Eastern Anatolian Journal of Science
Volume IX, Issue II, 2023, 27-36

Eastern Anatolian Journal of Science

A Large Class of Closed, Bounded and Convex Subsets in Köthe-Toeplitz Duals of Certain Generalized Difference Sequence Spaces with Fixed Point Property

Veysel NEZİR[1]* and Nizami MUSTAFA[2]

[1]Kafkas University, Faculty of Science and Letters, Department of Mathematics, Kars, Türkiye,

veyselnezir@yahoo.com

[2]Kafkas University, Faculty of Science and Letters, Department of Mathematics, Kars, Türkiye,

nizamimustafa@gmail.com

Abstract

In the present study, we consider the Köthe-Toeplitz duals for the 2nd order and 3rd order types difference sequence space generalizations by Et and Esi studied in 2000. We work on Goebel and Kuczumow analogy for those spaces to obtain large classes of closed, bounded and convex subsets satisfying the fixed point property. In the study, we also study some other Banach spaces in connection with the Köthe-Toeplitz duals for the 2nd order and 3rd order generalized difference sequence spaces.

Keywords: Fixed point property, nonexpansive mapping, Köthe-Toeplitz dual.

1. *Introduction*

When a Banach space satisfies the condition that every invariant nonexpansive mappings defined on any closed, bounded and convex (cbc) nonemtpy subset has a fixed point, then it is said that the space has the fixed point property for nonexpansive mappings. We need to note that distances between images of distant points under nonexpansive mapping cannot exceed the distances between the points taken. Researchers have considered categorizing Banach spaces with this property.

Received: 02.11.2023
Accepted: 31.12.2023
Published: 31.12.2023
**Corresponding author:* Veysel NEZİR, *PhD*
Kafkas University, Faculty of Science and Letters, Department of Mathematics, Kars Türkiye
E-mail: veyselnezir@yahoo.com
Cite this article as: V. Nezir and N. Mustafa, A Large Class of Closed, Bounded and Convex Subsets in Köthe-Toeplitz Duals of Certain Generalized Difference Sequence Spaces with Fixed Point Property, *Eastern Anatolian Journal of Science, Vol. 9, Issue 2, 27-36, 2023.*

Firstly, in (Browder 1965) it is found that Hilbert spaces have the property and the result was generalized in (Kirk 1965) to reflexive Banach spaces with normal structure.

Then, researchers have especially investigated nonreflexive classical Banach spaces and wondered if they can be renormable and falls in the same category with their equivalent norm while they fail to be members of the category with their usual norm but they were able to detect some nonreflexive Banach spaces which have equivalent norms and they become to have the fixed point property with those renormings. The first example was given by Lin (2008) for ℓ^1. Then even it has been asked if the same could have been done for c_0, but the answer still remains open. Since the researchers have considered trying to obtain the analogous results for well-known other classical nonreflexive Banach spaces, another experiment was done for Lebesgue integrable functions space $L_1[0,1]$ by Hernandes Lineares and Maria (2012) but they were able to obtain the positive answer when they restricted the nonexpansive mappings by assuming they were affine as well. One can say that there is no doubt most tries have been inspired by the ideas of the study (Goebel and Kuczumow 1979) where Goebel and Kuczumow proved that while ℓ^1 fails the fixed point property since one can easily find a cbc nonweakly compact subset there and a fixed point free invariant nonexpansive map, it is possible to find a very large class subsets in target such that invariant nonexpansive mappings defined on the members of the class have fixed points. In fact, it is easy to notice the traces of those ideas in (Lin 2008) work. Even Goebel and Kuczumow's work has inspired many other researchers to investigate if there exist more example of nonreflexive Banach spaces with large classes satisfying fixed point property. For example, Kaczor and Prus (2004) wanted to generalize Goebel and Kuczumow's findings by investigating if the same could be done for asymptotically nonexpansive mappings. Then, as their result, they proved that under affinity condition, asymptotically nonexpansive invariant mappings defined on a large class of cbc subsets in ℓ^1 can have fixed points. Moreover, in (Everest 2013) Kaczor and Prus's results were extended by having been found larger classes satisfying the fixed point property for affine asymptotically nonexpansive mappings. Thus, affinity condition become an easiness tool for their works. In fact, as an another well-known nonreflexive Banach space, Lebesgue space $L_1[0,1]$ was studied in (Hernandes Lineares and Japón 2012) and in their study they obtained an analogous result to (Lin 2008) as they showed that $L_1[0,1]$ can be renormed to have the fixed point property for affine nonexpansive mappings.

In this study we will investigate some Banach spaces analogous to ℓ^1. We aim to discuss the analogous results for Köthe-Toeplitz duals of certain generalized difference sequence spaces studied by Et and Esi (2000). We show that there exists a very large class of cbc subsets in those spaces with fixed point property for nonexpansive mappings. Thus, first we will recall the definition of Cesàro sequence spaces introduced by Shiue (1970) and next we will give Kızmaz's construction in (Kızmaz 1981) for difference sequence spaces since the dual space we work on is obtained from the generalizations of Kızmaz's idea which are derived differently by many researchers such as (Çolak 1989), (Et 1996), (Et and Çolak 1995), (Et and Esi 2000), (NgPeng-Nung and and LeePeng-Yee 1978), (Orhan 1983), and (Tripathy et. al. 2005). But we need to note that Et and Esi's work (Et and Esi 2000) and the further study by Et and Çolak (1995) used the new type of difference sequence definition from Çolak's work (Çolak 1989).

2. *Materials and Methods*

First we recall that (Shiue 1970) introduced the Cesàro sequence spaces written as

$$\mathrm{ces}_p = \left\{ (x_n)_n \subset \mathbb{R} \,\middle|\, \left(\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^p \right)^{1/p} < \infty \right\}$$

such that $\ell^p \subset \mathrm{ces}_p$ and

$$\mathrm{ces}_\infty = \left\{ x = (x_n)_n \subset \mathbb{R} \,\middle|\, \sup_n \frac{1}{n} \sum_{k=1}^{n} |x_k| < \infty \right\}$$

such that $\ell^\infty \subset ces_\infty$ where $1 \le p < \infty$. Their topological properties have been investigated and it has been seen that for $1 < p < \infty$, ces_p is a seperable reflexive Banach space. Furthermore, many researchers such as (Cui 1999), (Cui, Hudzik, and Li 2000) and (Cui, Meng, and Pluciennik 2000) were able to prove that for $1 < p < \infty$, Cesàro sequence space ces_p has the fixed point property.

Easiest way to show that was due to both reflexivity by the fact the space has normal structure when $1 < p < \infty$ (using the fact via (Kirk 1965)) and the space having the weak fixed point property because of its Garcia-Falset coefficient is less than 2 (see for example (Falset 1997)). A good reference about fixed point theory results for Cesàro sequence spaces can be a survey in (Chen et. al. 2001).

