Çankaya University Journal of Science and Engineering
Volume 13, No. 1 (2016) 001–015

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MV-Modules in View of Soft Set Theory

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Abstract: In this paper, the concept of a soft *MV*-module is introduced and some examples are provided. Then, different types of intersections and unions of the family of soft *MV*-modules are established. Moreover, the notions of soft MV-submodules and soft *MV*-module homomorphisms are introduced and some of their properties are studied.

Keywords: Soft *MV*-module, Soft set, *MV*-module.

1. Introduction

MV-algebra was introduced by Chang [6] in 1958 as the algebraic structure corresponding to the infinitely-many-valued Lukasiewicz logic. Since then, this structure has been developed from an algebraic point of view by many mathematicians. In 1986, Mundici proved that the category of *MV*-algebras is equivalent to the category of abelian l-groups with strong unit (see [7]). Noje and Bede [25] introduced the concept of vectorial *MV*-algebra and showed that the RGB model (the color model of a pixel on the screen) has the vectorial *MV*-algebra structure. Di Nola et al. [9] introduced the notion of an *MV*-module over a *PMV*-algebra and established that, for a fixed lu-ring (R,v), the category of lu-modules over (R,v) is equivalent to the category of *MV*-modules over $\Gamma(R,v)$. Forouzesh et al. [13] introduced the notion of prime A-ideals in *MV*-modules and studied about annihilators of A-ideals. They proved that, if $h: M \to N$ is an *MV*-module homomorphism, then all prime A-ideals of N and prime A-ideals of M that contain $ker(h)$ are in a one to one correspondence.

On the other hand, soft set theory was initiated by Molodtsov [23] in 1999 as a new mathematical tool for modeling the uncertainties arising from the parametrization of elements of a universe. He mentioned several directions for the applications of soft sets. In fact, before soft set theory, there have been some mathematical theories such as probability theory, fuzzy set theory, rough set theory, vague set theory, and interval mathematics theory for dealing with uncertainties. However, the superiority of the soft set theory compared with other mathematical tools, is its ability of

ISSN 1309 - 6788

parametrization. Maji et al. [21] studied several operations on the theory of soft sets. Some authors have discussed the applications of (fuzzy) soft sets in decision making problems (see [5,22]). Aktas and Cagman [3] compared soft sets to the related concepts of fuzzy sets and rough sets. They also introduced the notion of soft groups. After them, soft algebraic structures have been studied by many authors (see [1,10,14,16,19,27,29]). Jun [17] introduced and investigated soft BCK/BCI-algebras. Feng et al. [12] applied soft set theory to the study of semirings and initiated the notion called a soft semiring. Afkhami et al. [2] presented the concept of a soft nexus. Zhu [30] introduced the concept of soft BL-algebras. Moreover, by combination of fuzzy set theory with soft set theory, fuzzy soft algebraic structures were born. For example, Hadipour et al. [14] defined the notion of fuzzy soft BF-algebra and investigated the level subset, union and intersection, fuzzy soft image and fuzzy soft inverse image of them. Murali [24] introduced the concept of a fuzzy soft Γ-semiring and fuzzy soft k-ideal over a Γ-semiring and studied some of their algebraical properties. Ersoy et al. [11] introduced the concept of an idealistic fuzzy soft Γ-near-ring and derived some results on this structure.

The most soft algebraic structures are defined as follows: for a set of parameters E and a general algebra X, a pair (F,E) is called a soft general algebra over X if F is a mapping of E into the set of all subsets of the set X such that for each $e \in E$, $F(e)$ is the empty set or a subalgebra of X.

In this paper, at first, some definitions and results related to soft set and MV-modules is reviewed. Then, the notion of a soft MV-module is introduced and some examples are provided. Two examples of MV-modules that have no proper submodules are given in Section 3. Then, the extended intersection, restricted intersection, $\bigwedge$-intersection, extended union, restricted union and $\vee$-union of the family of soft MV-modules are established. Moreover, the notions of soft MV-module homomorphisms, soft isomorphic MV-modules and soft MV-submodules are introduced and some of their properties are studied. Also, it is shown that there is a one-to-one correspondence between the soft MV-submodules of two soft isomorphic MV-modules.

2. Preliminaries

Some definitions and results about soft set and MV-module are presented in this section.

Let U be an initial universe set and let E be a set of parameters. Molodtsov [23] defined the soft set in the following way:

Definition 1. A pair (F,E) is called a soft set (over U) if F is a mapping of E into the set of all subsets of the set U.

Definition 2. [26] Let (F,A) and (G,B) be two soft sets over U. Then,

(i) (F,A) is said to be a soft subset of (G,B), denoted by $(F,A) \subseteq (G,B)$, if $A \subseteq B$ and $F(a) \subseteq G(a)$ for all $a \in A$,

(ii) (F,A) and (G,B) are said to be soft equal, denoted by $(F,A) = (G,B)$, if $(F,A) \subseteq (G,B)$ and $(G,B) \subseteq (F,A)$.

