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PRESERVING PROPERTIES OF THE GENERALIZED BERNARDI-LIBERA-LIVINGSTON INTEGRAL OPERATOR DEFINED ON SOME SUBCLASSES OF STARLIKE FUNCTIONS

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Abstract. In this paper we study the properties of the image of some subclasses of starlike functions, through the generalized Bernardi - Libera - Livingston integral operator. A new subclass of functions with negative coefficients is introduced and we study some properties of this class.

1. Introduction

Let $U = \{z \in \mathbb{C} : |z| < 1\}$ be the unite disk in the complex plane $\mathbb{C}$. We denote by $\mathcal{A}$ the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

We say that f is starlike in U if $f : U \to \mathbb{C}$ is univalent and $f(U)$ is a starlike domain in $\mathbb{C}$ with respect to 0. It is well-known that $f \in \mathcal{A}$ is starlike in U if and only if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad \text{for all } z \in U.$$

The class of starlike functions is denoted by S^*. The function $f \in \mathcal{A}$ is convex in U if and only if $f : U \to \mathbb{C}$ is univalent and $f(U)$ is convex domain in $\mathbb{C}$. The function $f \in \mathcal{A}$ is convex if and only if

$$\operatorname{Re}\frac{zf''(z)}{f'(z)} + 1 > 0, \; z \in U.$$

The class of convex functions is denoted by $\mathcal{K}$.

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Let T denote a subclass of $\mathcal{A}$, consisting of functions f of the form

$$f(z) = z - \sum_{j=2}^{\infty} a_j z^j, \tag{1.1}$$

where $a_j \geq 0$, $j = 2, 3, ...$ and $z \in U$. A function $f \in T$ is called a function with negative coefficients. For the class T, the followings are equivalent [7]:

(i) $\sum\limits_{j=2}^{\infty} j a_j \leq 1$,
(ii) $f \in T \cap S$,
(iii) $f \in T^*$, where $T^* = T \cap S^*$.

In [1] the authors introduced the following subclass of analytic functions

$$S^{**} = \left\{ f \in \mathcal{A} : \left| 1 + \frac{z f''(z)}{f'(z)} \right| < \sqrt{\frac{5}{4}},\ z \in U \right\}. \tag{1.2}$$

In the same paper the authors has shown that the class S^{**} is a subclass of S^* and this class has the property that the composition of two starlike functions from S^{**} is in the class S^* of starlike functions.

In [2] the authors studied the following subclass of convex functions

$$S^{***} = \left\{ f \in \mathcal{A} : \left| 1 - \frac{z f''(z)}{f'(z)} \right| < \sqrt{\frac{5}{4}},\ z \in U \right\}. \tag{1.3}$$

In the same paper the authors has shown that the class S^{***} is a subclass of $\mathcal{K}$, has determined the order of starlikeness of the class S^{***} and have shown that if $f, g \in S^{***}$ then $f \circ g$ is starlike in $U(r_0)$, where $r_0 = \sup\{r > 0 | g(U(r)) \subset U\}$.

Now we consider the generalized Bernardi - Libera - Livingston integral operator

$$F(z) = L_p f(z) = \frac{p+1}{z^p} \int_0^z t^{p-1} f(t) dt, \tag{1.4}$$

where $f \in \mathcal{A}$ and $p > -1$. This operator was studied by Bernardi for $p \in \{1, 2, 3, ...\}$ and for $p = 1$ by Libera.

In this paper we study the properties of the image of the classes S^{**} and S^{***} by the generalized Bernardi-Libera-Livingston integral operator $L_p f(z)$. The subclass S^{***} is defined also for functions with negative coefficients and some other results are derived for this class.

2. Preliminaries

The following preliminary lemmas are necessary to prove our main results.

Definition 2.1. [3][4] Let f and g be analytic functions in U. We say that the function f is subordinate to the function g, if there exist a function w, which is analytic in U and for which $w(0) = 0$, $|w(z)| < 1$ for $z \in U$, such that $f(z) = g(w(z))$, for all $z \in U$. The function f is subordinate to g will be denoted by $f \prec g$.

Definition 2.2. [4] Let Q be the class of analytic functions q in U which has the property that are analytic and injective on $\overline{U} \backslash E(q)$, where

$$E(q) = \{\zeta \in \partial U : \lim_{z \longrightarrow \zeta} q(z) = \infty\},$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial U \backslash E(q)$.

