quad \forall \alpha, \beta \in \mathbb{R}_+^* \quad \alpha\beta = \inf_{\epsilon > 0} \left\{ \frac{1}{p} \left( \frac{\alpha}{\epsilon} \right)^p + \frac{1}{p^*} (\epsilon\beta)^{p^*} \right\},$$

we have

$$\begin{aligned}
 & \sup_{(a,b) \in S} f_{((x_i), (y_i^*))}(a, b) \\
 = & \sup_{(a,b) \in S} \sum_{i=1}^n \left( \frac{d_{i_p}(T)}{p} \|x_i\|^p + \frac{d_{i_p}(T)}{p^*} \|ay_i^*b\|_{S_{p^*}(\mathcal{K})}^{p^*} - |\langle T(x_i), y_i^* \rangle| \right) \\
 = & \frac{d_{i_p}(T)}{p} \sum_{i=1}^n \|x_i\|^p + \frac{d_{i_p}(T)}{p^*} \sup_{(a,b) \in S} \sum_{i=1}^n \|ay_i^*b\|_{S_{p^*}(\mathcal{K})}^{p^*} - \sum_{i=1}^n |\langle T(x_i), y_i^* \rangle| \\
 \geq & d_{i_p}(T) \left( \sum_{i=1}^n \|x_i\|^p \right)^{\frac{1}{p}} \sup_{(a,b) \in S} \left( \sum_{i=1}^n \|ay_i^*b\|_{S_{p^*}(\mathcal{K})}^{p^*} \right)^{\frac{1}{p^*}} - \sum_{i=1}^n |\langle T(x_i), y_i^* \rangle| \\
 \geq & 0 \text{ (by hypothesis see (3.1)),}
 \end{aligned}$$

and this for all  $f$  in the convex cone  $\mathcal{C}$ . Let  $\mathcal{C}'$  be the open set of  $l_\infty(S)$  such that  $\sup_{(a,b) \in S} f_{((x_i), (y_i^*))}(a, b) < 0$ . The sets  $\mathcal{C}$  and  $\mathcal{C}'$  are disjoint in  $l_\infty(S)$  which is

isomorphically isometric to  $C(\widehat{S})$  the space of all continuous functions on the Stone Cěch compactification  $\widehat{S}$  of  $S$  with values in the real. By Hahn-Banach theorem and Riesz representation theorem, there is a probability measure  $\lambda$  on  $\widehat{S}$  such that  $\lambda(f) \geq 0$  for all  $f$  in  $\mathcal{C}$ . Consequently, there are a set  $I$ , an ultrafilter  $\mathcal{U}$  on  $I$  and a family  $\{\lambda_\alpha\}_{\alpha \in I}$  of finitely supported probability measures on  $S$  such that

$$\begin{array}{ccc}
 \lambda_\alpha & \longrightarrow & \lambda \\
 \sigma(l_\infty^*(S), l_\infty(S)) & & 
 \end{array}$$

and

$$\forall f \in \mathcal{K}, \int_{\widehat{S}} f(a, b) d\lambda(a, b) = \lim_{\mathcal{U}} \int_S f(a, b) d\lambda_\alpha(a, b) \geq 0.$$

Particularly, if we take

$$f_{((x), (y^*))}(a, b) = \frac{d_{i_p}(T)}{p} \|x\|^p + \frac{d_{i_p}(T)}{p^*} \|ay^*b\|_{S_{p^*}(\mathcal{K})}^{p^*} - |\langle T(x), y^* \rangle|$$

we have

$$\begin{aligned}
 & \lim_{\mathcal{U}} \int_S f(a, b) d\lambda_\alpha(a, b) \\
 = & \frac{d_{i_p}(T)}{p} \|x\|^p + \frac{d_{i_p}(T)}{p^*} \lim_{\mathcal{U}} \int_S \|a_\alpha y^* b_\alpha\|_{S_{p^*}(\mathcal{K})}^{p^*} d\lambda_\alpha(a, b) - |\langle T(x), y^* \rangle| \\
 \geq & 0,
 \end{aligned}$$

$$(\lambda_\alpha = \sum_{k=1}^{n_\alpha} \lambda_{\alpha_k} \delta(a_{\alpha_k}, b_{\alpha_k}) \text{ with } \sum_{k=1}^{n_\alpha} \lambda_{\alpha_k} = 1 \text{ and } \lambda_{\alpha_k} \geq 0).$$

Whence by Pisier [9, Lemma 1.14]

$$\begin{aligned}
 & |\langle T(x), y^* \rangle| \\
 \leq & \frac{d_{i_p}(T)}{p} \|x\|^p + \frac{d_{i_p}(T)}{p^*} \lim_{\mathcal{U}} \sum_{k=1}^{n_\alpha} \lambda_{\alpha_k} \|a_{\alpha_k} y^* b_{\alpha_k}\|_{S_{p^*}(\mathcal{K})}^{p^*} \quad (a_{\alpha_k}, b_{\alpha_k} \geq 0) \\
 \leq & \frac{d_{i_p}(T)}{p} \|x\|^p + \frac{d_{i_p}(T)}{p^*} \lim_{\mathcal{U}} \left\| \left( \sum_{k=1}^{n_\alpha} \lambda_{\alpha_k} a_{\alpha_k}^{2p^*} \right)^{\frac{1}{2p^*}} \cdot y^* \cdot \left( \sum_{j=1}^{n_\alpha} \lambda_{\alpha_k} b_{\alpha_k}^{2p^*} \right)^{\frac{1}{2p^*}} \right\|_{S_{p^*}(\mathcal{K})}^{p^*} \\
 \leq & d_{i_p}(T) \left( \frac{1}{p} \|x\|^p + \frac{1}{p^*} \lim_{\mathcal{U}} \|a_\alpha y^* b_\alpha\|_{S_{p^*}(\mathcal{K})}^{p^*} \right).
 \end{aligned}$$