The next definition introduces three types of intersections and three types of unions of the family of soft sets over a common universe set:

Definition 3. [28] For a family $\{(F_i,A_i) \mid i \in \mathscr{I}\}$ of soft sets over U, we give some definitions as follows:

- The extended intersection of the family (F_i,A_i) is defined as the soft set
$$\widetilde{\bigcap}_{i\in\mathscr{I}}(F_i,A_i) = (H,C),$$
where $C = \bigcup_{i\in\mathscr{I}} A_i$ and $H(x) = \bigcap_{i\in I(x)} F_i(x)$ where $I(x) = \{i \in \mathscr{I} \mid x \in A_i\}$ for all $x \in C$.
- The restricted intersection of the family (F_i,A_i) is defined as the soft set
$$\widetilde{\sqcap}_{i\in\mathscr{I}}(F_i,A_i) = (H,C),$$
where $C = \bigcap_{i\in\mathscr{I}} A_i$ and $H(x) = \bigcap_{i\in\mathscr{I}} F_i(x)$ for all $x \in C$.
- The extended union of the family (F_i,A_i) is defined as the soft set
$$\widetilde{\bigcup}_{i\in\mathscr{I}}(F_i,A_i) = (H,C),$$
where $C = \bigcup_{i\in\mathscr{I}} A_i$, $H(x) = \bigcup_{i\in I(x)} F_i(x)$ and $I(x) = \{i \in \mathscr{I} \mid x \in A_i\}$ for all $x \in C$.
- The restricted union of the family (F_i,A_i) is defined as the soft set
$$\widetilde{\bigsqcup}_{i\in\mathscr{I}}(F_i,A_i) = (H,C),$$
where $C = \bigcap_{i\in\mathscr{I}} A_i \neq \emptyset$ and $H(x) = \bigcup_{i\in\mathscr{I}} F_i(x)$ for all $x \in C$.
- The $\wedge$-intersection of the family (F_i,A_i) is defined as the soft set
$$\widetilde{\bigwedge}_{i\in\mathscr{I}}(F_i,A_i) = (H,C),$$
where $C = \prod_{i\in\mathscr{I}} A_i$ and $H((a_i)_{i\in\mathscr{I}}) = \bigcap_{i\in\mathscr{I}} F_i(a_i)$ for all $(a_i)_{i\in\mathscr{I}} \in C$.
- The $\vee$-union of the family (F_i,A_i) is defined as the soft set
$$\widetilde{\bigvee}_{i\in\mathscr{I}}(F_i,A_i) = (H,C),$$
where $C = \prod_{i\in\mathscr{I}} A_i$ and $H((a_i)_{i\in\mathscr{I}}) = \bigcup_{i\in\mathscr{I}} F_i(a_i)$ for all $(a_i)_{i\in\mathscr{I}} \in C$.

Definition 4. [12] The support of the soft set (F,A) is denoted by $Supp(F,A)$ and is defined as $Supp(F,A) = \{x \in A \mid F(x) \neq \emptyset\}$. If $Supp(F,A) \neq \emptyset$, then the soft set (F,A) is called non-null.

Definition 5. [19] Let $f : X \to Y$ be a function. If (G,A) and (H,C) are non-null soft sets over X and Y respectively, then the functions $f(G)$ and $f^{-1}(H)$ are defined as follows:

$f(G) : A \to P(Y)$ defined by $f(G)(a) = f(G(a))$ for all $a \in Supp(G,A)$,

$f^{-1}(H) : C \to P(X)$ defined by $f^{-1}(H)(c) = f^{-1}(H(c))$ for all $c \in Supp(H,C)$.

Definition 6. [8] Let φ be be a mapping from a set X to a set Y and let f be a soft set of X over U. The function $\varphi(f) : Y \to P(U)$, defined by:

$$\varphi(f)(y) = \begin{cases} \bigcup\{f(x) \mid x \in X,\ \varphi(x) = y\} & \text{if } y \in \varphi(X) \\ \phi & \text{if } y \notin \varphi(X) \end{cases}$$

for all $y \in Y$, is a soft set called a soft image of f under φ.

Now we recall the definitions and some of the known results about the MV-algebra and MV-module:

Definition 7. [7] An MV-algebra is an algebra $(M, \oplus, ^*, 0_M)$ of type $(2,1,0)$ satisfying the following equations:

(MV1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,
(MV2) $x \oplus y = y \oplus x$,
(MV3) $x \oplus 0_M = x$,
(MV4) $x^{**} = x$,
(MV5) $x \oplus 0_M^* = 0_M^*$,
(MV6) $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$ for all $x,y,z \in M$.

Remark 2.1. [7] On each MV-algebra M, the constant 1_M and the operation $\odot$ and $\ominus$ are defined as follows:

1) $1_M =_{def} 0_M^*$,
2) $x \odot y =_{def} (x^* \oplus y^*)^*$,
3) $x \ominus y =_{def} x \oplus y^*$.

Theorem 1. [7] Let M be an MV-algebra and $x,y \in M$. Then the following conditions are equivalent

1) $x^* \oplus y = 1_M$,
2) $x \odot y^* = 0_M$,
3) $y = x \odot (y \ominus x)$,
4) there is $z \in M$ such that $x \oplus y = z$.

