ON α-KENMOTSU MANIFOLDS SATISFYING
SEMI-SYMMETRIC CONDITIONS

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ABSTRACT. The main purpose of this paper is to study α-Kenmotsu manifolds satisfying some semi-symmetric conditions where α is a smooth function defined by $d\alpha \wedge \eta = 0$ on M^{2n+1}. In particularly, projectively, conformally and concircularly semi-symmetric tensor fields are considered. The results related to the effects of semi-symmetric conditions are given. Finally, illustrating examples on α-Kenmotsu manifolds depending on α are constructed.

1. INTRODUCTION

A $(2n+1)$-dimensional differentiable manifold M of class C^∞ is said to have an almost contact structure if the structural group of its tangent bundle reduces to $U(n) \times 1$, (see [1], [5]); equivalently an almost contact structure is given by a triple (ϕ, ξ, η) satisfying certain conditions. Many different types of almost contact structures are defined in the literature (cosymplectic, almost cosymplectic, Sasakian, Quassi Sasakian, α-Kenmotsu, almost α-Kenmotsu,..., [4], [6]).

Manifolds known as Kenmotsu manifolds have been studied by K. Kenmotsu in 1972 [3]. The author set up one of the three classes of almost contact Riemannian manifolds whose automorphism group attains the maximum dimension [8]. A Kenmotsu manifold can be defined as a normal almost contact metric manifold such that $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$.

It is well known that Kenmotsu manifolds can be characterized through their Levi-Civita connection, by $(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X$, for any vector fields X and Y. Kenmotsu defined a structure closely related to the warped product which was characterized by tensor equations. He proved that such a manifold M^{2n+1} is locally a warped product $(-\varepsilon, +\varepsilon) \times_f N^{2n}$ being a Kaehlerian manifold and $f(t) = ce^t$ where c is a positive constant. Moreover, Kenmotsu showed locally symmetric Kenmotsu manifolds are of constant curvature -1 that means locally

symmetry is a strong restriction for Kenmotsu manifolds. Also, the author realized that if Kenmotsu structure satisfies the Nomizu's condition [9], i.e., $R \cdot R = 0$, then it has negative constant curvature and if Kenmotsu manifold is conformally flat, then the manifold is a space of constant negative curvature -1 for dimension greater than 3.

The notion of semi-symmetric manifold is defined by

$$(1.1) \qquad\qquad\qquad R(X,Y) \cdot R = 0,$$

for all vector fields X, Y on M, where $R(X,Y)$ acts as a derivation on R [9]. Such a space is called "semi-symmetric space" since the curvature tensor of (M,g) at a point $p \in M$, R_p; is the same as the curvature tensor of a symmetric space (that can change with the point of p). Thus locally symmetric spaces are obviously semi-symmetric, but the converse is not true [15], [16]. A complete intrinsic classification of these spaces was given by Szabó [10]. However, it is interesting to investigate the semi-symmetry of special Riemannian manifolds. Nomizu proved that if M^n is a complete, connected semi-symmetric hypersurfaces of an Euclidean space R^{n+1}, $n > 3$, i.e., $R \cdot R = 0$, then M^n is locally symmetric, i.e., $\nabla R = 0$. For the case of a compact Kaehlerian manifold, Ogawa proved that if it is semi-symmetric then it must be locally symmetric [11]. In the case of contact structures, Tanno showed that there exists no proper semi-symmetric or Ricci semi-symmetric K-contact manifold. These manifolds studied many authors [15], [16]. Furthermore, the conditions $R(X,Y) \cdot P = 0$, $R(X,Y) \cdot C = 0$ and $R(X,Y) \cdot \overline{C} = 0$ are called projectively semi-symmetric, conformally (Weyl) semi-symmetric and concircularly semi-symmetric respectively, where $R(X,Y)$ is considered as derivation of tensor algebra at each point of the manifold.

In recent years, Pastore and Dileo studied locally symmetric almost Kenmotsu manifolds. The authors showed that locally symmetric Kenmotsu manifold is a Kenmotsu manifold with constant sectional curvature $K = -1$, equivalently; $h = 0$. If the manifold M^{2n+1} does not have constant sectional curvature then, $h \neq 0$ and the rank of the manifold must be greater than 1.

More recently almost contact metric manifolds such that η is closed and $d\Phi = 2\alpha\eta \wedge \Phi$, where α is a smooth function on M^{2n+1} satisfying $d\alpha \wedge \eta = 0$ have been studied by Murathan et al. in [13], [14]. Such manifolds are called almost α-cosymplectic. A normal almost α-cosymplectic manifold is an α-cosymplectic manifold. As it will be remarked in further sections, one can obtain important information on the geometry of the manifold by the tensor $h = \frac{1}{2}\left(\mathcal{L}_\xi\phi\right)$ or also by $\phi \circ h$.

This paper is devoted to obtain some results on α-Kenmotsu manifolds by choosing a real value-function α instead of any real number α (constant function) with the help of some certain curvature tensor fields. For this reason, we have an α-Kenmotsu structure if there exists a normal almost contact metric structure (ϕ, ξ, η, g) such that $d\eta = 0$ and $d\Phi = 2\alpha\left(\eta \wedge \Phi\right)$ for any vector fields X, Y on M^{2n+1}, where α is a smooth function defined by $d\alpha \wedge \eta = 0$ on M^{2n+1}.

In this paper, the semi-symmetric conditions of α-Kenmotsu manifolds are investigated where α is a smooth function defined by $d\alpha \wedge \eta = 0$ on M^{2n+1}. In particularly, projectively, conformally (Weyl) and concircularly semi-symmetric tensor fields are considered. The results related to the effects of semi-symmetric conditions are given. Finally, illustrating examples on α-Kenmotsu manifolds depending on α are constructed.