After the introduction of Cesàro sequence spaces, Kızmaz (1981), denoting by $\ell^\infty(\triangle)$, $\mathrm{c}(\triangle)$ and $\mathrm{c}_0(\triangle)$, introduced difference sequence spaces for ℓ^∞, c and c_0 where they are the Banach spaces of bounded, convergent and null sequences, respectively. Here $\triangle$ represented the difference operator applied to the sequence $x = (x_n)_n$ with the rule given by $\triangle x = (x_k - x_{k+1})_k$. Kızmaz studied then Köthe-Toeplitz Duals and topological properties for them.

As earlier it was stated, Çolak was one of the researchers generalizing Kızmaz's (1981) ideas. In his work, Çolak (1989) obtained the generalized version of the difference sequence space in the following way by picking an arbitrary sequence of nonzero complex values $v = (v_n)_n$. The new difference operator is denoted by $\triangle_v$ and the difference sequence of a sequence $x = (x_n)_n$ is written as $\triangle_v x = (v_k x_k - v_{k+1} x_{k+1})_k$. Then, in their study, Et and Esi (2000) defined a generalized difference sequence space as below.

$$\triangle_v(\ell^\infty) = \{x = (x_n)_n \subset \mathbb{R} \mid \triangle_v x \in \ell^\infty\},$$
$$\triangle_v(\mathrm{c}) = \{x = (x_n)_n \subset \mathbb{R} \mid \triangle_v x \in \mathrm{c}\},$$
$$\triangle_v(\mathrm{c}_0) = \{x = (x_n)_n \subset \mathbb{R} \mid \triangle_v x \in \mathrm{c}_0\}.$$

Then, they also defined m^{th} order generalized type difference sequence for any $m \in \mathbb{N}$ given by

$$\triangle_v^0 x = (v_k x_k)_k,$$

$\triangle_v^m x = (\triangle_v^m x_k)_k = (\triangle_v^{m-1} x_k - \triangle_v^{m-1} x_{k+1})_k$ with $\triangle_v^m x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} v_{k+i} x_{k+i}$ for each $k \in \mathbb{N}$.

In fact, Et and Esi (2000) further generalized the above difference sequence spaces and Bektaş, Et and

Çolak (2004) not only found the Köthe-Toeplitz duals for them but also obtained the duals for the generalized types of Et and Esi's. In this study, we will consider the 2nd order and 3rd order types which have the following norms respectively:

$$\|x\|_v^{(2)} = |v_1x_1| + |v_2x_2| + \|\triangle_v^m x\|_\infty,$$
$$\|x\|_v^{(3)} = |v_1x_1| + |v_2x_2| + |v_3x_3| + \|\triangle_v^m x\|_\infty.$$

Then the corresponding Köthe-Toeplitz duals were obtained as in (Bektaş, Et and Çolak 2004) and (Et and Esi 2000) such that they are written as below:

$$U_1^2 := \{a = (a_n)_n \subset \mathbb{R} | (n^2 {v_n}^{-1} a_n)_n \in \ell^1\}$$
$$= \left\{ a = (a_n)_n \subset \mathbb{R}:\ \|a\|^{(2)} = \sum_{k=1}^{\infty} \frac{k^2|a_k|}{|v_k|} < \infty \right\}$$

and

$$U_1^3 := \{a = (a_n)_n \subset \mathbb{R} | (n^3 {v_n}^{-1} a_n)_n \in \ell^1\}$$
$$= \left\{ a = (a_n)_n \subset \mathbb{R}:\ \|a\|^{(3)} = \sum_{k=1}^{\infty} \frac{k^3|a_k|}{|v_k|} < \infty \right\}.$$

Note that $U_1^m \subset \ell^1$ if $k^m|{v_k}^{-1}| > 1$ for each $k \in \mathbb{N}$ and $\ell^1 \subset U_1^m$ if $k^m|{v_k}^{-1}| < 1$ for each $k \in \mathbb{N}$ and $m = 2, 3$.

In this study, we will also condiser two more Banach spaces which are closely related to the above ones. We will denote them by W_1^2 and W_1^3 and their definitions are as follow:

$$W_1^2 := \left\{ a = (a_n)_n \subset \mathbb{R} \middle| \left(\frac{a_n}{n^2 v_n}\right)_n \in \ell^1 \right\}$$
$$= \left\{ a = (a_k)_k \subset \mathbb{R}:\ \|a\|_{(2)} = \sum_{k=1}^{\infty} \frac{|a_k|}{k^2|v_k|} < \infty \right\}$$

and

$$W_1^3 := \left\{ a = (a_n)_n \subset \mathbb{R} \middle| \left(\frac{a_n}{n^3 v_n}\right)_n \in \ell^1 \right\}$$
$$= \left\{ a = (a_k)_k \subset \mathbb{R}:\ \|a\|_{(3)} = \sum_{k=1}^{\infty} \frac{|a_k|}{k^3|v_k|} < \infty \right\}.$$

Note that $W_1^m \subset \ell^1$ if $k^m|v_k| < 1$ for each $k \in \mathbb{N}$ and $\ell^1 \subset W_1^m$ if $k^m|v_k| > 1$ for each $k \in \mathbb{N}$ and $m = 2, 3$.

These Banach spaces in connection with the above Köthe-Toeplitz duals are types of degenerate Lorentz-Marcinkiewicz spaces. The reader is recommended to see for example (Lindenstrauss and Tzafriri 1977) about them.

We will need the below well-known preliminaries before giving our main results. (Goebel and Kirk 1990) may be suggested as a good reference for these fundamentals.

Definition 2.1. *Consider that* $(X, \|\cdot\|)$ *is a Banach space and let* C *be a non-empty cbc subset. Let* $: C \to C$ *be a mapping. We say that*

1. T *is an affine mapping if for every* $t \in [0,1]$ *and* $a, b \in C$, $T\big((1-t)a + tb\big) = (1-t)T(a) + t\,T(b)$.
2. T *is a nonexpansive mapping if for every* $a, b \in C$, $\| T(a) - T(b) \| \leq \| a - b \|$.

Then, we will easily obtain an anologous key lemma from the below lemma in the work (Goebel and Kuczumow 1979).

Lemma 2.2. *Let* $\{u_n\}$ *be a sequence in* ℓ^1 *converging to* u *in weak-star topology, then for every* $w \in \ell^1$,

$$r(w) = r(u) + \|w - u\|_1$$

where

$$r(w) = \limsup_n \|u_n - w\|_1.$$

Note that our scalar field in this study will be real numbers although (Çolak 1989) considers complex values of $v = (v_n)_n$ while introducing his structer of the difference sequence which is taken as the fundamental concept in this study.

3. *Results*

In this section, we will present our results. As earlier it has been mentioned in the first section, we investigate Goebel and Kuczumow's analogy for the spaces U_1^2, U_1^3, W_1^2 and W_1^3. We aim to show that there are large classes of cbc subsets in these spaces such that every nonexpansive invariant mapping defined on the subsets in the classes taken has a fixed point. Recall that the invariant mappings have the same domain and range.

Firstly, due to isometric isomorphism, using Lemma 2.2, we will provide the straight analogous result as a lemma below which will be a key step as in the works such as (Goebel and Kuczumow 1979) and (Everest 2013) and in fact the methods in the study (Everest 2013) will be our lead in this work.