Definition 8. [7] Let M be an MV-algebra and $x,y \in M$. We say that $x \leq y$ if x and y satisfy one of the equivalent conditions of Theorem 1.

Remark 2.2. [7] It follows that $\leq$ is a partial order, called the natural order of M. An MV-algebra, whose natural order is total is called an MV-chain. On each MV-algebra, the natural order determines a lattice structure.

Definition 9. [7] A subalgebra of an MV-algebra M is a subset S of M, containing the zero element of M, closed under the operations of M and equipped with the restriction to S of these operations.

Definition 10. [9] Let M be an MV-algebra. We define partial addition on M as follows: for any $x,y \in M$, $x+y$ is defined if and only if $x \leq y^*$ and in this case, $x+y = x \oplus y$.

Definition 11. [9] A product MV-algebra (or PMV-algebra, for short) is a structure $(A, \oplus, ^*, ., 0)$ where $(A, \oplus, ^*, 0)$ is an MV-algebra and . is a binary associative operation on A such that the following property is satisfied:

if $x+y$ is defined, then $x.z+y.z$ and $z.x+z.y$ are defined and $(x+y).z = x.z+y.z$, $z.(x+y) = z.x+z.y$, where $+$ is the partial addition on A. for all $x,y,z \in A$.

Definition 12. [9] Let A be a PMV-algebra. A unity for product is an element $e \in A$ such that $e.x = x.e = x$ for any $x \in A$. A PMV-algebra that has unity for product is called unital PMV-algebra.

Example 2.1. [9] The interval $[0,1]$ is a unital PMV-algebra, where $x \oplus y = min\{1, x+y\}$, $x^* = 1-x$ and $x.y = xy$ (real product) for each $x,y \in [0,1]$. This structure is called standard PMV-algebra.

The concept of MV-module is defined in the next definition:

Definition 13. [9] Let $(A, \oplus, ^*, ., 0)$ be a PMV-algebra and $(M, \oplus, ^*, 0)$ an MV-algebra. M is called a (left) MV-module over A if there is an external operation

$$\varphi : A \times M \to M, \qquad \varphi(\alpha, x) = \alpha x,$$

such that the following properties hold for any $x,y \in M$ and $\alpha, \beta \in A$:

(1) if $x+y$ is defined in M, then $\alpha x + \alpha y$ is defined and

$$\alpha(x+y) = \alpha x + \alpha y,$$

(2) if $\alpha + \beta$ is defined in A, then $\alpha x + \beta x$ is defined in M and

$$(\alpha + \beta)x = \alpha x + \beta x,$$

(3) $(\alpha.\beta)x = \alpha(\beta x)$.

M is called unital MV-module if A is unital PMV-algebra and M is MV-module over A such that

(4) $1_A x = x$ for any $x \in M$.

Example 2.2. [9] Let A be a PMV-algebra and M an MV-algebra. If $\alpha x = 0$ for any $x \in M$ and $\alpha \in A$, then M is an A-module.

Example 2.3. [9] Any MV-algebra M is a unital L_2-module, where $L_2 = \{0,1\}$ is the Boolean algebra with two elements.

Example 2.4. [9] Let Ω be a nonempty set. Then $\mathscr{A} = P(\Omega)$ is a unital PMV-algebra with $X \oplus Y = X \cup Y$, $X^* = \Omega - X$ and $X.Y = X \cap Y$ for each $X, Y \in \mathscr{A}$. If $\Lambda \subseteq \Omega$ and $M = P(\Lambda)$, then M becomes an MV-module over $\mathscr{A}$ with the external operation defined by $BX = B \cap X$ for any $B \in \mathscr{A}$ and $X \in M$.

Definition 14. [9] Let X and Y be two MV-modules over A. A function $f : X \to Y$ is called an MV-module homomorphism if it satisfies the following conditions, for every $x, y \in X$ and $a \in A$:

(i) $f(0_X) = 0_Y$,
(ii) $f(x \oplus y) = f(x) \oplus f(y)$,
(iii) $f(x^*) = (f(x))^*$,
(iv) $f(ax) = af(x)$.

Now, the concept of an MV-submodule can be defined:

Definition 15. Let M be an MV-module over A. A non-empty subset N of M is called an MV-submodule of M, if it is a MV-subalgebra of M and for each $x \in N$ and $a \in A$, $a.x \in N$.

Clearly, each MV-submodule is an MV-module.

3. Soft MV-module

In this section, the concept of a soft MV-module with the related examples is presented. Then, the unions and intersections of the family of MV-modules are investigated.

Definition 16. Let M be an MV-module over A and (F,E) be a non-null soft set over M. The soft set (F,E) is called a soft MV-module over M if $F(x)$ is an MV-submodule of M for each $x \in Supp(F,E)$.

Now we give some examples of soft MV-modules:

Example 3.1. Let $M = [0,1]$ be the standard MV-algebra. By Example 2.3, M is a unital L_2-module. Let $F : \mathbb{N} \to P(M)$ defined by $F(n) = L_{n+1}$ where $L_{n+1} = \{k/n \mid n \in \mathbb{N},\ k \in \mathbb{N} \cup \{0\},\ 0 \leq k \leq n\}$. Then $(F, \mathbb{N})$ is a soft MV-module over M.