2. **Preliminaries**

An almost contact manifold is an odd-dimensional manifold M^{2n+1} which carries a field ϕ of endomorphisms of the tangent spaces, a vector field ξ, called characteristic or Reeb vector field , and a 1-form η satisfying $\phi^2 = -I + \eta \otimes \xi$ and $\eta(\xi) = 1$, where $I : TM^{2n+1} \to TM^{2n+1}$ is the identity mapping. From the definition it follows also that $\phi\xi = 0$, $\eta \circ \phi = 0$ and that the $(1,1)$-tensor field ϕ has constant rank $2n$ [1]. An almost contact manifold $(M^{2n+1}, \phi, \xi, \eta)$ is said to be normal when the tensor field $N = [\phi, \phi] + 2d\eta \otimes \xi$ vanishes identically, $[\phi, \phi]$ denoting the Nijenhuis tensor of ϕ. It is known that any almost contact manifold $(M^{2n+1}, \phi, \xi, \eta)$ admits a Riemannian metric g such that

$$(2.1) \qquad\qquad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\,\eta(Y),$$

for any vector fields X, Y on M^{2n+1}. This metric g is called a compatible metric and the manifold M^{2n+1} together with the structure $(M^{2n+1}, \phi, \xi, \eta, g)$ is called an almost contact metric manifold. As an immediate consequence of (2.1), one has $\eta = g(., \xi)$. The 2-form Φ of M^{2n+1} defined by $\Phi(X, Y) = g(\phi X, Y)$, is called the fundamental 2-form of the almost contact metric manifold M^{2n+1}. Almost contact metric manifolds such that both η and Φ are closed are called almost cosymplectic manifolds and almost contact metric manifolds such that $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$ are almost Kenmotsu manifolds.

An almost contact metric manifold M^{2n+1} is said to be almost α-Kenmotsu if $d\eta = 0$ and $d\Phi = 2\alpha\eta \wedge \Phi$, α being a non-zero real constant. Geometrical properties and examples of almost α-Kenmotsu manifolds are studied in [4], [18], [3] and [6]. Given an almost Kenmotsu metric structure (ϕ, ξ, η, g), consider the deformed structure

$$\eta^{|} = \frac{1}{\alpha}\eta, \ \xi^{|} = \alpha\xi, \ \phi^{|} = \phi, \ g^{|} = \frac{1}{\alpha^2}g, \ \alpha \neq 0, \ \alpha \in \mathbb{R},$$

where α is a non-zero real constant. So we get an almost α-Kenmotsu structure $(\phi^{|}, \xi^{|}, \eta^{|}, g^{|})$. This deformation is called a homothetic deformation [4], [6]. It is important to note that almost α-Kenmotsu structures are related to some special local conformal deformations of almost cosymplectic structures, [18].

The conformal (Weyl) curvature tensor is a measure of the curvature of spacetime and differs from the Riemannian curvature tensor. It is the traceless component of the Riemannian tensor which has the same symmetries as the Riemannian tensor. The most important of its special property that it is invariant under conformal changes to the metric. Namely, if $g^* = kg$ for some positive scalar functions k, then the Weyl tensor satisfies the equation $W^* = W$. In other words, it is called conformal tensor.

Let M be a $(2n+1)$-dimensional Riemannian manifold with metric g. The Ricci operator Q of (M, g) is defined by $g(QX, Y) = S(X, Y)$, where S denotes the Ricci tensor of type $(0, 2)$ on M. Weyl constructed a generalized curvature tensor of type $(1, 3)$ on a Riemannian manifold which vanishes whenever the metric is (locally) conformally equivalent to a flat metric; for this reason he called it the conformal

curvature tensor of the metric. The Weyl conformal curvature tensor is defined by

$$(2.2) \qquad \begin{aligned} C(X,Y)Z &= R(X,Y)Z - \frac{1}{2n-1}\left[S(Y,Z)X - S(X,Z)Y\right.\\ &\quad + g(Y,Z)QX - g(X,Z)QY\big]\\ &\quad + \frac{r}{(2n)(2n-1)}\left[g(Y,Z)X - g(X,Z)Y\right], \end{aligned}$$

for any vector fields X,Y on M, where R, r are denoting the Riemannian curvature tensor and scalar curvature of M, respectively [17].

A necessary condition for a Riemannian manifold to be conformally flat is that the Weyl curvature tensor vanish. The Weyl tensor vanish identically for 2 dimensional case. In dimensions ≥ 4, it is generally nonzero. If the Weyl tensor vanishes in dimensions ≥ 4, then the metric is locally conformally flat. So there exists a local coordinate system in which the metric is proportional to a constant tensor. For the dimensions greater than 3, this condition is sufficient as well. But in dimension 3 the vanishing of the equation $c = 0$, that is,

$$c(X,Y) = (\nabla_X Q)Y - (\nabla_Y Q)X - \tfrac{1}{2(2n-1)}\left[(\nabla_X r)Y - (\nabla_Y r)X\right],$$

is a necessary and sufficient condition for the Riemannian manifold being conformally flat, where c is the divergence operator of C, for all vector fields X and Y on M. It should be noted that if the manifold is conformally flat and of dimension greater than 3, then $C = 0$ implies $c = 0$ [5].

Moreover, the concircular curvature tensor $\overline{C}$ and the projective curvature tensor of (M^{2n+1}, g) are defined as

$$(2.3) \qquad \overline{C}(X,Y)Z = R(X,Y)Z - \frac{r}{2n(2n+1)}(g(Y,Z)X - g(X,Z)Y)$$

$$(2.4) \qquad P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}\left[S(Y,Z)X - S(X,Z)Y\right],$$

respectively, where S is the Ricci tensor, $r = tr(S)$ is the scalar curvature and $X, Y, Z \in \chi(M^n)$, $\chi(M^n)$ being the Lie algebra of vector fields of M^{2n+1}.

Moreover, an α-Kenmotsu manifold satisfies the following relations

$$(2.5) \qquad\qquad\qquad \nabla_X \xi = -\alpha \phi^2 X,$$

$$(2.6) \qquad\qquad (\nabla_X \eta)(Y) = \alpha\left[g(X,Y) - \eta(X)\eta(Y)\right],$$

$$(2.7) \qquad\qquad (\nabla_X \phi)Y = -\alpha\left[g(X,\phi Y)\xi + \eta(Y)\phi X\right],$$

for any vector fields X, Y on M^{2n+1}.