Lemma 2.1. *[Miller-Mocanu] Let* $q \in Q$, *with* $q(0) = a$, *and let* $p(z) = a + a_n z^n + \dots$ *be analytic in* U *with* $p(z) \not\equiv a$ *and* $n \geq 1$. *If* $p \not\prec q$, *then there are two points* $z_0 = r_0 e^{i\theta_0} \in U$, *and* $\zeta_0 \in \partial U \backslash E(q)$ *and a real number* $m \in [n, \infty)$ *for which* $p(U_{r_0}) \subset q(U)$,
(i) $p(z_0) = q(\zeta_0)$
(ii) $z_0 p'(z_0) = m \zeta_0 q'(\zeta_0)$
(iii) $\operatorname{Re} \frac{z_0 p''(z_0)}{p'(z_0)} + 1 \geq m \operatorname{Re} \left(\frac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)} + 1 \right)$.

The following result is a particular case of Lemma 2.1.

Lemma 2.2 (Miller-Mocanu)**.** *Let* $p(z) = 1 + a_n z^n + \dots$ *be analytic in* U *with* $p(z) \not\equiv 1$ *and* $n \geq 1$.
If $p(z) \not\prec q(z) = M \frac{Mz+1}{M+z}$ *then there is a point* $z_0 \in U$, *and* $\zeta_0 \in \partial U \backslash E(q)$ *and a real number* $m \in [n, \infty)$ *for which* $p(U_{r_0}) \subset q(U)$, *such that*
(i) $p(z_0) = q(\zeta_0)$, *where* $\zeta_0 = e^{i\theta}$
(ii) $z_0 p'(z_0) = m e^{i\theta} M \frac{M^2 - 1}{(M + e^{i\theta})^2}$,
(iii) $\operatorname{Re} z_0^2 p''(z_0) + z_0 p'(z_0) \leq 0$.

3. Main Results

Theorem 3.1. *Let*

$$F(z) = L_p f(z) = \frac{p+1}{z^p} \int_0^z t^{p-1} f(t) dt.$$

If $p \geq \sqrt{\frac{5}{4}}$ *and* $f \in S^{**}$, *then* $F \in S^{**}$.

Proof.

$$z^p F(z) = (p+1) \int_0^z f(t) t^{p-1} dt. \tag{3.1}$$

Differentiating the relation (3.1) we obtain

$$p z^{p-1} F(z) + z^p F'(z) = (p+1) f(z) z^{p-1}. \tag{3.2}$$

Dividing with z^{p-1} the relation (3.2) we get

$$pF(z) + zF'(z) = (p+1) f(z). \tag{3.3}$$

Now differentiating (3.3) we obtain

$$(p+1) F'(z) + zF''(z) = (p+1) f'(z), \tag{3.4}$$

which is equivalent to

$$F'(z) \left[p + 1 + \frac{zF''(z)}{F'(z)} \right] = (p+1) f'(z). \tag{3.5}$$

We note $u = u(z) = 1 + \dfrac{zF''(z)}{F'(z)}$ and we obtain

$$F'(z)(p + u) = (p+1) f'(z). \tag{3.6}$$

Differentiating the above relation we get

$$F''(z)(p+u)+F'(z)u'=(p+1)f''(z) \tag{3.7}$$

Next we divide the relation (3.7) with (3.6) and results

$$\frac{F''(z)}{F'(z)}+\frac{u'}{p+u}=\frac{f''(z)}{f'(z)} \tag{3.8}$$

Multiplied the relation (3.8) with z and adding 1 to each side we get

$$u+\frac{zu'}{p+u}=1+\frac{zf''(z)}{f'(z)}.$$

The condition $\left|1+\frac{zf''(z)}{f'(z)}\right|<\sqrt{\frac{5}{4}}$ which is necessary to a holomorphic function to be in the class S^{**} is equivalent with

$$\left|u(z)+\frac{zu'(z)}{p+u(z)}\right|<\sqrt{\frac{5}{4}}=M. \tag{3.9}$$

To finish the proof we must to demonstrate that

$$\left|M\frac{Me^{i\theta}+1}{M+e^{i\theta}}+\frac{me^{i\theta}M\dfrac{M^2-1}{(M+e^{i\theta})^2}}{p+M\dfrac{Me^{i\theta}+1}{M+e^{i\theta}}}\right|\geq M. \tag{3.10}$$

Dividing (3.10) by M we get

$$\left|\frac{Me^{i\theta}+1}{M+e^{i\theta}}+\frac{me^{i\theta}(M^2-1)}{p(M+e^{i\theta})^2+M(M+e^{i\theta})(Me^{i\theta}+1)}\right|\geq 1. \tag{3.11}$$