Example 3.2. Let $A_m = \{k/m^n \mid n \in \mathbb{N},\ k \in \mathbb{N} \cup \{0\},\ 0 \leq k \leq m^n\}$ for each $m \in \mathbb{N}$. Then A_m is a PMV-algebra with the operations of the standard PMV-algebra (Example 2.1). Indeed A_m is a PMV-subalgebra of the standard PMV-algebra. Now the standard MV-algebra $M = [0,1]$ is an A_4-module and A_2 and $A_Q = Q \cap [0,1]$ are the submodules of M. Thus, if we consider $F : \mathbb{R} \to P(M)$ by

$$F(x) = \begin{cases} A_Q & \text{if x is rational}, \\ A_2 & \text{if x is irrational}, \end{cases}$$

then $(F, \mathbb{R})$ is a soft MV-module over M.

Example 3.3. Let X be a non-empty set and $F_X = \{\mu \mid \mu : X \to [0,1]\}$ be the set of all fuzzy subsets of X. Let $A = [0,1]$ be standard PMV-algebra. Then, $(F_X, \oplus, ^*, 0)$ is an MV-module over A, where 0 is the empty fuzzy subset and their operations are defined as follows:

$$(\mu \oplus \gamma)(x) = \min\{1, \mu(x) + \gamma(x)\}, \quad \mu(x)^* = 1 - \mu(x) \text{ for all } x \in X \text{ and } \mu, \gamma \in F_X,$$
$$(a\mu)(x) = a\mu(x) \text{ for all } x \in X, \mu \in F_X \text{ and } a \in A.$$

Let C be a set of all constant functions in F_X i.e.,

$$C = \{\mu \in F_X \mid \exists c \in [0,1] \text{ such that } \mu(x) = c \text{ for all } x \in X\}.$$

Let $F : \mathbb{N} \to P(M)$ defined by

$$F(n) = \begin{cases} \emptyset & \text{if n=3k} \\ C & \text{if n=3k+1} \\ F_X & \text{if n=3k+2}. \end{cases}$$

Then, $(F, \mathbb{N})$ is a soft MV-module over F_X.

Definition 17. Let M be an MV-module over A and (F, E) be a non-null soft set over M. The soft set (F, E) is called a whole soft MV-module over M if $F(x) = M$ for all $x \in Supp(F, E)$.

Theorem 2. Let M and $\mathscr{A}$ be the same as in Example 2.4 and (F, E) be a soft MV-module over M. Then (F, E) is a whole soft MV-module over M.

Proof. It suffices to show that M does not have any proper submodule. Let N be a submodule of M. Thus N is a MV-subalgebra of M, therefore, $\phi, \Lambda \in N$. Now let $B \in M$. Thus, $B \in \mathscr{A}$ and, since N is a submodule of M, we have $B.\Lambda = B \cap \Lambda = B \in N$. Hence $N = M$. ■

The famous algebraic structures such as groups, rings, fields, modules, MV-algebras, etc. have at least two subalgebras. Indeed, they have at least one proper subalgebra. However, in the proof of the previous theorem, we see that the MV-module may not have any proper MV-submodule. The next example, faces the same situation as well.

Example 3.4. Let $A = [0,1]$ be the standard PMV-algebra and $M_n = [0,n]$ for each $n \in \mathbb{N}$. Then, by the operations

$$x \oplus y = min\{n, x+y\}, x^* = n - x \text{ and } a.x = ax \text{ (real product)}$$

for each $x, y \in [0,n]$ and $a \in [0,1]$, M_n is an MV-module over A that does not have any proper MV-submodule. Because if N is an MV-submodule of M_n, then $0, n \in N$ and for each $a \in A$, $an \in N$. Now, if $x \in M_n$, then $x/n \in A$. Therefore, $(x/n)n = x \in N$ and $N = M_n$. Hence the only soft MV-module over M_n is a whole soft MV-module.

In the previous example, it must be noted that, if $n, m \in \mathbb{N}$ and $m < n$, then M_m is not an MV-submodule of M_n. Indeed, the operations of M_m and M_n are not the same.

We see three types of intersections for a family of soft sets in Definition 3. Now, in the three following theorems, we show that each type of intersections of a nonempty family of soft MV-modules is a soft MV-module, if these are non-null.

Theorem 3. Let M be an MV-module and $\{(F_i, A_i) \mid i \in \mathscr{I}\}$ be a nonempty family of soft MV-modules over M. Then, the extended intersection $\widetilde{\bigcap}_{i \in \mathscr{I}} (F_i, A_i)$ is a soft MV-module over M if it is non-null.