3. Basic Curvature Properties

By using the properties of Riemannian curvature tensor, the following relations are obtained on α-Kenmotsu manifolds

$$(3.1) \qquad\qquad R(X,Y)\xi = \left[\alpha^2 + \xi(\alpha)\right](\eta(X)Y - \eta(Y)X),$$

$$(3.2) \qquad\qquad R(X,\xi)\xi = \left[\alpha^2 + \xi(\alpha)\right](\eta(X)\xi - X),$$

$$(3.3) \qquad\qquad R(\xi,X)Y = \left[\alpha^2 + \xi(\alpha)\right](\eta(Y)X - g(X,Y)\xi),$$

$$(3.4) \qquad g(R(\xi, X)Y, \xi) = \left[\alpha^2 + \xi(\alpha)\right](-g(X, Y) + \eta(X)\eta(Y)),$$

$$(3.5) \qquad S(X, \xi) = -2n\left[\alpha^2 + \xi(\alpha)\right]\eta(X),$$

$$(3.6) \qquad S(\xi, \xi) = -2n(\alpha^2 + \xi(\alpha)),$$

$$(3.7) \qquad S(\phi X, \phi Y) = S(X, Y) + 2n\left[\alpha^2 + \xi(\alpha)\right]\eta(X)\eta(Y),$$

where α is a smooth function such that $d\alpha \wedge \eta = 0$, for any vector fields X, Y on M^{2n+1}. In these formulas, R is the Riemannian curvature tensor and S the Ricci tensor of M^{2n+1}.

Remark 3.1. In [7], the above curvature properties are obtained for $\alpha \in \mathbb{R}$, $\alpha \neq 0$.

4. Semi-symmetric α-Kenmotsu manifolds

In this section, we give some results about α-Kenmotsu manifolds in the light of [12] and [10]. Thus we have following:

Theorem 4.1. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an α-Kenmotsu manifold. If M^{2n+1} is semi-symmetric, there exists no constant curvature on M^{2n+1}.*

Proof. Assume that M^{2n+1} is semi-symmetric, i.e., $R \cdot R = 0$ which is equivalent to

$$(4.1) \qquad \begin{aligned} 0 &= R(X, \xi)R(U, V)W - R(R(X, \xi)U, V)W \\ &\quad - R(U, R(X, \xi)V)W - R(U, V)R(X, \xi)W, \end{aligned}$$

for all vector fields X, U, V and W on M^{2n+1}. At first, let us define α as a constant function on M^{2n+1}. Then taking $U = \xi$ in (4.1) with the help of (3.2) and (3.3), we have

$$(4.2) \qquad \begin{aligned} R(X, \xi)R(\xi, V)W &= -\alpha^4\eta(X)g(V, W)\xi + \alpha^4 g(V, W)X \\ &\quad + \alpha^4\eta(W)g(X, V)\xi - \alpha^4\eta(V)\eta(W)X, \end{aligned}$$

$$(4.3) \qquad \begin{aligned} R(R(X, \xi)\xi, V)W &= -\alpha^4\eta(X)g(V, W)\xi + \alpha^4\eta(X)\eta(W)V \\ &\quad - \alpha^2 R(X, V)W, \end{aligned}$$

$$(4.4) \qquad \begin{aligned} R(\xi, R(X, \xi)V)W &= \alpha^2\eta(W)g(X, V)\xi - \alpha^2\eta(V)\eta(W)X \\ &\quad - \alpha^2\eta(W)g(X, V)\xi + \alpha^2\eta(V)g(X, W)\xi \end{aligned}$$

and

$$(4.5) \qquad \begin{aligned} R(\xi, V)R(X, \xi)W &= -\alpha^4\eta(V)g(X, W)\xi + \alpha^4\eta(W)g(X, V)\xi \\ &\quad + \alpha^4 g(X, W)V - \alpha^4\eta(X)\eta(W)V. \end{aligned}$$

Taking into account (4.2), (4.3), (4.4), (4.5) and using (4.1) we obtain

$$\begin{aligned} 0 &= \alpha^4 g(V, W)X + \alpha^2 R(X, V)W - \alpha^4 g(X, W)V \\ &\quad - \alpha^4\eta(V)\eta(W)X + \alpha^2\eta(V)\eta(W)X \\ &\quad - \alpha^2\eta(V)g(X, W)\xi + \alpha^4\eta(V)g(X, W)\xi. \end{aligned}$$

Therefore, there exists no real constant k depending on α such that

$$R(X, V)W = k\left[-g(X, W)V + g(V, W)X\right].$$

Now, suppose that α is a smooth function defined by $d\alpha \wedge \eta = 0$ on M^{2n+1}. In this case, (4.1) takes the form

$$
\begin{aligned}
& F^2(\alpha)g(V,W)X + F(\alpha)R(X,V)W - F^2(\alpha)g(X,W)V \\
(4.6) \quad = \quad & F^2(\alpha)\eta(V)\eta(W)X - F(\alpha)\eta(V)\eta(W)X \\
& + F(\alpha)\eta(V)g(X,W)\xi - F^2(\alpha)\eta(V)g(X,W)\xi,
\end{aligned}
$$

for all vector fields X, V and W on M^{2n+1} where $F(\alpha) = \left[\alpha^2 + \xi(\alpha)\right].$

Then simplifying (4.6) we have

$$
\begin{aligned}
R(X,V)W \quad = \quad & F(\alpha)\left[g(X,W)V - g(V,W)X\right] \\
& + (F^2(\alpha) - F(\alpha))\eta(V)\left[\eta(W)X - g(X,W)\xi\right].
\end{aligned}
$$

Thus we find that M^{2n+1} exists no constant curvature. Consequently, we have the following results: □

Corollary 4.1. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an semi-symmetric α-Kenmotsu manifold. If α is parallel along the vector field ξ, i.e., $F(\alpha) = \alpha^2$, there exists no constant curvature on M^{2n+1} where α is a real constant for $\alpha \neq 0$.*

Corollary 4.2. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a semi-symmetric Kenmotsu manifold. Then M^{2n+1} is of constant curavture -1.*

Theorem 4.2. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an α-Kenmotsu manifold and α is parallel along the vector field ξ. If M^{2n+1} is locally symmetric, then M^{2n+1} is of a space of negative constant curvature with $-\alpha^2$.*

Proof. With the help of the hypothesis, we have

$$
R(X,Y)\xi = \alpha^2\left[\eta(X)Y - \eta(Y)X\right].
$$

Taking covariant derivative of both sides of the above equation with respect to the vector field Z, then we obtain
(4.7)

$$
\begin{aligned}
& = \quad \nabla_Z R(X,Y)\xi - R(\nabla_Z X, Y)\xi - R(X, \nabla_Z Y)\xi - R(X,Y)\nabla_Z\xi \\
(\nabla_Z R)(X,Y)\xi \quad = \quad & \alpha^2\left[\eta(\nabla_Z X)Y + g(X, \nabla_Z\xi)Y + \eta(X)\nabla_Z Y - \eta(\nabla_Z Y)X\right. \\
& \left. - g(Y, \nabla_Z\xi)X - \eta(Y)\nabla_Z X\right] - \alpha^2\left[\eta(\nabla_Z Y)X - \eta(Y)\nabla_Z X\right] \\
& - \alpha^2\left[\eta(X)\nabla_Z Y - \eta(\nabla_Z Y)X\right] + \alpha R(X,Y)\phi^2 Z \\
& = \quad \alpha^3\left[g(X,Z)Y - g(Y.Z)X\right] - \alpha R(X,Y)Z.
\end{aligned}
$$