Using once again the identity (3.5). Fix  $\epsilon > 0$ . Replacing  $x$  by  $\frac{1}{\epsilon}x$ ,  $y^*$  by  $\epsilon y^*$  and taking the infimum over all  $\epsilon > 0$ , we find that

$$\begin{aligned} & |\langle T(x), y^* \rangle| \\ &= |\langle T(\frac{1}{\epsilon}x), \epsilon y^* \rangle| \\ &\leq d_{l_p}(T) \left( \frac{1}{p} \left\| \frac{x}{\epsilon} \right\|^p + \frac{1}{p^*} \lim_{\mathcal{U}} \|a_\alpha \epsilon y^* b_\alpha\|_{S_{p^*}(\mathcal{K})}^{p^*} \right) \\ &\leq d_{l_p}(T) \left( \frac{1}{p} \left( \frac{\|x\|}{\epsilon} \right)^p + \frac{1}{p^*} \left( \lim_{\mathcal{U}} \|a_\alpha \epsilon y^* b_\alpha\|_{S_{p^*}(\mathcal{K})}^{p^*} \right)^{\frac{1}{p^*}} \right)^{p^*} \\ &\leq d_{l_p}(T) \|x\| \left( \lim_{\mathcal{U}} \|a_\alpha y^* b_\alpha\|_{S_{p^*}(\mathcal{K})}^{p^*} \right)^{\frac{1}{p^*}}. \end{aligned}$$

This implies that

$$|\langle T(x), y^* \rangle| \leq d_{l_p}(T) \|x\| \lim_{\mathcal{U}} \|a_\alpha y^* b_\alpha\|_{S_{p^*}(\mathcal{K})}.$$

Conversely, consider  $n \in \mathbb{N}$ . Let  $x_1, \dots, x_n \in X$  and  $y_1^*, \dots, y_n^* \in Y^*$ . We have by (3.4)

$$|\langle T(x_i), y_i^* \rangle| \leq C \|x_i\| \lim_{\mathcal{U}} \|a_\alpha y_i^* b_\alpha\|_{S_{p^*}(\mathcal{K})},$$

for all  $1 \leq i \leq n$ . Thus

$$\begin{aligned} & \sum_{i=1}^n |\langle T(x_i), y_i^* \rangle| \\ &\leq C \sum_{i=1}^n (\|x_i\| \lim_{\mathcal{U}} \|a_\alpha y_i^* b_\alpha\|_{S_{p^*}(\mathcal{K})}) \\ \text{(by Hölder)} &\leq C \left( \sum_{i=1}^n \|x_i\|^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^n \lim_{\mathcal{U}} \|a_\alpha y_i^* b_\alpha\|_{S_{p^*}(\mathcal{K})}^{p^*} \right)^{\frac{1}{p^*}} \\ &\leq C \left( \sum_{i=1}^n \|x_i\|^p \right)^{\frac{1}{p}} \sup_{a, b \in B_{S_{2p^*}^+}(\mathcal{K})} \left\| \left( \|a y_i^* b\|_{S_{p^*}(\mathcal{K})} \right)_{1 \leq i \leq n} \right\|_{l_{p^*}^n}. \end{aligned}$$

This implies that  $T \in \mathcal{D}_{l_p}(X, Y)$  and  $d_{l_p}(T) \leq C$ . This proves the converse and finishes the proof of the theorem.

**Proposition 5.** Consider  $1 \leq p_1, p_2 < \infty$  such that  $p_1 \leq p_2$ . If  $T \in \mathcal{D}_{l_{p_2}}(X, Y)$  then  $T \in \mathcal{D}_{l_{p_1}}(X, Y)$  and

$$d_{l_{p_1}}(T) \leq d_{l_{p_2}}(T).$$

*Proof.* The desired result follows immediatly by combining inequality (3.3) and Mezrag [7, Lemma 2].  $\square$

By the equalities (2.1), (2.2) and (2.3) we have,

**Proposition 6.** Let  $X$  be a Banach space and  $(\Omega, \mathcal{A}, \mu)$  be measure space. We have

$$\mathcal{D}_{l_p}(X, L_1(\Omega, \mathcal{A}, \mu)) = \mathcal{D}_p(X, L_1(\Omega, \mathcal{A}, \mu)), \quad (1 < p \leq \infty)$$

and

$$\mathcal{D}_{l_2}(X, L_2(\Omega, \mathcal{A}, \mu)) = \mathcal{D}_2(X, L_2(\Omega, \mathcal{A}, \mu)).$$

**Open Questions.** In this paper, we have introduced the concept of strongly  $l_p$ -summing sublinear operators in the non commutative case and characterize this class of operators by given the extension of the Pietsch domination theorem. Moreover, some properties have been proved. Apparently, there are many problems

left unsolved. To mention a few, if  $T$  be a sublinear operator from a Banach space  $X$  into a quantum complete Banach lattice  $Y$ . We denote by

$$\nabla T = \{u \in L(X, Y) : u \leq T \text{ (i.e., } \forall x \in X, u(x) \leq T(x))\}.$$

the subdifferential of  $T$ . Then, by [1, Proposition 2.3] the set  $\nabla T$  is not empty,  $T(x) = \sup \{u(x) : u \in \nabla T\}$  and the sup is attained. Does  $u \in \mathcal{D}_{l_p}(X, Y)$  for any  $u$  in  $\nabla T$  imply  $T \in \mathcal{D}_{l_p}(X, Y)$ ? We do not know if the converse is true.

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