Proof. Let $\{(F_i, A_i) \mid i \in \mathscr{I}\}$ be a nonempty family of soft MV-modules over M. By Definition 3, we can write $\widetilde{\bigcap}_{i \in \mathscr{I}} (F_i, A_i) = (H, C)$ where $C = \bigcup_{i \in \mathscr{I}} A_i$, $H(x) = \bigcap_{i \in I(x)} F_i(x)$ and $I(x) = \{i \in \mathscr{I} | x \in A_i\}$. Let (H, C) be non-null and $x \in Supp(H, C)$. Then, $\bigcap_{i \in I(x)} F_i(x) \neq \emptyset$, and so for all $i \in I(x)$, we have $F_i(x) \neq \emptyset$. Since $\{(F_i, A_i) \mid i \in \mathscr{I}\}$ is a nonempty family of soft MV-modules over M, it follows that $F_i(x)$ is an MV-submodule of M for each $i \in I(x)$. Since the intersection of any family of submodules is a submodule, so $H(x)$ is an MV-submodule of M. Hence, $\widetilde{\bigcap}_{i \in \mathscr{I}} (F_i, A_i) = (H, C)$ is a soft MV-module over M. ■

Theorem 4. Let M be an MV-module and $\{(F_i, A_i) \mid i \in \mathscr{I}\}$ be a nonempty family of soft MV-modules over M. Then, the restricted intersection $\widetilde{\sqcap}_{i \in \mathscr{I}} (F_i, A_i)$ is a soft MV-module over M if it is non-null.

Proof. The proof is similar to the proof of Theorem 3. ■

Theorem 5. Let M be an MV-module and $\{(F_i, A_i) \mid i \in \mathscr{I}\}$ be a nonempty family of soft MV-modules over M. Then, the $\bigwedge$-intersection $\widetilde{\bigwedge}_{i \in \mathscr{I}} (F_i, A_i)$ is a soft MV-module over M if it is non-null.

Proof. The proof is similar to the proof of Theorem 3. ■

The following example shows that the union of two soft MV-modules is not necessary a soft MV-module.

Example 3.5. Let $M = [0,1]$ and $(F,\mathbb{N})$ be the same as in Example 3.1 and $G : \mathbb{N} \to P(M)$ defined by $G(n) = L_{n+2}$. Then $(F,\mathbb{N})$ and $(G,\mathbb{N})$ are soft MV-modules over M, but $(F,\mathbb{N})\tilde{\cup}(G,\mathbb{N})$ is not a soft MV-module because $F(1) \cup G(1) = L_2 \cup L_3$ is not a submodule of M.

Lemma 1. Let M be an MV-module over A and $\{N_i \mid i \in \mathscr{I}\}$ be a nonempty family of MV-submodules of M. If for all $i, j \in \mathscr{I}$, $N_i \subseteq N_j$ or $N_j \subseteq N_i$, then $\bigcup_{i\in\mathscr{I}} N_i$ is a MV-submodule of M.

Proof. Let $N = \bigcup_{i\in\mathscr{I}} N_i$. Clearly $0_M \in N$. Let $x,y \in N$ and $a \in A$. Thus, there exist $i, j \in \mathscr{I}$ such that $x \in N_i$ and $y \in N_j$. By assumption, $x,y \in N_i$ or $x,y \in N_j$ and so $x \oplus y \in N_i$ or $x \oplus y \in N_j$. Hence, $x \oplus y \in N$. Also $x^* \in N_i$ and $ax \in N_i$, which implies that $x^* \in N$ and $ax \in N$. Therefore N is an MV-submodule of M. ■

Theorem 6. Let M be an MV-module and $\{(F_i,A_i) \mid i \in \mathscr{I}\}$ be a family of soft MV-modules over M such that $F_i(x_i) \subseteq F_j(x_j)$ or $F_j(x_j) \subseteq F_i(x_i)$ for all $i, j \in \mathscr{I}$ and $x_i \in A_i$. Then, the restricted union $\widetilde{\bigcup}_{i\in\mathscr{I}}(F_i,A_i)$ is a soft MV-module over M if it is non-null.

Proof. By Definition 3, we can write $\widetilde{\bigsqcup}_{i\in\mathscr{I}}(F_i,A_i) = (H,C)$, where $C = \bigcap_{i\in\mathscr{I}} A_i \neq \emptyset$ and $H(x) = \bigcup_{i\in\mathscr{I}} F_i(x)$ for all $x \in C$. Let $x \in Supp(H,C)$. Thus, for some $i \in \mathscr{I}$, $F_i(x) \neq \emptyset$. For each $j \in \mathscr{I}$ if $F_j(x) \neq \emptyset$ then $F_j(x)$ is an MV-submodule of M. Hence, by Lemma 1, $H(x)$ is an MV-submodule of M and so $\widetilde{\bigcup}_{i\in\mathscr{I}}(F_i,A_i)$ is a soft MV-module over M. ■

Theorem 7. Let M be an MV-module and $\{(F_i,A_i) \mid i \in \mathscr{I}\}$ be a family of soft MV-modules over M such that $F_i(x_i) \subseteq F_j(x_j)$ or $F_j(x_j) \subseteq F_i(x_i)$ for all $i, j \in \mathscr{I}$ and $x_i \in A_i$ and $x_j \in A_j$. Then, the $\vee$-union $\widetilde{\bigvee}_{i\in\mathscr{I}}(F_i,A_i)$ is a soft MV-module over M if it is non-null.