Then (4.7) holds

$$
R(X,Y)Z = -\alpha^2\left[g(Y,Z)X - g(X,Z)Y\right]
$$

under the locally symmetry condition for $\alpha \neq 0$. This means that the manifold is of negative constant curvature defined by $k = -\alpha^2$. This completes the proof.   □

Corollary 4.3. *For a Kenmotsu manifold the following conditions are equivalent:*

i) M^{2n+1} is of constant curvature -1,
ii) M^{2n+1} is locally symmetric,
iii) M^{2n+1} is semi-symmetric,
iv) $R(X,\xi) \cdot R = 0$ for any vector field X on M^{2n+1}.

5. Semi-Ricci symmetric α-Kenmotsu manifolds

In this section, we suppose that α-Kenmotsu manifolds which satisfy the following condition

$$(5.1) \qquad\qquad (R(X,Y) \cdot S)(Z,U) = 0.$$

defined by

$$
\begin{aligned}
(R(X,Y) \cdot S)(Z,U) \;=\;& R(X,Y)S(Z,U) - S(R(X,Y)Z,U) \\
& -S(Z,R(X,Y)U),
\end{aligned}
$$

(5.2)

for any vector fields X, Y, Z and U on M^{2n+1}.

Theorem 5.1. *Let* $(M^{2n+1}, \phi, \xi, \eta, g)$ *be an α-Kenmotsu manifold. If M^{2n+1} holds $R \cdot S = 0$, then M^{2n+1} is an Einstein space with $S = -2n\alpha^2 g$.*

Proof. Our assumption is equivalent to

$$S(R(X,\xi)Z,U) + S(Z,R(X,\xi)U) = 0,$$

for all vector fields X, Z and U on M^{2n+1}.
Putting $U = \xi$ we get

$$(5.3) \qquad F(\alpha)\left[g(X,Z)S(\xi,\xi) - \eta(Z)S(X,\xi) + \eta(X)S(Z,\xi) - S(X,Z)\right] = 0,$$

where $F(\alpha) = \left[\alpha^2 + \xi(\alpha)\right]$. Taking into account (3.5) and (3.6) in (5.3) we obtain

$$2nF^2(\alpha)g(X,Z) + F(\alpha)S(X,Z) = 0.$$

It follows that

$$S(X,Z) = -2nF(\alpha)g(X,Z).$$

Therefore M^{2n+1} is an Einstein space with $S = -2n\alpha^2 g$. $\qquad\qquad \square$

Remark 5.1. It is note that $R \cdot R = 0 \subset R \cdot S = 0$. Thus $R \cdot R = 0$ implies $R \cdot S = 0$ for $\alpha = 1$. Then we can state the following result:

Corollary 5.1. *A semi-symmetric Kenmotsu manifold is an Einstein space with $S = -2ng$.*

Remark 5.2. For a Kenmotsu manifold the following conditions are equivalent:

i) M^{2n+1} is an Einstein space with $S = -2ng$,
ii) M^{2n+1} is locally Ricci-symmetric,
iii) M^{2n+1} is semi-Ricci symmetric,
iv) $R(X,\xi) \cdot S = 0$ for any vector field X on M^{2n+1}.

6. α-Kenmotsu manifolds with $R(X,Y) \cdot C = 0$

In this section, we consider conformally semi-symmetric condition on α-Kenmotsu manifolds. Kenmotsu proved that Kenmotsu manifolds is conformallt flat if and only if the manifold is of constant curvature -1.

Now, we give some results satisfying $R \cdot C = 0$ on α-Kenmotsu manifolds:

Theorem 6.1. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a semi-conformally symmetric α-Kenmotsu manifold. Then M^{2n+1} is conformally flat if and only if it is cosymplectic manifold $(\alpha = 0)$.*

Proof. Assume that M^{2n+1} is semi-conformally symmetric. Then it is equivalent to

$$(6.1) \qquad \begin{aligned} 0 \;=\; & R(X,\xi)C(U,V)W - C(R(X,\xi)U,V)W \\ & -C(U,R(X,\xi)V)W - C(U,V)R(X,\xi)W. \end{aligned}$$

By the help of (2.2) we have

$$\begin{aligned} g(C(X,Y)Z,\xi) \;=\; & \left[-F(\alpha) + \frac{2nF(\alpha)}{2n-1} + \frac{r}{2n(2n-1)} \right] \eta(X)g(Y,Z) \\ & + \left[F(\alpha) - \frac{2nF(\alpha)}{2n-1} - \frac{r}{2n(2n-1)} \right] \eta(Y)g(X,Z) \\ (6.2) \qquad & + \frac{1}{2n-1} \left[\eta(Y)S(X,Z) - \eta(X)S(Y,Z) \right]. \end{aligned}$$

Hereafter we take $G = -F(\alpha) + \frac{2nF(\alpha)}{2n-1} + \frac{r}{2n(2n-1)}$ for shortening. Then putting $X = \xi$ in (6.2) we obtain

$$\begin{aligned} (6.3) \qquad g(C(\xi,Y)Z,\xi) \;=\; & G\left[g(Y,Z) - \eta(Y)\eta(Z) \right] \\ & + \frac{1}{2n-1} \left[-2nF(\alpha)\eta(Y)\eta(Z) - S(Y,Z) \right]. \end{aligned}$$