Lemma 3.1. *Let* $\{u_n\}$ *be a sequence in a Banach space* Z *which is a member of the spaces* U_1^2, U_1^3, W_1^2 *or* W_1^3 such that $\|.\|$ denotes the norm for each space and assume $\{u_n\}$ *converges to* u *in weak-star topology, then for every* $w \in Z$,

$$r(w) = r(u) + \|w - u\|$$

where $r(w) = \limsup_n \|u_n - w\|$.

Then we prove the following theorems as our main results.

Theorem 3.2. *Fix $b \in (0,1)$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence defined by $f_1 := b\, v_1\, e_1$, $f_2 := \frac{b\, v_2\, e_2}{2^2}$, and $f_n := \frac{v_n}{n^2} e_n$ for all integers $n \geq 3$ where the sequence $(e_n)_{n\in\mathbb{N}}$ is the canonical basis of both c_0 and ℓ^1. Then, consider the cbc subset $E^{(2)} = E_b{}^{(2)}$ of U_1^2 by*

$$E^{(2)} := \left\{\sum_{n=1}^{\infty} t_n f_n : \forall n \in \mathbb{N}, t_n \geq 0 \text{ and } \sum_{n=1}^{\infty} t_n = 1\right\}.$$

Then, $E^{(2)}$ has the fixed point property for $\|\,.\,\|^{(2)}$-nonexpansive mappings.

Proof. Fix $b \in (0,1)$. Let $T: E^{(2)} \to E^{(2)}$ be a nonexpansive mapping. Then, there exists a sequence so called aproximate fixed point sequence $\left(u^{(n)}\right)_{n\in\mathbb{N}} \in E^{(2)}$ such that $\left\|Tu^{(n)} - u^{(n)}\right\|^{(2)} \underset{n}{\to} 0$. Due to isometric isomorphism U_1^2 shares common geometric properties with ℓ^1 and so both U_1^2 and its predual have same fixed point theory facts to ℓ^1 and c_0, respectly. Thus, considering that on bounded subsets the weak star topology on ℓ^1 is equivalent to the coardinate-wise convergence topology, and c_0 is separable, in U_1^2, the unit closed ball is weak*-sequentially compact due to Banach-Alaoglu theorem. Then we can say that we may denote the weak* closure of the set $E^{(2)}$ by

$$C^{(2)} := \overline{E^{(2)}}^{w^*}$$
$$= \left\{\sum_{n=1}^{\infty} t_n\ f_n : \text{each } t_n \geq 0 \text{ and } \sum_{n=1}^{\infty} t_n \leq 1\right\}$$

and without loss of generality, we may pass to a subsequence if necessary, and get a weak* limit $u \in C^{(2)}$ of $u^{(n)}$. Then, by Lemma 3.1, we have a function $r: U_1^2 \to [0,\infty)$ defined by

$$r(w) = \limsup_n \left\|u^{(n)} - w\right\|^{(2)}, \qquad \forall w \in U_1^2$$

such that *for every* $w \in U_1^2$,

$$r(w) = r(u) + \|u - w\|^{(2)}.$$

Case 1: $u \in E^{(2)}$.

Then, $r(Tu) = r(u) + \|Tu - u\|^{(2)}$ and

$$\begin{aligned} r(Tu) &= \limsup_n \left\|Tu - u^{(n)}\right\|^{(2)} \\ &\leq \limsup_n \left\|Tu - T\left(u^{(n)}\right)\right\|^{(2)} \\ &\quad + \limsup_n \left\|u^{(n)} - T\left(u^{(n)}\right)\right\|^{(2)} \\ &\leq \limsup_n \left\|u - u^{(n)}\right\|^{(2)} + 0 \\ &= r(u). \end{aligned} \tag{3.2.1}$$

Thus, $r(Tu) = r(u) + \|Tu - u\|^{(2)} \leq r(u)$ and so $\|Tu - u\|^{(2)} = 0$. Therefore, $Tu = u$.

Case 2: $u \in C^{(2)} \backslash E^{(2)}$.

Then, we may find scalars satisfying $\sum_{n=1}^{\infty} \delta_n < 1$ *and* $\forall n \in \mathbb{N}, \delta_n \geq 0$ such that $u = \sum_{n=1}^{\infty} \delta_n f_n$. Then, let $\gamma := 1 - \sum_{n=1}^{\infty} \delta_n$ and for $\alpha \in \left[\frac{-\delta_1}{\gamma}, \frac{\delta_2}{\gamma} + 1\right]$ define

$$h_\alpha := (\delta_1 + \alpha\gamma) f_1 + (\delta_2 + (1-\alpha)\gamma) f_2 + \sum_{n=3}^{\infty} \delta_n f_n.$$

Then,

$$\begin{aligned} \|h_\alpha - u\|^{(2)} &= \left\|\alpha b \gamma v_1 e_1 + (1-\alpha)\gamma \frac{b\, v_2 e_2}{2^2}\right\|^{(2)} \\ &= \mathrm{b}|\alpha|\gamma + b|1-\alpha|\gamma. \end{aligned}$$

$\|h_\alpha - u\|^{(2)}$ is minimized for $\alpha \in [0,1]$ and its minimum value would be $b\gamma$.

Now fix $w \in E^{(2)}$. Then, we may find scalars t_n satisfying $\forall n \in \mathbb{N}, t_n \geq 0$ and $\sum_{n=1}^{\infty} t_n = 1$ such that $w = \sum_{n=1}^{\infty} t_n f_n$.
Then,

$$\begin{aligned} \|\mathrm{w} - u\|^{(2)} &= \left\|\sum_{k=1}^{\infty} t_k f_k - \sum_{k=1}^{\infty} \delta_k f_k\right\|^{(2)} \\ &= \mathrm{b}|t_1 - \delta_1| + \mathrm{b}|t_2 - \delta_2| + \sum_{k=3}^{\infty} |t_k - \delta_k| \\ &= \mathrm{b}|t_1 - \delta_1| + \mathrm{b}|t_2 - \delta_2| + b\sum_{k=3}^{\infty} |t_k - \delta_k| \\ &\quad + (1-b)\sum_{k=3}^{\infty} |t_k - \delta_k| \\ &\geq \mathrm{b}\left|\sum_{k=1}^{\infty} t_k - \delta_k\right| + (1-b)\sum_{k=3}^{\infty} |t_k - \delta_k| \\ &= \mathrm{b}\left|\sum_{k=1}^{\infty} t_k - \sum_{k=1}^{\infty} \delta_k\right| + (1-b)\sum_{k=3}^{\infty} |t_k - \delta_k| \\ &= \mathrm{b}|1 - (1-\gamma)| + (1-b)\sum_{k=3}^{\infty} |t_k - \delta_k|. \end{aligned}$$

Hence,

$$\|\mathrm{w} - u\|^{(2)} \geq b\gamma + (1-b)\sum_{k=3}^{\infty} |t_k - \delta_k| \geq b\gamma$$

and the equality is obtained if and only if $(1-b)\sum_{k=3}^{\infty} |t_k - \delta_k| = 0$; that is, we have $\|\mathrm{w} - u\|^{(2)} =$

$b\gamma$ if and only if $t_k = \delta_k$ for every $k \geq 3$; or say, $\|\mathrm{w} - u\|^{(2)} = b\gamma$ if and only if $\mathrm{w} = h_\alpha$ for some $\alpha \in [0,1]$.