Proof. The proof is similar to the proof of Theorem 6. ■

Theorem 8. Let M be an MV-module and $\{(F_i,A_i) \mid i \in \mathscr{I}\}$ be a nonempty family of soft MV-modules over M such that $A_i \cap A_j = \emptyset$ for all $i, j \in \mathscr{I}$ $(i \neq j)$. Then, the extended union $\widetilde{\bigcup}_{i\in\mathscr{I}}(F_i,A_i)$ is a soft MV-module over M if it is non-null.

Proof. The proof is straightforward. ■

Theorem 9. Let X,Y be MV-modules over A and $f : X \to Y$ be an MV-module homomorphism.

(i) If (H,C) is a soft MV-module over Y, then $(f^{-1}(H),C)$ is a soft MV-module over X.
(ii) If (G,E) is a soft MV-module over X, then $(f(G),E)$ is a soft MV-module over Y.

Proof. (i) For each $c \in C$, $H(c)$ is the subset of Y and so $f^{-1}(H(c))$ is the subset of X. Hence, we have $f^{-1}(H) : C \to P(X)$ such that $f^{-1}(H)(c) = f^{-1}(H(c))$. Thus $(f^{-1}(H),C)$ is a soft set over X. Now let $c \in Supp(H,C)$. Since $H(c)$ is an MV-submodule of Y, $0_Y \in H(c)$. On the other hand,

$f(0_X) = 0_Y$, thus $0_X \in f^{-1}(H(c))$ and so $(f^{-1}(H), C)$ is non-null. Now let $x, t \in f^{-1}(H(c))$ and $a \in A$. Thus, $f(x), f(t) \in H(c)$ and since $H(c)$ is a submodule, $f(x) \oplus f(t) \in H(c)$ and $(f(x))^* \in H(c)$. Hence, $f(x \oplus t) \in H(c)$ and $f(x^*) \in H(c)$. Therefore, $x \oplus t \in f^{-1}(H(c))$ and $x^* \in f^{-1}(H(c))$. Also we have $ax \in f^{-1}(H(c))$, because $af(x) = f(ax) \in H(c)$. Hence, $f^{-1}(H(c))$ is a submodule of X and thus $(f^{-1}(H), C)$ is a soft MV-module over X.

(ii) For each $e \in E$, $G(e)$ is the subset of X. Therefore, $f(G(e))$ is the subset of Y. Hence, we have $f(G) : A \to P(Y)$ such that $f(G)(e) = f(G(e))$. Thus $(f(G), E)$ is a soft set over Y. Now let $e \in Supp(G, E)$. We have $0_Y \in f(G(e))$ (because $G(e)$ is a submodule of X). Now let $y, z \in f(G(e))$ and $a \in A$. Then, there exist $x, t \in G(e)$ such that $z = f(x)$ and $y = f(t)$. Now $z \oplus y = f(x) \oplus f(t) = f(x \oplus t)$ and since $(x \oplus t) \in G(e)$, $z \oplus y \in f(G(e))$. Also $z^* = f(x^*) \in f(G(e))$ and $ay \in f(G(e))$, because $ay = af(t) = f(at)$ and $at \in G(e)$. Hence, $f(G(e))$ is a submodule of Y and thus $(f(G), E)$ is a soft MV-module over Y. ■

Theorem 10. Let $f : X \to Y$ be an MV-module homomorphism. Let (G, E) and (H, C) be two non-null soft sets over X and Y, respectively.

(i) If f is onto and (G, E) is a whole soft MV-module over X, then $(f(G), E)$ is a whole soft MV-module over Y.

(ii) If $H(c) = f(X)$ for all $c \in Supp(H, C)$, then $(f^{-1}(H), C)$ is a whole soft MV-module over X.

Proof. The proof is straightforward. ■

Theorem 11. Let (F, X) be a soft MV-module over M. Let $\phi : X \to Y$ be an injective function and $\phi(f)$ be a soft image of f under ϕ. Then, $(\phi(f), Y)$ is a soft MV-module over M, if it is non-null.

Proof. Since ϕ is injective, we have $\phi(f)(x) = f(\phi^{-1}(x))$ for all $x \in Supp(\phi(f), Y)$. Thus, $(\phi(f), Y)$ is a soft MV-module over M. ■

The concept of soft homomorphism has been defined by many authors ([1,3,12,19,27]). This concept is used here for soft MV-modules.

Definition 18. Let (F, E) and (G, B) be two soft MV-modules over M and N, respectively. Let $f : M \to N$ and $g : E \to B$ be two functions. Then, we say that (f, g) is a soft MV-module homomorphism if the following conditions are satisfied:

(1) f is an MV-module homomorphism from M to N.

(2) $f(F(e)) = G(g(e))$ for all $e \in E$.

We say that (F, E) is soft homomorphic to (G, B) if there exist a soft MV-module homomorphism (f, g) between (F, E) and (G, B) such that f and g are both surjective.

Moreover, if f is an isomorphism and g is a bijective function, then we say (f,g) is a soft isomorphism and (F,E) is soft isomorphic to (G,B). This is denoted by $(F,E) \simeq (G,B)$.

Theorem 12. The relation $\simeq$ is an equivalence relation on soft MV-modules.