The (3.11) is equivalent with

$$\left|\frac{M+e^{-i\theta}}{M+e^{i\theta}}+m\frac{M^2-1}{p(M+e^{i\theta})^2+M(M+e^{i\theta})(Me^{i\theta}+1)}\right|\geq 1. \tag{3.12}$$

The (3.12) inequality is equivalent with

$$\left|1+m\frac{M^2-1}{p(M+e^{i\theta})(M+e^{-i\theta})+M(Me^{i\theta}+1)(M+e^{-i\theta})}\right|\geq 1. \tag{3.13}$$

The real part of

$$W=\frac{m(M^2-1)}{p(M+e^{i\theta})(M+e^{-i\theta})+M(Me^{i\theta}+1)(M+e^{-i\theta})}$$

is positive if and only if

$$V=\operatorname{Re}[p(M+e^{i\theta})(M+e^{-i\theta})+M(Me^{i\theta}+1)(M+e^{-i\theta})]>0.$$

On the other hand we have

$$V=p(M^2+1+2M\cos\theta)+M[(M^2+1)\cos\theta+2M]\geq(p-M)(M-1)^2.$$

Thus the inequality $p\geq M$ implies $\operatorname{Re}W\geq 0$, and we get $|1+W|\geq 1$.
This inequality contradicts (3.9) and the proof is done. □

Theorem 3.2. *Let*

$$F(z) = L_p f(z) = \frac{p+1}{z^p} \int_0^z t^{p-1} f(t)dt, \ \ p > -2.$$

If $f \in S^{***}$ then $F \in S^{***}$.

Proof.

$$z^p F(z) = (p+1) \int_0^z f(t) t^{p-1} dt. \tag{3.14}$$

Differentiating the relation (3.14) we obtain

$$pz^{p-1}F(z) + z^p F'(z) = (p+1) f(z) z^{p-1}. \tag{3.15}$$

Dividing with z^{p-1} the relation (3.15) we get

$$pF(z) + zF'(z) = (p+1) f(z). \tag{3.16}$$

Now differentiating (3.16) we obtain

$$(p+1)F'(z) + zF''(z) = (p+1) f'(z),$$

which is equivalent to

$$F'(z) \left[p + 1 + \frac{zF''(z)}{F'(z)} \right] = (p+1) f'(z). \tag{3.17}$$

We note $v = v(z) = 1 - \dfrac{zF''(z)}{F'(z)}$ and we obtain

$$F'(z)(p + 2 - v) = (p+1) f'(z). \tag{3.18}$$

Differentiating the above relation we get

$$F''(z)(p+2-v) + F'(z)(-v)' = (p+1) f''(z) \tag{3.19}$$

Next we divide the relation (3.19) with (3.18) and results

$$\frac{F''(z)}{F'(z)} - \frac{v'}{p+2-v} = \frac{f''(z)}{f'(z)} \tag{3.20}$$

Multiplied the relation (3.20) with $-z$ and adding 1 to each side we get

$$v + \frac{zv'}{p+2-v} = 1 - \frac{zf''(z)}{f'(z)}.$$

The condition $\left| 1 - \dfrac{zf''(z)}{f'(z)} \right| < \sqrt{\dfrac{5}{4}}$ which is necessary to a holomorphic function to be in the class S^{***} is equivalent with

$$\left| v(z) + \frac{zv'(z)}{p+2-v(z)} \right| < \sqrt{\frac{5}{4}} = M. \tag{3.21}$$

We have to prove that

$$|v(z)| < \frac{\sqrt{5}}{2} = M. \tag{3.22}$$

The following equivalence holds

$$v(0) = 1 \text{ and } |v(z)| < M \Leftrightarrow$$

$$v(z) \prec M\frac{zM+1}{M+z}. \tag{3.23}$$

Now we have

$$q(z) = M\frac{zM+1}{M+z} \text{ and } q'(z) = M\frac{M^2-1}{(M+z)^2}.$$

If the subordination (3.23) does not hold, then according to the Miller-Mocanu lemma there are two complex numbers $\zeta_0 = e^{i\theta} \in \partial U$ and $z_0 \in U$, and a real number $m \in [1, \infty)$ such that