Then, there exists a continuous function $\rho: [0,1] \to E^{(2)}$ defined by $\rho(\alpha) = h_\alpha$ and $\Lambda \coloneqq \rho([0,1])$ is a compact convex subset and so $\|\mathrm{w} - u\|^{(2)}$ achieves its minimum value at $\mathrm{w} = h_\alpha$ and for any $\mathrm{h}_\alpha \in \Lambda$, we get

$$\begin{aligned} r(h_\alpha) &= r(u) + \|h_\alpha - u\|^{(2)} \\ &\leq r(u) + \|Th_\alpha - u\|^{(2)} \\ &= r(Th_\alpha) = \limsup_n \left\|Th_\alpha - u^{(n)}\right\|^{(2)} \end{aligned}$$

then same as the inequality (3.2.1), we get

$$\begin{aligned} r(h_\alpha) &\leq \limsup_n \left\|Th_\alpha - T\left(u^{(n)}\right)\right\|^{(2)} \\ &\quad + \limsup_n \left\|u^{(n)} - T\left(u^{(n)}\right)\right\|^{(2)} \\ &\leq \limsup_n \left\|h_\alpha - u^{(n)}\right\|^{(2)} \\ &\quad + \limsup_n \left\|u^{(n)} - T\left(u^{(n)}\right)\right\|^{(2)} \\ &\leq \limsup_n \left\|h_\alpha - u^{(n)}\right\|^{(2)} + 0 \\ &= r(h_\alpha). \end{aligned}$$

Hence, $r(h_\alpha) \leq r(Th_\alpha) \leq r(h_\alpha)$ and so $r(Th_\alpha) = r(h_\alpha)$.

Therefore,

$r(u) + \|Th_\alpha - u\|^{(2)} = r(u) + \|h_\alpha - u\|^{(2)}$.

Thus, $\|Th_\alpha - u\|^{(2)} = \|h_\alpha - u\|^{(2)}$ and so $Th_\alpha \in \Lambda$ but this shows $T(\Lambda) \subseteq \Lambda$ and using Schauder's Fixed Point Theorem (Schauder 1930) easily we get the result T has a fixed point since T is continuous; thus, h_α is the unique minimizer of $\|\mathrm{w} - u\|^{(2)} : w \in E^{(2)}$ and $Th_\alpha = h_\alpha$.

Therefore, $E^{(2)}$ has the fixed point property for nonexpansive mappings.

Theorem 3.3. *Fix $b \in (0,1)$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence defined by $f_1 := b\, v_1\, e_1$, $f_2 := \frac{b\, v_2\, e_2}{2^3}$, and $f_n := \frac{v_n}{n^3} e_n$ for all integers $n \geq 3$ where the sequence $(e_n)_{n\in\mathbb{N}}$ is the canonical basis of both c_0 and ℓ^1. Then, consider the cbc subset $E^{(3)} = {E_b}^{(3)}$ of U_1^3 by*

$$E^{(3)} := \left\{ \sum_{n=1}^{\infty} t_n f_n : \forall n \in \mathbb{N}, t_n \geq 0 \text{ and } \sum_{n=1}^{\infty} t_n = 1 \right\}.$$

Then, $E^{(3)}$ has the fixed point property for $\|\,.\,\|^{(3)}$-nonexpansive mappings.

Proof. Fix $b \in (0,1)$. Let $T: E^{(3)} \to E^{(3)}$ be a nonexpansive mapping. Then, there exists a sequence so called aproximate fixed point sequence $\left(u^{(n)}\right)_{n\in\mathbb{N}} \in E^{(3)}$ such that $\left\|Tu^{(n)} - u^{(n)}\right\|^{(3)} \underset{n}{\to} 0$. Due to isometric isomorphism U_1^3 shares common geometric properties with ℓ^1 and so both U_1^3 and its predual have same fixed point theory facts to ℓ^1 and c_0, respectly. Thus, considering that on bounded subsets the weak star topology on ℓ^1 is equivalent to the coardinate-wise convergence topology, and c_0 is separable, in U_1^3, the unit closed ball is weak*-sequentially compact due to Banach-Alaoglu theorem. Then we can say that we may denote the weak* closure of the set $E^{(3)}$ by

$$\begin{aligned} C^{(3)} &:= \overline{E^{(3)}}^{w^*} \\ &= \left\{ \sum_{n=1}^{\infty} t_n\ f_n : \text{each } t_n \geq 0 \text{ and } \sum_{n=1}^{\infty} t_n \leq 1 \right\} \end{aligned}$$

and without loss of generality, we may pass to a subsequence if necessary, and get a weak* limit $u \in C^{(3)}$ of $u^{(n)}$. Then, by Lemma 3.1, we have a function $r: U_1^3 \to [0,\infty)$ defined by

$$r(w) = \limsup_n \left\|u^{(n)} - w\right\|^{(3)}, \ \forall w \in U_1^3$$

such that *for every $w \in U_1^3$*,

$$r(w) = r(u) + \|u - w\|^{(3)}.$$

Case 1:$u \in E^{(3)}$.

Then, $r(Tu) = r(u) + \|Tu - u\|^{(3)}$ and

$$\begin{aligned} r(Tu) &= \limsup_n \left\|Tu - u^{(n)}\right\|^{(3)} \\ &\leq \limsup_n \left\|Tu - T\left(u^{(n)}\right)\right\|^{(3)} \\ &\quad + \limsup_n \left\|u^{(n)} - T\left(u^{(n)}\right)\right\|^{(3)} \\ &\leq \limsup_n \left\|u - u^{(n)}\right\|^{(3)} + 0 \\ &= r(u). \end{aligned} \tag{3.3.1}$$

Thus, $r(Tu) = r(u) + \|Tu - u\|^{(3)} \leq r(u)$ and so $\|Tu - u\|^{(3)} = 0$. Therefore, $Tu = u$.

Case 2: $u \in C^{(3)} \backslash E^{(3)}$.

Then, we may find scalars satisfying $\sum_{n=1}^{\infty} \delta_n < 1$ *and* $\forall n \in \mathbb{N}, \delta_n \geq 0$ such that $u = \sum_{n=1}^{\infty} \delta_n f_n$.

Then, let $\gamma := 1 - \sum_{n=1}^{\infty} \delta_n$ and for $\alpha \in \left[\frac{-\delta_1}{\gamma}, \frac{\delta_2}{\gamma} + 1\right]$ define

$$h_\alpha := (\delta_1 + \alpha\gamma) f_1 + (\delta_2 + (1 - \alpha)\gamma) f_2 + \sum_{n=3}^{\infty} \delta_n f_n.$$

Then,

$$\|h_\alpha - u\|^{(3)} = \left\|\alpha b\gamma v_1 e_1 + (1-\alpha)\gamma \frac{b\, v_2 e_2}{2^3}\right\|^{(3)}$$
$$= \mathrm{b}|\alpha|\gamma + b|1-\alpha|\gamma.$$

$\|h_\alpha - u\|^{(3)}$ is minimized for $\alpha \in [0,1]$ and its minimum value would be $b\gamma$.