Proof. The proof is straightforward. ∎

Example 3.6. Let Ω be a nonempty set and $\phi \neq \Gamma \subseteq \Lambda \subseteq \Omega$. Let $\mathscr{A} = P(\Lambda)$, $M = P(\Lambda)$ and $N = P(\Gamma)$. Then, M and N become MV-modules over $\mathscr{A}$ analogous to Example 2.4. Let $(F,\mathbb{R})$ and $(G,\mathbb{Z})$ be soft MV-modules over M and N, respectively. Then by Theorem 2, these are whole soft MV-modules. Thus $F(x) = M$ and $G(n) = N$ for all $x \in \mathbb{R}$ and $n \in \mathbb{Z}$. Now we define $g : \mathbb{R} \to \mathbb{Z}$ by $g(x) = [x]$ and $f : M \to N$ by $f(Y) = Y \cap \Gamma$. Then g is a surjective function and f is an MV-module epimorphism. Thus, (f,g) is a soft MV-module homomorphism and $(F,\mathbb{R})$ is soft homomorphic to $(G,\mathbb{Z})$.

Example 3.7. Consider $M = [0,1]$ and $N = [0,2]$ as MV-modules over L_2. Suppose $F : \mathbb{N} \to P(M)$ by $F(n) = L_{n+1}$ and $G : \mathbb{N} - \{1\} \to P(N)$ by $G(n) = 2L_n = \{2x \mid x \in L_n\}$. Thus, $(F,\mathbb{N})$ and $(G,\mathbb{N} - \{1\})$ are soft MV-modules over M and N respectively. Now let $f : M \to N$ by $f(x) = 2x$ and $g : \mathbb{N} \to \mathbb{N} - \{1\}$ by $g(n) = n+1$. Then (f,g) is a soft isomorphism and $(F,\mathbb{N}) \simeq (G,\mathbb{N} - \{1\})$.

Theorem 13. Let (F,E) and (G,B) be two soft MV-modules over M and N, respectively and (F,E) be soft homomorphic to (G,B). If (F,E) is a whole soft MV-module, then (G,B) is a whole soft MV-module.

Proof. Suppose that (F,E) is a whole soft MV-module and (F,E) is soft homomorphic to (G,B). Thus, there exists a surjective function $g : E \to B$ and an MV-module epimorphism $f : M \to N$. Hence, $f(M) = N$ and $F(e) = M$ for all $e \in E$. Now let $b \in B$. Since g is onto, there is $e \in E$ such that $g(e) = b$. So $G(b) = G(g(e)) = f(F(e)) = f(M) = N$. Therefore, (G,B) is a whole soft MV-module. ∎

Theorem 14. Let (F,E) and (G,B) be two soft MV-modules over M and N, respectively and (f,g) be a soft homomorphism between them. If (G,B) is a whole soft MV-module, then f is an epimorphism.

Proof. By hypothesis, we have $G(g(e)) = N$ and so $f(F(e)) = N$ for all $e \in E$. On the other hand, since $F(e) \subseteq M$, we have $N = f(F(e)) \subseteq f(M) \subseteq N$. Thus, $f(M) = N$ and so f is an epimorphism.
∎

Theorem 15. Let (F,E) and (G,B) be two soft MV-modules over M and N, respectively and (f,g) be a soft homomorphism between them. If (G,B) is a whole soft MV-module and f is monomorphism, then (F,E) is a whole soft MV-module.

Proof. Suppose that $e \in E$. By hypothesis, we have $G(g(e)) = N$ and so $f(F(e)) = N$. On the other hand, by the previous theorem, f is onto and so $f(M) = N$. Now, since f is one-to-one, we have $F(e) = M$. Hence, (F,E) is a whole soft MV-module. ■

Example 3.6, shows that the converse of the previous theorem does not hold.

4. Soft Submodule

In this section, the concept of soft MV-submodules is defined and some properties about this concept are investigated.

Definition 19. Let (F,A) and (G,B) be two soft MV-modules over M. Then (G,B) is called a soft MV-submodule of (F,A), denoted by $(G,B) \leq (F,A)$, if $B \subseteq A$ and $G(b) \subseteq F(b)$ for all $b \in Supp(G,B)$.

Clearly, if $(G,B) \leq (F,A)$, then $Supp(G,B) \subseteq Supp(F,A)$.

Example 4.1. Let A_m and A_Q be the same as in Example 3.2 and $M = [0,1]$. Let $F : \mathbb{R} \to P(M)$ and $G : \mathbb{N} \to P(M)$ be defined by $F(x) = A_Q$ and $G(n) = A_2$. Then, $(G,\mathbb{N})$ is a soft MV-submodule of $(F,\mathbb{R})$ over M.

Theorem 16. Let (F,A) and (G,B) be two soft MV-submodules over M. If $(G,B) \subseteq (F,A)$, then $(G,B) \leq (F,A)$.

Proof. The proof is straightforward. ■

The next theorems states that each type of intersection of soft MV-submodules is again a soft MV-submodule:

Theorem 17. Let M be an MV-module and $\{(F_i,A_i) \mid i \in \mathscr{I}\}$ be a nonempty family of soft MV-submodules of (F,A) over M.