Applying (6.2) and (6.3) to (6.1) by the help of (3.3), it follows that

$$\begin{aligned} g(R(X,\xi)C(U,V)W,\xi) \;=\; & F(\alpha)g(C(U,V)W,X) \\ & -F(\alpha)\eta(X)\left[G\left(\eta(U)g(V,W) - \eta(V)g(U,W) \right) \right. \\ (6.4) \qquad & \left. + \tfrac{1}{2n-1}\left(\eta(V)S(U,W) - \eta(U)S(V,W) \right) \right], \end{aligned}$$

$$\begin{aligned} g(C(R(X,\xi)U,V)W,\xi) \;=\; & -F(\alpha)\eta(U)\left[G\left(\eta(X)g(V,W) - \eta(V)g(X,W) \right) \right. \\ & \left. + \tfrac{1}{2n-1}\left(\eta(V)S(X,W) - \eta(X)S(V,W) \right) \right] \\ (6.5) \qquad & + F(\alpha)g(X,U)\left[G\left(g(V,W) - \eta(V)\eta(W) \right) \right. \\ & \left. - \tfrac{1}{2n-1}\left(2nF(\alpha)\eta(V)\eta(W) + S(V,W) \right) \right], \end{aligned}$$

$$\begin{aligned} g(C(U,R(X,\xi)V)W,\xi) \;=\; & F(\alpha)\eta(V)\left[G\left(\eta(X)g(U,W) - \eta(U)g(X,W) \right) \right. \\ & \left. + \tfrac{1}{2n-1}\left(\eta(U)S(X,W) - \eta(X)S(U,W) \right) \right] \\ (6.6) \qquad & - F(\alpha)g(X,V)\left[G\left(g(U,W) - \eta(U)\eta(W) \right) \right. \\ & \left. - \tfrac{1}{2n-1}\left(2nF(\alpha)\eta(U)\eta(W) + S(U,W) \right) \right], \end{aligned}$$

and

$$\begin{aligned} g(C(U,V)R(X,\xi)W,\xi) \;=\; & -F(\alpha)\eta(W)\left[G\left(\eta(U)g(X,V) - \eta(V)g(X,U) \right) \right. \\ (6.7) \qquad & \left. + \tfrac{1}{2n-1}\left(\eta(V)S(X,U) - \eta(U)S(V,X) \right) \right], \end{aligned}$$

where $g(C(X,Y)\xi,\xi) = 0$.

Then taking into account (6.4-6.7) in (6.1) yield

(6.8)

$F(\alpha)g(C(U,V)W,X) - F(\alpha)\eta(X)\left[G\left(\eta(U)g(V,W) - \eta(V)g(U,W)\right)\right.$

$+\frac{1}{2n-1}\left(\eta(V)S(U,W) - \eta(U)S(V,W)\right)\right] + F(\alpha)\eta(U)\left[G\left(\eta(X)g(V,W) - \eta(V)g(X,W)\right)\right.$

$+\frac{1}{2n-1}\left(\eta(V)S(X,W) - \eta(X)S(V,W)\right)\right] - F(\alpha)g(X,U)\left[G\left(g(V,W) - \eta(V)\eta(W)\right)\right.$

$-\frac{1}{2n-1}\left(2nF(\alpha)\eta(V)\eta(W) + S(V,W)\right)\right] - F(\alpha)\eta(V)\left[G\left(\eta(X)g(U,W) - \eta(U)g(X,W)\right)\right.$

$+\frac{1}{2n-1}\left(\eta(U)S(X,W) - \eta(X)S(U,W)\right)\right] + F(\alpha)g(X,V)\left[G\left(g(U,W) - \eta(U)\eta(W)\right)\right.$

$-\frac{1}{2n-1}\left(2nF(\alpha)\eta(U)\eta(W) + S(U,W)\right)\right] + F(\alpha)\eta(W)\left[G\left(\eta(U)g(X,V) - \eta(V)g(X,U)\right)\right.$

$+\frac{1}{2n-1}\left(\eta(V)S(X,U) - \eta(U)S(V,X)\right)\right] = 0$

Let $\{E_i,\ i = 1, 2, \ldots, 2n+1\}$ be an orthonormal basis of the tangent space at the points of M^{2n+1}. Thus using (2.2) we get

$$(6.9) \qquad\qquad \sum_{i=1}^{2n+1} g(C(E_i, Y)Z, E_i) = 0.$$

Putting $X = U = E_i$ in (6.8), summarizing for $1 \leq i \leq 2n+1$ and taking into consideration (6.9), we obtain

$$(6.10) \qquad\qquad S(V,W) = (2n-1)Gg(V,W) - E\eta(V)\eta(W),$$

where E is a function defined by $E = \left(\frac{r}{2n} + F(\alpha)(2n+1)\right)$.

At last, using (6.10), (6.8) reduces to

$$
\begin{aligned}
g(C(U,V)W,X) &= \left(\frac{E - 2nF(\alpha)}{2n-1} - G\right) g(X,U)\eta(V)\eta(W) \\
(6.11) &\quad + \left(G - \frac{E + 2nF(\alpha)}{2n-1}\right) g(X,V)\eta(U)\eta(W).
\end{aligned}
$$

Then contracting with respect to U and X, we have

$$
\begin{aligned}
0 &= (2n+1)\left(\frac{E - 2nF(\alpha)}{2n-1} - G\right)\eta(V)\eta(W) \\
&\quad + \left(G - \frac{E + 2nF(\alpha)}{2n-1}\right)\eta(V)\eta(W),
\end{aligned}
$$

which implies $F(\alpha) = 0$. It follows that α is constant and $\alpha = 0$ (cosymplectic case). This completes the proof. $\qquad\square$

7. α-Kenmotsu manifolds with $R(X,Y) \cdot P = 0$

In this section, we study projectively semi-symmetric condition on α-Kenmotsu manifolds. First we consider projectively flat α-Kenmotsu manifolds. Then we investigate the effects of the tensor products $R \cdot P = 0$ on α-Kenmotsu manifolds.

Thus we obtain the following results:

Theorem 7.1. *A projectively flat α-Kenmotsu manifold is an Einstein space.*

Proof. Suppose that $P = 0$. Then from (2.4), we have

$$(7.1) \qquad\qquad R(X,Y)Z = \frac{1}{2n}\left[S(Y,Z)X - S(X,Z)Y\right].$$

From (7.1), we get

(7.2) $$R(X,Y,Z,W) = \frac{1}{2n}\left[S(Y,Z)g(X,W) - S(X,Z)g(Y,W)\right],$$

where $R(X,Y,Z,W) = g(R(X,Y)Z,W)$.