Now fix $w \in E^{(3)}$. Then, we may find scalars t_n satisfying $\forall n \in \mathbb{N}, t_n \geq 0$ and $\sum_{n=1}^{\infty} t_n = 1$ such that $w = \sum_{n=1}^{\infty} t_n f_n$.

Then,

$$\|\mathrm{w} - u\|^{(3)} = \left\|\sum_{k=1}^{\infty} t_k f_k - \sum_{k=1}^{\infty} \delta_k f_k\right\|^{(3)}$$
$$= \mathrm{b}|t_1 - \delta_1| + \mathrm{b}|t_2 - \delta_2| + \sum_{k=3}^{\infty} |t_k - \delta_k|$$
$$= \mathrm{b}|t_1 - \delta_1| + \mathrm{b}|t_2 - \delta_2| + b\sum_{k=3}^{\infty} |t_k - \delta_k|$$
$$+(1-b)\sum_{k=3}^{\infty} |t_k - \delta_k|$$
$$\geq \mathrm{b}\left|\sum_{k=1}^{\infty} t_k - \delta_k\right| + (1-b)\sum_{k=3}^{\infty} |t_k - \delta_k|$$
$$= \mathrm{b}\left|\sum_{k=1}^{\infty} t_k - \sum_{k=1}^{\infty} \delta_k\right| + (1-b)\sum_{k=3}^{\infty} |t_k - \delta_k|$$
$$= \mathrm{b}|1-(1-\gamma)| + (1-b)\sum_{k=3}^{\infty} |t_k - \delta_k|.$$

Hence,

$$\|\mathrm{w} - u\|^{(3)} \geq b\gamma + (1-b)\sum_{k=3}^{\infty} |t_k - \delta_k| \geq b\gamma$$

and the equality is obtained if and only if $(1-b)\sum_{k=3}^{\infty} |t_k - \delta_k| = 0$; that is, we have $\|\mathrm{w} - u\|^{(2)} = b\gamma$ if and only if $t_k = \delta_k$ for every $k \geq 3$; or say, $\|\mathrm{w} - u\|^{(3)} = b\gamma$ if and only if $\mathrm{w} = h_\alpha$ for some $\alpha \in [0,1]$.

Then, there exists a continuous function $\rho: [0,1] \to E^{(3)}$ defined by $\rho(\alpha) = h_\alpha$ and $\Lambda \coloneqq \rho([0,1])$ is a compact convex subset and so $\|\mathrm{w} - u\|^{(3)}$ achieves its minimum value at $\mathrm{w} = h_\alpha$ and for any $\mathrm{h}_\alpha \in \Lambda$, we get

$$r(h_\alpha) = r(u) + \|h_\alpha - u\|^{(3)}$$
$$\leq r(u) + \|Th_\alpha - u\|^{(3)}$$
$$= r(Th_\alpha)$$
$$= \limsup_n \left\|Th_\alpha - u^{(n)}\right\|^{(3)}$$

then same as the inequality (3.3.1), we get

$$r(h_\alpha) \leq \limsup_n \left\|Th_\alpha - T\left(u^{(n)}\right)\right\|^{(3)}$$
$$+\limsup_n \left\|u^{(n)} - T\left(u^{(n)}\right)\right\|^{(3)}$$
$$\leq \limsup_n \left\|h_\alpha - u^{(n)}\right\|^{(3)}$$
$$+\limsup_n \left\|u^{(n)} - T\left(u^{(n)}\right)\right\|^{(3)}$$
$$\leq \limsup_n \left\|h_\alpha - u^{(n)}\right\|^{(3)} + 0$$
$$= r(h_\alpha).$$

Hence, $r(h_\alpha) \leq r(Th_\alpha) \leq r(h_\alpha)$ and so $r(Th_\alpha) = r(h_\alpha)$.

Therefore,

$$r(u) + \|Th_\alpha - u\|^{(3)} = r(u) + \|h_\alpha - u\|^{(3)}.$$

Thus, $\|Th_\alpha - u\|^{(3)} = \|h_\alpha - u\|^{(3)}$ and so $Th_\alpha \in \Lambda$ but this shows $T(\Lambda) \subseteq \Lambda$ and using Schauder's Fixed Point Theorem (Schauder 1930) easily we get the result T has a fixed point since T is continuous; thus, h_α is the unique minimizer of $\|\mathrm{w} - u\|^{(3)} : w \in E^{(3)}$ and $Th_\alpha = h_\alpha$.

Therefore, $E^{(3)}$ has the fixed point property for nonexpansive mappings.

Theorem 3.4. *Fix $b \in (0,1)$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence defined by $f_1 := b\, v_1\, e_1$, $f_2 := 2^2 b\, v_2\, e_2$, and $f_n := n^2 v_n e_n$ for all integers $n \geq 3$ where the sequence $(e_n)_{n\in\mathbb{N}}$ is the canonical basis of both c_0 and ℓ^1. Then, consider the cbc subset $E_{(2)} = E_{b(2)}$ of W_1^2 by*

$$E_{(2)} := \left\{\sum_{n=1}^{\infty} t_n f_n : \forall n \in \mathbb{N}, t_n \geq 0 \text{ and } \sum_{n=1}^{\infty} t_n = 1\right\}.$$

Then, $E_{(2)}$ has the fixed point property for $\|\,.\,\|_{(2)}$-nonexpansive mappings.

Proof. Fix $b \in (0,1)$. Let $T: E_{(2)} \to E_{(2)}$ be a nonexpansive mapping. Then, there exists a sequence so called aproximate fixed point sequence $\left(u^{(n)}\right)_{n\in\mathbb{N}} \in E_{(2)}$ such that $\left\|Tu^{(n)} - u^{(n)}\right\|_{(2)} \underset{n}{\to} 0.$ Due to isometric isomorphism W_1^2 shares common geometric properties with ℓ^1 and so both W_1^2 and its predual have same fixed point theory facts to ℓ^1 and c_0, respectly. Thus, considering that on bounded subsets the weak star topology on ℓ^1 is equivalent to the coardinate-wise convergence topology, and c_0 is separable, in W_1^2, the unit closed ball is weak*-sequentially compact due to Banach-Alaoglu theorem. Then we can say that we may denote the weak* closure of the set $E_{(2)}$ by

$$C_{(2)} := \overline{E_{(2)}}^{w^*}$$
$$= \left\{ \sum_{n=1}^{\infty} t_n \ f_n : each\ t_n \geq 0\ and\ \sum_{n=1}^{\infty} t_n \leq 1 \right\}$$

and without loss of generality, we may pass to a subsequence if necessary, and get a weak* limit $u \in C_{(2)}$ of $u^{(n)}$. Then, by Lemma 3.1, we have a function $r: W_1^2 \to [0,\infty)$ defined by

$$r(w) = \limsup_n \left\| u^{(n)} - w \right\|_{(2)}, \qquad \forall w \in W_1^2$$

such that *for every* $w \in W_1^2$,

$$r(w) = r(u) + \|u - w\|_{(2)}.$$

Case 1: $u \in E_{(2)}$.