$$v(z_0) = M\frac{Me^{i\theta}+1}{e^{i\theta}+M}$$

and

$$z_0v'(z_0) = me^{i\theta}M\frac{M^2-1}{(M+e^{i\theta})^2}.$$

Thus

$$\left|v(z_0) + \frac{z_0v'(z_0)}{p+2-v(z_0)}\right| = \left|M\frac{Me^{i\theta}+1}{e^{i\theta}+M} + m\frac{e^{i\theta}M\dfrac{M^2-1}{(M+e^{i\theta})^2}}{p+2-M\dfrac{Me^{i\theta}+1}{e^{i\theta}+M}}\right|$$

$$= \left|M\frac{Me^{i\theta}+1}{e^{i\theta}+M} + m\frac{e^{i\theta}M(M^2-1)}{(p+2)(M+e^{i\theta})^2 - M(Me^{i\theta}+1)(M+e^{i\theta})}\right|$$

$$= M\left|\frac{M+e^{-i\theta}}{M+e^{i\theta}} + m\frac{M^2-1}{(p+2)(M+e^{i\theta})^2 - M(Me^{i\theta}+1)(M+e^{i\theta})}\right| \geq M.$$

Dividing the above inequality by $M\frac{M+e^{-i\theta}}{M+e^{i\theta}}$ we obtain

$$\left|1+m\frac{M^2-1}{(p+2)(M+e^{i\theta})(M+e^{-i\theta}) - M(Me^{i\theta}+1)(M+e^{-i\theta})}\right| \geq 1. \tag{3.24}$$

If we prove that

$$\operatorname{Re} m\frac{M^2-1}{(p+2)(M+e^{i\theta})(M+e^{-i\theta}) - M(Me^{i\theta}+1)(M+e^{-i\theta})} > 0, \tag{3.25}$$

then the inequality (3.24) holds. The (3.25) inequality is true if and only if

$$Q = \operatorname{Re}[(p+2)(M+e^{i\theta})(M+e^{-i\theta}) - M(Me^{i\theta}+1)(M+e^{-i\theta})] > 0.$$

It is easly seen that

$$Q = (p+2)(M^2+2M\cos\theta+1) - M(2M+M^2\cos\theta+\cos\theta)$$

$$= (p+2)(M^2+1) - 2M^2 + 2(p+2)M\cos\theta - M(M^2\cos\theta+\cos\theta).$$

Since

$$\frac{2M^2+M(M^2+1)\cos\theta}{M^2+1+2M\cos\theta} \geq \frac{2M^2-M(M^2+1)}{M^2+1-2M} = -M,$$

it follows that the inequality $p+2 \geq -M$ implies $Q > 0$ and consequently (3.25) holds.

The inequality (3.25) contradicts the subordination (3.23) and consequently the inequality (3.22) holds. □

In the followings we define the class S^{***} for functions with negative coefficients.

Definition 3.1. The function $f \in T$ belongs to the class $TS^{***} = S^{***} \cap T$ if

$$\left|1 - \frac{zf''(z)}{f'(z)}\right| < \sqrt{\frac{5}{4}},\ z \in U.$$

Below we give a coefficient delimitation theorem for the class TS^{***}.

Theorem 3.3. *The function $f \in T$ belongs to the class TS^{***} if and only if*

$$\sum_{j=2}^{\infty} j\Big(j - 2 + \frac{\sqrt{5}}{2}\Big)a_j < \frac{\sqrt{5}}{2} - 1. \tag{3.26}$$

Proof. It is easly seen that the inequality (3.26) is equivalent to

$$\frac{1 + \sum\limits_{j=2}^{\infty} j(j-2)a_j}{1 - \sum\limits_{j=2}^{\infty} ja_j} < \frac{\sqrt{5}}{2}.$$

On the other hand we have

$$\left|1 - \frac{zf''(z)}{f'(z)}\right| = \left|1 + \frac{\sum\limits_{j=2}^{\infty} j(j-1)a_j z^{j-1}}{1 - \sum\limits_{j=2}^{\infty} ja_j z^{j-1}}\right| = \left|\frac{1 + \sum\limits_{j=2}^{\infty} j(j-2)a_j z^{j-1}}{1 - \sum\limits_{j=2}^{\infty} ja_j z^{j-1}}\right|$$

$$\leq \frac{1 + \sum\limits_{j=2}^{\infty} j(j-2)a_j |z|^{j-1}}{1 - \sum\limits_{j=2}^{\infty} ja_j |z|^{j-1}} \leq \frac{1 + \sum\limits_{j=2}^{\infty} j(j-2)a_j}{1 - \sum\limits_{j=2}^{\infty} ja_j} < \frac{\sqrt{5}}{2},$$

which implies $f \in TS^{***}$.