Putting $W = \xi$ in (7.2), we obtain

(7.3) $$\eta(R(X,Y)Z) = \frac{1}{2n}\left[S(Y,Z)\eta(X) - S(X,Z)\eta(Y)\right].$$

Again taking $X = \xi$ in (7.3) and using (3.2) and (3.5), we have

(7.4) $$S(Y,Z) \quad = \quad -2nF(\alpha)g(Y,Z).$$

According to the assumption, using (7.4), the manifold is an Einstein space with $S = -2n\alpha^2 g$. $\qquad\square$

Theorem 7.2. *If in an α-Kenmotsu manifold $M^{2n+1}, n > 0$, the relation $R(X,Y) \cdot P = 0$ holds, then the manifold is projectively flat.*

Proof. Using (2.1) and (3.1) in (2.4), we get

$$\eta(P(X,Y)Z) \quad = \quad -F(\alpha)\eta(X)g(Y,Z) + F(\alpha)\eta(Y)g(X,Z)$$
(7.5) $$-\frac{1}{2n}\left[\eta(X)S(Y,Z) - \eta(Y)S(X,Z)\right].$$

Putting $Z = \xi$ in (7.5), we obtain

(7.6) $$\eta(P(X,Y)\xi) = 0$$

Again taking $X = \xi$ in (7.5), we have

(7.7) $$\eta(P(\xi,Y)Z) = -F(\alpha)g(Y,Z) - \frac{1}{2n}S(Y,Z)$$

Now, we consider the condition $R(X,Y) \cdot P$ which is defined as

$$(R(X,Y)P)(U,V)Z \quad = \quad R(X,Y) \cdot P(U,V)Z - P(R(X,Y)U,V)Z$$
(7.8) $$-P(U,R(X,Y)V)Z - P(U,V)R(X,Y)Z.$$

As it has been supposed that $R(X,Y) \cdot P = 0$, thus we have

$$0 \quad = \quad R(X,Y) \cdot P(U,V)Z - P(R(X,Y)U,V)Z$$
(7.9) $$-P(U,R(X,Y)V)Z - P(U,V)R(X,Y)Z.$$

Putting $X = \xi$ and applying inner product with respect to ξ in (7.9). From this, it follows that

$$0 \quad = \quad -P(U,V,Z,Y) + \eta(Y)\eta(P(U,V)Z) - \eta(U)\eta(P(Y,V)Z)$$
(7.10) $$+g(Y,U)\eta(P(\xi,V)Z) - \eta(V)\eta(P(U,Y)Z)$$
$$+g(Y,V)\eta(P(U,\xi)Z) - \eta(Z)\eta(P(U,V)Y),$$

where $P(U,V,Z,Y) = g(P(U,V)Z,Y)$.

Putting $Y = U$ in (7.10), we get

$$0 \quad = \quad -P(U,V,Z,U) + g(U,U)\eta(P(\xi,V)Z) - \eta(V)\eta(P(U,U)Z)$$
(7.11) $$+g(U,V)\eta(P(U,\xi)Z) - \eta(Z)\eta(P(U,V)U).$$

Let $\{E_i\}$, $i = 1, \ldots, 2n + 1$ be an orthonormal basis of the tangent space at any point. Then the sum for $1 \le i \le 2n + 1$ of the relation (7.11) for $U = E_i$ yields

$$\eta(P(\xi, V)Z) = -\left(\frac{1}{2n(2n+1)}\right) S(V, Z) - \left(\frac{F(\alpha)}{2n+1}\right) g(V, Z)$$
$$(7.12) \qquad\qquad + \left(F(\alpha) + \frac{r}{2n(2n+1)}\right) \eta(V)\eta(Z)$$

From (7.7) and (7.12), we have

$$(7.13) \qquad S(V, Z) = -2nF(\alpha)g(V, Z) - \left((2n+1)F(\alpha) + \frac{r}{2n}\right)\eta(V)\eta(Z).$$

Taking $Z = \xi$ in (7.13) and using (3.5) we get

$$(7.14) \qquad\qquad\qquad r = -2n(2n+1)F(\alpha).$$

Now using (7.5), (7.12), (7.13) and (7.14) in (7.10), we obtain

$$(7.15) \qquad\qquad\qquad P(U, V, Z, Y) = 0.$$

From (7.15) it follows that

$$(7.16) \qquad\qquad\qquad P(U, V)Z = 0.$$

Therefore, an α-Kenmotsu manifold under consideration is projectively flat. Hence, we can state the next theorem: □

Theorem 7.3. *An α-Kenmotsu manifold M^{2n+1}, $n > 0$, satisfying $R(X, Y) \cdot P = 0$ is an η-Einstein manifold and also it is a manifold of constant curvature*

$$r = -2n(2n+1)F(\alpha),$$

where $F(\alpha) = \left[\alpha^2 + \xi(\alpha)\right]$.

8. α-**Kenmotsu manifolds with** $R(X, Y) \cdot \overline{C} = 0$

In this section, we study concircularly semi-symmetric condition on α-Kenmotsu manifolds. At first, we examine concircularly flat α-Kenmotsu manifolds. Next, we investigate the tensor products $R \cdot \overline{C} = 0$ on α-Kenmotsu manifolds.

Thus we obtain the following results:

Theorem 8.1. *If an α-Kenmotsu manifold is concircularly flat, then it is a manifold of constant curvature $r = -2n(2n+1)F(\alpha)$.*

Proof. Assume that $\overline{C} = 0$. Then from (2.3), we have

$$(8.1) \qquad R(X, Y)Z = \frac{r}{2n(2n+1)} \left[g(Y, Z)X - g(X, Z)Y\right].$$

From (8.1), we get

$$R(X, Y, Z, W) = \frac{r}{2n(2n+1)} \left[g(Y, Z)g(X, W)\right.$$
$$(8.2) \qquad\qquad\qquad \left. - g(X, Z)g(Y, W)\right],$$

where $R(X, Y, Z, W) = g(R(X, Y)Z, W)$.