Then, $r(Tu) = r(u) + \|Tu - u\|_{(2)}$ and

$$\begin{aligned} r(Tu) &= \limsup_n \left\| Tu - u^{(n)} \right\|_{(2)} \\ &\leq \limsup_n \left\| Tu - T\left(u^{(n)}\right) \right\|_{(2)} \\ &\quad + \limsup_n \left\| u^{(n)} - T\left(u^{(n)}\right) \right\|_{(2)} \\ &\leq \limsup_n \left\| u - u^{(n)} \right\|_{(2)} + 0 \\ &= r(u). \end{aligned} \qquad (3.4.1)$$

Thus, $r(Tu) = r(u) + \|Tu - u\|_{(2)} \leq r(u)$ and so $\|Tu - u\|_{(2)} = 0$. Therefore, $Tu = u$.

Case 2: $u \in C^{(2)} \backslash E_{(2)}$.

Then, we may find scalars satisfying $\sum_{n=1}^{\infty} \delta_n < 1\ and\ \forall n \in \mathbb{N}, \delta_n \geq 0$ such that $u = \sum_{n=1}^{\infty} \delta_n f_n$. Then, let $\gamma := 1 - \sum_{n=1}^{\infty} \delta_n$ and for $\alpha \in \left[\frac{-\delta_1}{\gamma}, \frac{\delta_2}{\gamma} + 1\right]$ define

$$h_\alpha := (\delta_1 + \alpha\gamma) f_1 + (\delta_2 + (1-\alpha)\gamma) f_2 + \sum_{n=3}^{\infty} \delta_n f_n.$$

Then,

$$\begin{aligned} \|h_\alpha - u\|_{(2)} &= \|\alpha b \gamma v_1 e_1 + (1-\alpha)\gamma 2^2 b\ v_2 e_2\|_{(2)} \\ &= \mathrm{b}|\alpha|\gamma + b|1-\alpha|\gamma. \end{aligned}$$

$\|h_\alpha - u\|_{(2)}$ is minimized for $\alpha \in [0,1]$ and its minimum value would be $b\gamma$.

Now fix $w \in E_{(2)}$. Then, we may find scalars t_n satisfying $\forall n \in \mathbb{N}, t_n \geq 0$ and $\sum_{n=1}^{\infty} t_n = 1$ such that $w = \sum_{n=1}^{\infty} t_n f_n$.

Then,

$$\begin{aligned} \|\mathrm{w} - u\|_{(2)} &= \left\| \sum_{k=1}^{\infty} t_k f_k - \sum_{k=1}^{\infty} \delta_k f_k \right\|_{(2)} \\ &= \mathrm{b}|t_1 - \delta_1| + \mathrm{b}|t_2 - \delta_2| + \sum_{k=3}^{\infty} |t_k - \delta_k| \\ &= \mathrm{b}|t_1 - \delta_1| + \mathrm{b}|t_2 - \delta_2| + b \sum_{k=3}^{\infty} |t_k - \delta_k| \\ &\quad + (1-b) \sum_{k=3}^{\infty} |t_k - \delta_k| \\ &\geq \mathrm{b} \left| \sum_{k=1}^{\infty} t_k - \delta_k \right| + (1-b) \sum_{k=3}^{\infty} |t_k - \delta_k| \\ &= \mathrm{b} \left| \sum_{k=1}^{\infty} t_k - \sum_{k=1}^{\infty} \delta_k \right| + (1-b) \sum_{k=3}^{\infty} |t_k - \delta_k| \\ &= \mathrm{b}|1 - (1-\gamma)| + (1-b) \sum_{k=3}^{\infty} |t_k - \delta_k|. \end{aligned}$$

Hence,

$$\|\mathrm{w} - u\|_{(2)} \geq b\gamma + (1-b) \sum_{k=3}^{\infty} |t_k - \delta_k| \geq b\gamma$$

and the equality is obtained if and only if $(1-b)\sum_{k=3}^{\infty} |t_k - \delta_k| = 0$; that is, we have $\|\mathrm{w} - u\|_{(2)} = b\gamma$ if and only if $t_k = \delta_k$ for every $k \geq 3$; or say, $\|\mathrm{w} - u\|_{(2)} = b\gamma$ if and only if $\mathrm{w} = h_\alpha$ for some $\alpha \in [0,1]$.

Then, there exists a continuous function $\rho: [0,1] \to \mathrm{E}_{(2)}$ defined by $\rho(\alpha) = h_\alpha$ and $\Lambda := \rho([0,1])$ is a compact convex subset and so $\|w - u\|_{(2)}$ achieves its minimum value at $\mathrm{w} = h_\alpha$ and for any $\mathrm{h}_\alpha \in \Lambda$, we get

$$\begin{aligned} r(h_\alpha) &= r(u) + \|h_\alpha - u\|_{(2)} \\ &\leq r(u) + \|Th_\alpha - u\|_{(2)} \\ &= r(Th_\alpha) \\ &= \limsup_n \left\| Th_\alpha - u^{(n)} \right\|_{(2)} \end{aligned}$$

then same as the inequality (3.4.1), we get

$$\begin{aligned} r(h_\alpha) &\leq \limsup_n \left\| Th_\alpha - T\left(u^{(n)}\right) \right\|_{(2)} \\ &\quad + \limsup_n \left\| u^{(n)} - T\left(u^{(n)}\right) \right\|_{(2)} \\ &\leq \limsup_n \left\| h_\alpha - u^{(n)} \right\|_{(2)} \\ &\quad + \limsup_n \left\| u^{(n)} - T\left(u^{(n)}\right) \right\|_{(2)} \\ &\leq \limsup_n \left\| h_\alpha - u^{(n)} \right\|_{(2)} + 0 \\ &= r(h_\alpha). \end{aligned}$$

Hence, $r(h_\alpha) \leq r(Th_\alpha) \leq r(h_\alpha)$ and so $r(Th_\alpha) = r(h_\alpha)$.

Therefore,

$$r(u) + \|Th_\alpha - u\|_{(2)} = r(u) + \|h_\alpha - u\|_{(2)}.$$

Thus, $\|Th_\alpha - u\|_{(2)} = \|h_\alpha - u\|_{(2)}$ and so $Th_\alpha \in \Lambda$ but this shows $T(\Lambda) \subseteq \Lambda$ and using Schauder's Fixed Point Theorem (Schauder 1930) easily we get the result T has a fixed point since T is continuous; thus, h_α is the unique minimizer of $\|w-u\|_{(2)}\ : w \in E_{(2)}$ and $Th_\alpha = h_\alpha$.

Therefore, $E_{(2)}$ has the fixed point property for nonexpansive mappings.