(i) The extended intersection $\widetilde{\bigcap}_{i\in\mathscr{I}}(F_i,A_i)$ is a soft MV-submodule of (F,A) over M, if it is non-null.

(ii) The restricted intersection $\widetilde{\sqcap}_{i\in\mathscr{I}}(F_i,A_i)$ is a soft MV-submodule of (F,A) over M, if it is non-null.

(iii) The restricted intersection $\widetilde{\sqcap}_{i\in\mathscr{I}}(F_i,A_i)$ is a soft MV-submodule of (F_k,A_k) over M for all $k \in \mathscr{I}$, if it is non-null.

Proof. (i) Let $\widetilde{\bigcap}_{i\in\mathscr{I}}(F_i,A_i)=(H,C)$ be non-null. Thus, by Theorem 3, it is a soft MV-module over M and by Definition 3, $C\subseteq A$ and $H(x)\subseteq F(x)$ for all $c\in Supp(H,C)$. Therefore, it is a soft MV-submodule of (F,A).

The proofs of (ii) and (iii) are similar to the proof of (i). ■

Note that the extended intersection $\widetilde{\bigcap}_{i\in\mathscr{I}}(F_i,A_i)$ is not a soft MV-submodule of (F_k,A_k) in general. Indeed, by Definition 3, it is not a soft subset of (F_k,A_k).

Theorem 18. Let M be an MV-submodule and $\{(F_i,A_i)\mid i\in\mathscr{I}\}$ be a family of soft MV-submodules of (F,A) over M such that $F_i(x_i)\subseteq F_j(x_j)$ or $F_j(x_j)\subseteq F_i(x_i)$ for all $i,j\in\mathscr{I}$ and $x_i\in A_i$ and $x_j\in A_j$. Then, the restricted union $\widetilde{\bigcup}_{i\in\mathscr{I}}(F_i,A_i)$ is a soft MV-submodule of (F,A) over M, if it is non-null.

Proof. By Theorem 6, the proof is straightforward. ■

The effect of a MV-module homomorphism on the soft MV-submodules is given below:

Theorem 19. Let $f:X\to Y$ be an MV-module homomorphism.

(i) Let (F,A) and (G,B) be two soft MV-modules over X. If $(G,B)\leq(F,A)$, then $(f(G),B)\leq(f(F),A)$.

(ii) Let (H,C) and (K,D) be two soft MV-modules over Y. If $(H,C)\leq(K,D)$, then $(f^{-1}(H),C)\leq(f^{-1}(K),D)$.

Proof. (i) Since $(G,B)\leq(F,A)$, we have $G(x)\subseteq F(x)$ for all $x\in B$. Thus, $f(G)(x)=f(G(x))\subseteq f(F)(x)=f(F(x))$. Hence, $(f(G),B)\subseteq(f(F),A)$. On the other hand, by Theorem 9, $(f(G),B)$ and $(f(F),A)$ are two soft MV-modules over Y. Thus, by Theorem 16, we have $(f(G),B)\leq(f(F),A)$.

(ii) The proof is similar to the proof of (i). ■

There is a one-to-one correspondence between the soft MV-submodules of two soft isomorphic MV-modules. More precisely, we have the following theorem:

Theorem 20. Let (F,E) and (G,B) be two soft MV-modules over M and N, respectively. Let $(F,E)\simeq(G,B)$. Then, for each soft MV-submodule (F_1,E_1) of (F,E), there exists a soft MV-submodule (G_1,B_1) of (G,B) such that $(F_1,E_1)\simeq(G_1,B_1)$.

Proof. Let $(F,E)\simeq(G,B)$ via soft isomorphism (f,g). Let $(F_1,E_1)\leq(F,E)$. We define the function $G_1:g(E_1)\to P(N)$ by $G_1(b)=f(F_1(g^{-1}(b)))$ for all $b\in g(E_1)$. Let $B_1=g(E_1)$. Since $F_1(e)\leq F(e)$ and $f(F(e))=G(g(e))$ we have:

$$G_1(b)=f(F_1(g^{-1}(b)))\subseteq f(F(g^{-1}(b)))=G(g(g^{-1}(b)))=G(b)$$

Thus, $G_1(b) \leq G(b)$. Hence $(G_1,B_1) \leq (G,B)$. Nowm we consider $g_1 = g\mid_E : E_1 \to B_1$. Clearly, g_1 is a bijective function. We also have $G_1(g_1(e)) = G_1(g(e)) = f(F_1(e))$ for each $e \in E_1$. Thus, $(F_1,E_1) \simeq (G_1,B_1)$ via soft isomorphism (f,g_1). ■

5. Conclusion

Soft sets were introduced by Molodtsov as a new mathematical tool in order to deal with uncertainties. This concept was applied to algebraic structures such as fuzzy sets. In this paper, we defined the notion of a soft MV-module and then we focused on soft MV-submodules, the soft MV-module homomorphism and different types of intersections and unions of the family of soft MV-modules.

Acknowledgements

The authors extremely appreciate the referees for providing many valuable comments and helpful suggestions which helped in improving the presentation of this paper.

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