Putting $W = \xi$ in (8.2), we obtain

$$(8.3) \qquad \left(\frac{r}{2n(2n+1)} + F(\alpha)\right) \left[\eta(X)g(Y, Z) - \eta(Y)g(X, Z)\right] = 0.$$

Then using (8.3), we have

(8.4) $$\frac{r}{2n(2n+1)} + F(\alpha) = 0.$$

From (8.4), the proof is completed. It is note that the cosymplectic case ($\alpha = 0$) exists if and only if $r = 0$. $\qquad\square$

Theorem 8.2. *If in an α-Kenmotsu manifold $M^{2n+1}, n > 0$, the relation $R(X,Y)\cdot \overline{C} = 0$ holds, then the manifold is concircularly flat.*

Proof. In view of (2.1) and (3.1) in (2.3), we get

(8.5) $$\eta(\overline{C}(X,Y)Z) = \left(F(\alpha) + \frac{r}{2n(2n+1)} \right) (\eta(Y)g(X,Z) - \eta(X)g(Y,Z)).$$

Putting $Z = \xi$ in (8.5), we obtain

(8.6) $$\eta(\overline{C}(X,Y)\xi) = 0$$

Again taking $X = \xi$ in (8.5), we have

(8.7) $$\eta(\overline{C}(\xi,Y)Z) = \left(F(\alpha) + \frac{r}{2n(2n+1)} \right) (\eta(Y)\eta(Z) - g(Y,Z)).$$

Now, we consider the tensor product $R(X,Y) \cdot \overline{C}$ which is defined by

$$(R(X,Y)\overline{C})(U,V)Z = R(X,Y) \cdot \overline{C}(U,V)Z - \overline{C}(R(X,Y)U,V)Z$$
(8.8) $$-\overline{C}(U,R(X,Y)V)Z - \overline{C}(U,V)R(X,Y)Z.$$

Assume that $R(X,Y) \cdot \overline{C} = 0$. Thus we have

$$0 = R(X,Y) \cdot \overline{C}(U,V)Z - \overline{C}(R(X,Y)U,V)Z$$
(8.9) $$-\overline{C}(U,R(X,Y)V)Z - \overline{C}(U,V)R(X,Y)Z.$$

Then putting $X = \xi$ and applying inner product with respect to ξ in (8.9).
From this, it follows that

$$0 = -\overline{C}(U,V,Z,Y) + \eta(Y)\eta(\overline{C}(U,V)Z) - \eta(U)\eta(\overline{C}(Y,V)Z)$$
(8.10) $$+g(Y,U)\eta(\overline{C}(\xi,V)Z) - \eta(V)\eta(\overline{C}(U,Y)Z)$$
$$+g(Y,V)\eta(\overline{C}(U,\xi)Z) - \eta(Z)\eta(\overline{C}(U,V)Y),$$

where $\overline{C}(U,V,Z,Y) = g(\overline{C}(U,V)Z,Y)$.
Putting $Y = U$ in (8.10), we get

$$0 = -\overline{C}(U,V,Z,U) + g(U,U)\eta(\overline{C}(\xi,V)Z) - \eta(V)\eta(\overline{C}(U,U)Z)$$
(8.11) $$+g(U,V)\eta(\overline{C}(U,\xi)Z) - \eta(Z)\eta(\overline{C}(U,V)U).$$

Let $\{E_i\}$, $i = 1,\ldots,2n+1$ be an orthonormal basis of the tangent space at any point. Then the sum for $1 \leq i \leq 2n+1$ of the relation (8.11) for $U = E_i$ yields

$$\eta(P(\xi,V)Z) = \frac{1}{(2n+1)}S(V,Z) - \left(\frac{2F(\alpha)n + r}{2n(2n+1)} \right) g(V,Z)$$
(8.12) $$+ \left(F(\alpha) + \frac{r}{2n(2n+1)} \right) \eta(V)\eta(Z)$$

From (8.7) and (8.12), we have

(8.13) $$S(V,Z) = -2nF(\alpha)g(V,Z).$$

Taking into account (8.4), (8.5), (8.7) and (8.13) in (8.10), we obtain

(8.14) $\overline{C}(U, V, Z, Y) = 0.$

From (8.14) it follows that

(8.15) $\overline{C}(U, V)Z = 0.$

Therefore, an α-Kenmotsu manifold is projectively flat with $R(X, Y) \cdot \overline{C} = 0$. As we know, in general, a concircularly flat Riemannian manifold is Einstein and so, in particular, a concircularly α-Kenmotsu manifold is Einstein. Hence, we can state the next theorem: □

Theorem 8.3. *An α-Kenmotsu manifold M^{2n+1}, $n > 0$, satisfying $R(X, Y) \cdot \overline{C} = 0$ is an Einstein manifold and also a manifold of constant curvature*

$$r = -2n(2n + 1)F(\alpha),$$

where $F(\alpha) = \left[\alpha^2 + \xi(\alpha) \right].$

9. EXAMPLES

9.1. Example in Three Dimensional when α is a constant function

We consider the 3-dimensional manifold $M^3 = \left\{ (x, y, z) \in \mathbb{R}^3 \right\}$, where (x, y, z) are the standart coordinates in $\mathbb{R}^3$. The vector fields are

$$
\begin{aligned}
e_1 &= f_1(z)\frac{\partial}{\partial x} + f_2(z)\frac{\partial}{\partial y}, \\
e_2 &= -f_2(z)\frac{\partial}{\partial x} + f_1(z)\frac{\partial}{\partial y}, \\
e_3 &= \frac{\partial}{\partial z},
\end{aligned}
$$

where f_1, f_2 are given by

$$
\begin{aligned}
f_1(z) &= c_2 e^{-\alpha z}, \\
f_2(z) &= c_1 e^{-\alpha z},
\end{aligned}
$$

with $c_1^2 + c_2^2 \neq 0$, $\alpha \neq 0$ for constants c_1, c_2 and α. It is obvious that $\{e_1, e_2, e_3\}$ are linearly independent at each point of M^3. Let g be the Riemannian metric defined by

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1, \qquad g(e_1, e_2) = g(e_1, e_3) = g(e_2, e_3) = 0$$

and given by the tensor product

$$g = (f_1^2 + f_2^2)^{-1}(dx \otimes dx + dy \otimes dy) + dz \otimes dz.$$

Let η be the 1-form defined by $\eta(X) = g(X, e_3)$ for any vector field X on M^3 and ϕ be the $(1, 1)$ tensor field defined by $\phi(e_1) = e_2$, $\phi(e_2) = -e_1$, $\phi(e_3) = 0$. Then using linearity of g and ϕ, we have

$$\phi^2 X = -X + \eta(X)e_3, \qquad \eta(e_3) = 1, \qquad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for any vector fields on M^3.