Theorem 3.5. *Fix* $b \in (0,1)$. *Let* $(f_n)_{n\in\mathbb{N}}$ *be a sequence defined by* $f_1 := b\, v_1\, e_1$, $f_2 := 2^3 b\, v_2\, e_2$, *and* $f_n := n^3 v_n e_n$ *for all integers* $n \geq 3$ *where the sequence* $(e_n)_{n\in\mathbb{N}}$ *is the canonical basis of both* c_0 *and* ℓ^1. *Then, consider the cbc subset* $E_{(3)} = E_{b(3)}$ *of* W_1^3 *by*

$$E_{(3)} := \left\{ \sum_{n=1}^{\infty} t_n f_n : \forall n \in \mathbb{N}, t_n \geq 0 \text{ and } \sum_{n=1}^{\infty} t_n = 1 \right\}.$$

Then, $E_{(3)}$ *has the fixed point property for* $\|.\|_{(3)}$*-nonexpansive mappings.*

Proof. Fix $b \in (0,1)$. Let $T: E_{(3)} \to E_{(3)}$ be a nonexpansive mapping. Then, there exists a sequence so called aproximate fixed point sequence $\left(u^{(n)}\right)_{n\in\mathbb{N}} \in E_{(3)}$ such that $\left\|Tu^{(n)} - u^{(n)}\right\|_{(3)} \underset{n}{\to} 0$. Due to isometric isomorphism W_1^2 shares common geometric properties with ℓ^1 and so both W_1^3 and its predual have same fixed point theory facts to ℓ^1 and c_0, respectly. Thus, considering that on bounded subsets the weak star topology on ℓ^1 is equivalent to the coardinate-wise convergence topology, and c_0 is separable, in W_1^3, the unit closed ball is weak*-sequentially compact due to Banach-Alaoglu theorem. Then we can say that we may denote the weak* closure of the set $E_{(3)}$ by

$$\begin{aligned} C_{(3)} &:= \overline{E_{(3)}}^{w^*} \\ &= \left\{ \sum_{n=1}^{\infty} t_n\ f_n : \text{each } t_n \geq 0 \text{ and } \sum_{n=1}^{\infty} t_n \leq 1 \right\} \end{aligned}$$

and without loss of generality, we may pass to a subsequence if necessary, and get a weak* limit $u \in C_{(3)}$ of $u^{(n)}$. Then, by Lemma 3.1, we have a function $r: W_1^3 \to [0,\infty)$ defined by

$$r(w) = \limsup_n \left\|u^{(n)} - w\right\|_{(3)}, \qquad \forall w \in W_1^3$$

such that *for every* $w \in W_1^3$,

$$r(w) = r(u) + \|u - w\|_{(3)}.$$

Case 1: $u \in E_{(3)}$.

Then, $r(Tu) = r(u) + \|Tu - u\|_{(3)}$ and

$$\begin{aligned} r(Tu) &= \limsup_n \left\|Tu - u^{(n)}\right\|_{(3)} \\ &\leq \limsup_n \left\|Tu - T\left(u^{(n)}\right)\right\|_{(3)} \\ &\quad + \limsup_n \left\|u^{(n)} - T\left(u^{(n)}\right)\right\|_{(3)} \\ &\leq \limsup_n \left\|u - u^{(n)}\right\|_{(3)} + 0 \\ &= r(u). \end{aligned} \tag{3.5.1}$$

Thus, $r(Tu) = r(u) + \|Tu - u\|_{(3)} \leq r(u)$ and so $\|Tu - u\|_{(3)} = 0$. Therefore, $Tu = u$.

Case 2: $u \in C^{(3)} \setminus E_{(3)}$.

Then, we may find scalars satisfying $\sum_{n=1}^{\infty} \delta_n < 1$ and $\forall n \in \mathbb{N}, \delta_n \geq 0$ such that $u = \sum_{n=1}^{\infty} \delta_n f_n$. Then, let $\gamma := 1 - \sum_{n=1}^{\infty} \delta_n$ and for $\alpha \in \left[\frac{-\delta_1}{\gamma}, \frac{\delta_2}{\gamma} + 1\right]$ define

$$h_\alpha := (\delta_1 + \alpha\gamma) f_1 + (\delta_2 + (1-\alpha)\gamma) f_2 + \sum_{n=3}^{\infty} \delta_n f_n.$$

Then,

$$\begin{aligned} \|h_\alpha - u\|_{(3)} &= \|\alpha b \gamma v_1 e_1 + (1-\alpha)\gamma 2^3 b\, v_2\, e_2\|_{(3)} \\ &= \mathrm{b}|\alpha|\gamma + b|1-\alpha|\gamma. \end{aligned}$$

$\|h_\alpha - u\|_{(3)}$ is minimized for $\alpha \in [0,1]$ and its minimum value would be $b\gamma$.

Now fix $w \in E_{(3)}$. Then, we may find scalars t_n satisfying $\forall n \in \mathbb{N}, t_n \geq 0$ and $\sum_{n=1}^{\infty} t_n = 1$ such that $w = \sum_{n=1}^{\infty} t_n f_n$.

Then,

$$\begin{aligned} \|w - u\|_{(3)} &= \left\| \sum_{k=1}^{\infty} t_k f_k - \sum_{k=1}^{\infty} \delta_k f_k \right\|_{(3)} \\ &= \mathrm{b}|t_1 - \delta_1| + \mathrm{b}|t_2 - \delta_2| + \sum_{k=3}^{\infty} |t_k - \delta_k| \\ &= \mathrm{b}|t_1 - \delta_1| + \mathrm{b}|t_2 - \delta_2| + b\sum_{k=3}^{\infty} |t_k - \delta_k| \\ &\quad + (1-b)\sum_{k=3}^{\infty} |t_k - \delta_k| \\ &\geq \mathrm{b}\left|\sum_{k=1}^{\infty} t_k - \delta_k\right| + (1-b)\sum_{k=3}^{\infty} |t_k - \delta_k| \\ &= \mathrm{b}\left|\sum_{k=1}^{\infty} t_k - \sum_{k=1}^{\infty} \delta_k\right| + (1-b)\sum_{k=3}^{\infty} |t_k - \delta_k| \\ &= \mathrm{b}|1 - (1-\gamma)| + (1-b)\sum_{k=3}^{\infty} |t_k - \delta_k|. \end{aligned}$$

Hence,

$$\|w-u\|_{(3)} \geq b\gamma + (1-b)\sum_{k=3}^{\infty} |t_k - \delta_k| \geq b\gamma$$

and the equality is obtained if and only if $(1-b)\sum_{k=3}^{\infty} |t_k-\delta_k| = 0$; that is, we have $\|w-u\|_{(3)} = b\gamma$ if and only if $t_k = \delta_k$ for every $k \geq 3$; or say, $\|w-u\|_{(3)} = b\gamma$ if and only if $w = h_\alpha$ for some $\alpha \in [0,1]$.