To prove the reciproc implication let suppose $\left|1 - \frac{zf''(z)}{f'(z)}\right| < \frac{\sqrt{5}}{2}$, where $z \in U$. The above inequality is equivalent to

$$\left|\frac{1 + \sum\limits_{j=2}^{\infty} j(j-2)a_j z^{j-1}}{1 - \sum\limits_{j=2}^{\infty} ja_j z^{j-1}}\right| < \frac{\sqrt{5}}{2}.$$

If we put $z \to 1$, then it follows that

$$\left| \frac{1+\sum_{j=2}^{\infty} j(j-2)a_j}{1-\sum_{j=2}^{\infty} ja_j} \right| < \frac{\sqrt{5}}{2}.$$

□

Next we prove that the class TS^{***} is closed under convolution with convex functions.

Theorem 3.4. *Let $f \in T$ be of the form (1.1) and $\phi(z) = z - \sum_{j=2}^{\infty} b_j z^j$ convex in U, where $b_j \geq 0$ for $j \in \{2, 3, ...\}$. If $f \in TS^{***}$ then $f * \phi \in TS^{***}$.*

Proof. Let

$$(f * \phi)(z) = z - \sum_{j=2}^{\infty} a_j b_j z^j.$$

Suppose $f \in TS^{***}$. Then by Theorem 3.3 we have

$$\sum_{j=2}^{\infty} j\Big(j-2+\frac{\sqrt{5}}{2}\Big)a_j < \frac{\sqrt{5}}{2} - 1. \tag{3.27}$$

To finish our proof, we must to show

$$\sum_{j=2}^{\infty} j\Big(j-2+\frac{\sqrt{5}}{2}\Big)a_j b_j < \frac{\sqrt{5}}{2} - 1.$$

Since $\phi \in T$ the above inequality is equivalent to

$$\sum_{j=2}^{\infty} j\Big(j-2+\frac{\sqrt{5}}{2}\Big)a_j |b_j| < \frac{\sqrt{5}}{2} - 1. \tag{3.28}$$

Because ϕ is convex, by the coefficient delimitation theorem for convex functions we have $|b_j| \leq 1$, for $j = 2, 3, ...$
Then from (3.28) we get

$$\sum_{j=2}^{\infty} j\Big(j-2+\frac{\sqrt{5}}{2}\Big)a_j |b_j| \leq \sum_{j=2}^{\infty} j\Big(j-2+\frac{\sqrt{5}}{2}\Big)a_j < \frac{\sqrt{5}}{2} - 1,$$

and the proof is done. □

Theorem 3.5. *Let*

$$F(z) = L_p f(z) = \frac{p+1}{z^p} \int_0^z t^{p-1} f(t)dt, \ p \in (-1, 0].$$

If $f \in TS^{***}$, then $F \in TS^{***}$.

Proof. Let $f \in TS^{***}$ be a function of the form $f(z) = z - \sum_{j=2}^{\infty} a_j z^j$. Then according to Theorem 3.3 we have

$$f \in TS^{***} \Leftrightarrow \frac{1 + \sum_{j=2}^{\infty} j(j-2)a_j}{1 - \sum_{j=2}^{\infty} j a_j} < \frac{\sqrt{5}}{2} \Leftrightarrow \sum_{j=2}^{\infty} j\Big(j - 2 + \frac{\sqrt{5}}{2}\Big)a_j < \frac{\sqrt{5}}{2} - 1.$$

On the other hand we have

$$F(z) = z - \sum_{j=2}^{\infty} A_j z^j,$$

where $A_j = a_j \cdot \frac{1+p}{j+p}$ and $j \geq 2$.
According to Theorem 3.3, the function F belongs to the class TS^{***} if and only if

$$\sum_{j=2}^{\infty} j\Big(j - 2 + \frac{\sqrt{5}}{2}\Big)A_j < \frac{\sqrt{5}}{2} - 1. \tag{3.29}$$

The inequality (3.29) easily follows because $j + p > 1 + p$, where $p \in (-1, 0]$ and we get

$$jA_j\left(j - 2 + \frac{\sqrt{5}}{2}\right) = ja_j \frac{1+p}{j+p}\left(j - 2 + \frac{\sqrt{5}}{2}\right) < ja_j(1+p)\left(j - 2 + \frac{\sqrt{5}}{2}\right) < \frac{\sqrt{5}}{2} - 1.$$

□

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