Let ∇ be the Levi-Civita connection with respect to the metric g. Then we get

$$[e_1, e_3] = \alpha e_1, \qquad [e_2, e_3] = \alpha e_2, \qquad [e_1, e_2] = 0.$$

It follows that the structure of (ϕ, ξ, η, g) can easily be obtained. So it is sufficient to check that the only non-zero components of the second fundamental form Φ are

$$\Phi(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) = -\Phi(\frac{\partial}{\partial y}, \frac{\partial}{\partial x}) = -\frac{1}{f_1^2 + f_2^2} = -\frac{e^{2\alpha z}}{c_1^2 + c_2^2}.$$

Thus, we get

$$\Phi = -\frac{2e^{2\alpha z}}{c_1^2 + c_2^2}(dx \wedge dy),$$

and the exterior derivation of Φ is given by

$$d\Phi = -\frac{4\alpha e^{2\alpha z}}{c_1^2 + c_2^2}(dx \wedge dy \wedge dz).$$

Since $\eta = dz$, it implies $d\Phi = 2\alpha\eta \wedge \Phi$ on M^3 and it is also note that Nijenhuis torsion tensor of ϕ vanishes.

9.2. Example in Three Dimensional when α is a smooth function

Let us denote the standart coordinates of $\mathbb{R}^3(x, y, z)$ and consider 3-dimensional manifold $M \subset \mathbb{R}^3$ defined by

$$M = \left\{(x, y, z) \in \mathbb{R}^3 : z \neq 0\right\}.$$

The vector fields are

$$e_1 = e^{z^3}\frac{\partial}{\partial x}, \quad e_2 = e^{z^3}\frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}.$$

It is clear that $\{e_1, e_2, e_3\}$ are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1, \qquad g(e_1, e_2) = g(e_1, e_3) = g(e_2, e_3) = 0,$$

and given by the tensor product

$$g = \frac{1}{e^{2z^3}}(dx \otimes dx + dy \otimes dy) + dz \otimes dz.$$

Let η be the 1-form defined by $\eta(X) = g(X, e_3)$ for any vector field X on M and ϕ be the $(1,1)$ tensor field defined by $\phi(e_1) = e_2$, $\phi(e_2) = -e_1$, $\phi(e_3) = 0$. Then using linearity of g and ϕ, we have

$$\phi^2 X = -X + \eta(X)e_3, \qquad \eta(e_3) = 1, \qquad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for any vector fields on M.

Let ∇ be the Levi-Civita connection with respect to the metric g. Then we get

$$[e_1, e_3] = -3z^2 e_1, \qquad [e_2, e_3] = -3z^2 e_2, \qquad [e_1, e_2] = 0.$$

It follows that the structure of (ϕ, ξ, η, g) can easily be obtained. So it is sufficient to check that the only non-zero components of the second fundamental form Φ are

$$\Phi(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) = -\Phi(\frac{\partial}{\partial y}, \frac{\partial}{\partial x}) = -\frac{1}{e^{2z^3}},$$

and hence

(9.1) $$\Phi = -\frac{1}{e^{2z^3}}(dx \wedge dy),$$

where $\Phi(e_1, e_2) = -1$ and otherwise $\Phi(e_i, e_j) = 0$ for $i \leq j$. Thus the exterior derivation of Φ is given by

(9.2) $$d\Phi = 6z^2 e^{-2z^3}(dx \wedge dy \wedge dz).$$

Since $\eta = dz$, by the help of 9.1 and 9.2, we have

$$d\Phi = -6z^2 \left(\eta \wedge \Phi \right),$$

where α defined $\alpha(z) = -3z^2$. Moreover, it can be noted that Nijenhuis torsion tensor of ϕ vanishes. Hence, the manifold is an α-Kenmotsu.

References

[1] D. E. Blair, Riemannian Geometry of Contact and Symplectic Manifolds, Progress in Mathematics, Birkhâuser, Boston, 2002.

[2] J. B. Jun, U. C. De and G. Pathak, On Kenmotsu Manifolds, J. Korean Math. Soc., 42 (3)(2005), 435-445.

[3] K. Kenmotsu, A Class of Contact Riemannian Manifold, Tohôku Math. J., 24(1972), 93-103.

[4] T. W. Kim, H. K. Pak, Canonical Foliations of Certain Classes of Amost Contact Metric Structures, Acta Math. Sinica, 21(4)(2005), 841-846.

[5] K. Yano, M. Kon, Structures on Manifolds, Series in Pure Mathematics, 3. World Scientific Publishing, Singapore, 1984.

[6] Z. Olszak, On Amost Cosymplectic Manifolds, Kodai Math, 4 (2)(1981), 239-250.

[7] H. Öztürk, N. Aktan and C. Murathan, On α-Kenmotsu Manifolds Satisfying Certain Conditions, Applied Sciences, 12(2010), 115-126.

[8] S. Tanno, The Automorphism Groups of Almost Contact Riemannian Manifolds, Tôhoku Math. J., 21(1969), 21-38.

[9] K. Nomizu, On Hypersurfaces Satisfying a Certain Condition on the Curvature Tensor, Tôhoku Mat. J., 20(1968), 46-69.

[10] Z. I. Szabó, Structure Theorem on Riemannian Spaces Satisfying $R.R = 0$, Journal of Differential Geo., 17(1982), 531-582.

[11] Y. Ogawa, A Condition for a Compact Kaehlerian Space to be Locally Symmetric, Nat. Sci. Rep. Ochanomizu Univ., 28(1977), 21-23.

[12] S. Tanno, Isometric Immersion of Sasakian Manifolds in Spheres, Kodai Math. Sem. Rep., 21(1969), 448-458.

[13] H. Öztürk, N. Aktan, C. Murathan and A. T. Vanlı, Almost α-Cosymplectic f-Manifolds, The Journal of Alexandru Ioan Cuza University, 60 (1)(2014), 211-226.

[14] N. Aktan, M. Yıldırım and C. Murathan, Almost f-Cosymplectic Manifolds, Mediterranean J. Math., 11(2014), 775-787.

[15] C. S. Bagewadi, Venkatesha, Some Curvature Tensors on a Trans-Sasakian Manifold, Turkish J. Math., 31(2007), 111-121.

[16] G. Calvaruso, D. Perrone, Semi-Symmetric Contact Metric Three-Manifolds, Yokohama Math. J., 49(2002), 149-161.

[17] P. Dacko, Z. Olszak, On Conformally Flat Almost Cosymplectic Manifolds with Kaehlerian Leaves, Rend. Sem. Math. Univ. Pol. Torino, 56(1998), 89-103.

[18] I. Vaisman, Conformal Changes of Almost Contact Metric Manifolds, Lecture Notes in Math., Berlin-Heidelberg-New York, 792(1980), 435-443.

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