Then, there exists a continuous function $\rho: [0,1] \to E_{(3)}$ defined by $\rho(\alpha) = h_\alpha$ and $\Lambda \coloneqq \rho([0,1])$ is a compact convex subset and so $\|w-u\|_{(3)}$ achieves its minimum value at $w = h_\alpha$ and for any $h_\alpha \in \Lambda$, we get

$$\begin{aligned} r(h_\alpha) &= r(u) + \|h_\alpha - u\|_{(3)} \\ &\leq r(u) + \|Th_\alpha - u\|_{(3)} \\ &= r(Th_\alpha) \\ &= \limsup_n \left\|Th_\alpha - u^{(n)}\right\|_{(3)} \end{aligned}$$

then same as the inequality (3.5.1), we get

$$\begin{aligned} r(h_\alpha) &\leq \limsup_n \left\|Th_\alpha - T\left(u^{(n)}\right)\right\|_{(3)} \\ &\quad + \limsup_n \left\|u^{(n)} - T\left(u^{(n)}\right)\right\|_{(3)} \\ &\leq \limsup_n \left\|h_\alpha - u^{(n)}\right\|_{(3)} \\ &\quad + \limsup_n \left\|u^{(n)} - T\left(u^{(n)}\right)\right\|_{(3)} \\ &\leq \limsup_n \left\|h_\alpha - u^{(n)}\right\|_{(3)} + 0 \\ &= r(h_\alpha). \end{aligned}$$

Hence, $r(h_\alpha) \leq r(Th_\alpha) \leq r(h_\alpha)$ and so $r(Th_\alpha) = r(h_\alpha)$. Therefore, $r(u) + \|Th_\alpha - u\|_{(3)} = r(u) + \|h_\alpha - u\|_{(3)}$. Thus, $\|Th_\alpha - u\|_{(3)} = \|h_\alpha - u\|_{(3)}$ and so $Th_\alpha \in \Lambda$ but this shows $T(\Lambda) \subseteq \Lambda$ and using Schauder's Fixed Point Theorem (Schauder 1930) easily we get the result T has a fixed point since T is continuous; thus, h_α is the unique minimizer of $\|w-u\|_{(3)} : w \in E_{(3)}$ and $Th_\alpha = h_\alpha$.

Therefore, $E_{(3)}$ has the fixed point property for nonexpansive mappings.

4. Discussion

In this study, we have considered the Köthe-Toeplitz duals for the 2nd order and 3rd order types difference sequence space generalizations by Et and Esi (2000). We have studied Goebel and Kuczumow (1979) analogy for those spaces and showed that there exist large classes of cbc subsets in those Köthe-Toeplitz duals with fixed point property for nonexpansive mappings. We have also another study in preparation to get larger classes for mth order types. Furthermore, the first author has another study in preparation to investigate Kaczor and Prus (2004) analogy for the 2nd order and 3rd order types difference sequence space generalizations by Et and Esi (2000), which is to look for large classes of cbc subsets satisfying the fixed point property for asymptotically nonexpansive mappings. These spaces we have studied are analogous Banach spaces to ℓ^1. There are many Banach spaces analogous to ℓ^1 and Goebel-Kuczumow analogy or Kaczor-Prus analogy might be investigated by researchers.

References

BEKTAŞ, Ç. A., ET, M., & ÇOLAK, R. (2004). Generalized difference sequence spaces and their dual spaces. *Journal of Mathematical Analysis and Applications*, *292*(2): 423-432.

BROWDER, F. E. (1965). Fixed-point theorems for noncompact mappings in Hilbert space. *Proceedings of the National Academy of Sciences*, 53(6): 1272-1276.

CHEN, S., CUI, Y., HUDZIK, H., & SIMS, B. (2001). Geometric properties related to fixed point theory in some Banach function lattices. In *Handbook of metric fixed point theory*. Springer, Dordrecht, 339-389.

CUI, Y. (1999). Some geometric properties related to fixed point theory in Cesàro spaces. *Collectanea Mathematica*, 277-288.

CUI, Y., HUDZIK, H., & LI, Y. (2000). On the Garcfa-Falset Coefficient in Some Banach Sequence Spaces. In *Function Spaces*. CRC Press, 163-170.

CUI, Y., MENG, C., & PŁUCIENNIK, R. (2000). Banach—Saks Property and Property (β) in Cesàro Sequence Spaces. *Southeast Asian Bulletin of Mathematics*, *24*(2):201-210.

ÇOLAK, R. (1989). On some generalized sequence spaces. *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics*, 38: 35-46.

ET, M. (1996). On some generalized Cesàro difference sequence spaces. *İstanbul University Science Faculty The Journal Of Mathematics, Physics and Astronomy*, *55*:221-229.

ET, M., & ÇOLAK, R. (1995). On some generalized difference sequence spaces. *Soochow journal of mathematics*, *21*(4):377-386.

ET, M., & ESI, A. (2000). On Köthe-Toeplitz duals of generalized difference sequence spaces. *Bull. Malays. Math. Sci. Soc*, *23*(1): 25-32.

EVEREST, T. M. (2013). *Fixed points of nonexpansive maps on closed, bounded, convex sets in l^1* (Doctoral dissertation, University of Pittsburgh).

FALSET, J. G. (1997). The fixed point property in Banach spaces with the NUS-property. *Journal of Mathematical Analysis and Applications*, *215*(2):532-542.

GOEBEL, K., & KIRK, W. A. (1990). *Topics in metric fixed point theory* (No. 28). Cambridge university press.

GOEBEL, K., & KUCZUMOW, T. (1979). Irregular convex sets with fixed-point property for nonexpansive mappings. *Colloquium Mathematicum*, 2(40): 259-264.

KACZOR, W., & PRUS, S. (2004). Fixed point properties of some sets in l^1. In *Proceedings of the International Conference on Fixed Point Theory and Applications, 11p*.

KIZMAZ, H. (1981). On certain sequence spaces. *Canadian mathematical bulletin*, *24*(2):169-176.

KIRK, W. A. (1965). A fixed point theorem for mappings which do not increase distances. *The American mathematical monthly*, *72*(9):1004-1006.

LIN, P. K. (2008). There is an equivalent norm on ℓ^1 that has the fixed point property. *Nonlinear Analysis: Theory, Methods & Applications*, *68*(8):2303-2308.

HERNÁNDEZ-LINARES, C. A., & JAPÓN, M. A. (2012). Renormings and fixed point property in non-commutative L1-spaces II: Affine mappings. *Nonlinear Analysis: Theory, Methods & Applications*, *75*(13):5357-5361.

LINDENSTRAUSS, J., & TZAFRIRI, L. (1977). *Classical Banach Spaces. Pt. 1. Sequence Spaces*. Springer.

NGPENG-NUNG, N. N., & LEEPENG-YEE, L. Y. (1978). Cesàro sequence spaces of nonabsolute type. *Commentationes mathematicae*, *20*(2):429–433.

ORHAN, C. (1983). Casaro Differance Sequence Spaces and Related Matrix Transformations. *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics*, *32*:55-63.

SCHAUDER, J. (1930). Der fixpunktsatz in funktionalraümen. *Studia Mathematica*, *2*(1), 171-180.

SHIUE, J. S. (1970). On the Cesaro sequence spaces. *Tamkang J. Math*, *1*(1):19-25.

TRIPATHY, B. C., ESI, A., & TRIPATHY, B. (2005). On new types of generalized difference Cesaro sequence spaces. *Soochow Journal of Mathematics*, *31*(3):333-340.