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NEW HERMITE-HADAMARD TYPE INEQUALITIES FOR CONVEX FUNCTIONS ON A RECTANGULAR BOX

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Abstract. In this paper some Hermite-Hadamard type inequalities for convex functions of three variables on a rectangular box in $\mathbb{R}^3$ are given.

1. Introduction

Let $I = [a, b]$, $a < b$, be an interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ a convex function. The following double inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

is known in the literature as Hermite-Hadamard inequality for convex functions (see for example [5]). In resent years there have been many extensions, generalizations, refinements and similar results of Hetmite-Hadamard inequality, see [1-16] and references therein.

In [11] Dragomir consider an inequality of Hetmite-Hadamard type for convex functions on the co-ordinates on a rectangle from the plane $\mathbb{R}^2$. A function $f : \Delta := [a, b] \times [c, d] \subset \mathbb{R}^2 \to \mathbb{R}$, $a < b$ and $c < d$, is called convex on the co-ordinates if partial mappings, $f_y : [a, b] \to \mathbb{R}$ defined as $f_y(u) = f(u, y)$, and $f_x : [c, d] \to \mathbb{R}$ defined as $f_x(v) = f(x, v)$, are convex where defined for every $y \in [c, d]$ and $x \in [a, b]$. Clearly every convex function is co-ordinated convex. Furthermore, there exists co-ordinated convex function which is not convex (see [11]). Since then several important generalizations introduced on this category, see [14,15,18-20] and references therein.

On the other hand, M.E. Özdemir [15] defined convex functions on a rectangular box $\Omega := [a, b] \times [c, d] \times [e, f]$ in $\mathbb{R}^3$ as follows: A function $g : \Omega \to \mathbb{R}$ is said to be

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convex on the co-ordinates on Ω if for every $(x,y,z)\in\Omega$, the partial mapping,

$$\begin{aligned} &g_z:[a,b]\times[c,d]\to\mathbb{R}, && g_z(u,v)=g(u,v,z), && z\in[e,f],\\ &g_y:[a,b]\times[e,f]\to\mathbb{R}, && g_y(u,w)=g(u,y,w), && y\in[c,d],\\ &g_x:[c,d]\times[e,f]\to\mathbb{R}, && g_x(v,w)=g(x,v,w), && x\in[a,b], \end{aligned}$$

are convex. Recall the following inequality of Hermite-Hadamard type for co-ordinated convex function on a rectangular box in $\mathbb{R}^3$ from [15].

Theorem 1.1. *Suppose that $g:\Omega\to\mathbb{R}$ is convex function. Then one has the inequalities:*

$$\begin{aligned} & g\left(\frac{a+b}{2},\frac{c+d}{2},\frac{e+f}{2}\right)\\ &\le \frac{1}{3}\left[\frac{1}{b-a}\int_a^b g\left(x,\frac{c+d}{2},\frac{e+f}{2}\right)dx+\frac{1}{d-c}\int_c^d g\left(\frac{a+b}{2},y,\frac{e+f}{2}\right)dy\right.\\ &\quad\left.+\frac{1}{f-e}\int_e^f g\left(\frac{a+b}{2},\frac{c+d}{2},z\right)dz\right]\\ &\le \frac{1}{(b-a)(d-c)(f-e)}\iiint_\Omega g(x,y,z)dxdydz\\ &\le \frac{1}{6}\left[\frac{1}{(b-a)(d-c)}\iint_{\Delta_1}\big[g(x,y,e)+g(x,y,f)\big]dydx\right.\\ &\quad+\frac{1}{(b-a)(f-e)}\iint_{\Delta_2}\big[g(x,c,z)+g(x,d,z)\big]dzdx\\ &\quad\left.+\frac{1}{(d-c)(f-e)}\iint_{\Delta_3}\big[g(a,y,z)+g(b,y,z)\big]dzdy\right]\\ &\le \frac{1}{8}\Big(g(a,c,e)+g(a,d,e)+g(b,c,e)+g(b,d,e)\\ &\quad+g(a,c,f)+g(a,d,f)+g(b,c,f)+g(b,d,f)\Big), \end{aligned}$$

where $\Omega:=[a,b]\times[c,d]\times[e,f]\subseteq\mathbb{R}^3$, $\Delta_1=[a,b]\times[c,d]$, $\Delta_2=[a,b]\times[e,f]$ and $\Delta_3=[c,d]\times[e,f]$.

The main purpose of this paper is to establish new Hermite-Hadamard type inequalities of convex functions of 3-variables on the co-ordinates.

2. MAIN RESULT

Throughout this section Γ is a rectangular box in $\mathbb{R}^3$ defined by $\Gamma: I_1\times I_2\times I_3$ where $I_1:=[a_1,b_1]$, $I_2:=[c_1,d_1]$ and $I_3:=[e_1,f_1]$ are intervals in $\mathbb{R}$ with $a_1<b_1$, $c_1<d_1$ and $e_1<f_1$. Moreover, $\Omega:=[a,b]\times[c,d]\times[e,f]$ is a rectangular box contained in Γ° where, $a,b\in I_1^\circ$(the interior of I_1), $c,d\in I_2^\circ$ and $e,f\in I_3^\circ$ such that $a<b$, $c<d$, $e<f$.
To reach our goal, we need the following lemma.

Lemma 2.1. *Let $g : \Gamma \to \mathbb{R}$ be a mapping on Γ. Suppose that g is third partial differentiable on Ω. If $\frac{\partial^3 g}{\partial t \partial s \partial r} \in L_1(\Omega)$, then the following equality holds:*

$$
\begin{aligned}
&\frac{1}{8}\Big(g(a,c,e)+g(a,d,e)+g(b,c,e)+g(b,d,e) \\
&+g(a,c,f)+g(a,d,f)+g(b,c,f)+g(b,d,f)\Big) \\
&-\frac{1}{(b-a)(d-c)(f-e)}\iiint_{\Omega} g(x,y,z)dxdydz
\end{aligned}
\tag{2.1}
$$

$$
\begin{aligned}
&+\frac{1}{2}\Bigg[\frac{1}{(b-a)(d-c)}\iint_{\Delta_1}\big[g(x,y,e)+g(x,y,f)\big]dxdy \\
&+\frac{1}{(b-a)(f-e)}\iint_{\Delta_2}\big[g(x,c,z)+g(x,d,z)\big]dxdz \\
&+\frac{1}{(d-c)(f-e)}\iint_{\Delta_3}\big[g(a,y,z)+g(b,y,z)\big]dydz\Bigg] \\
&-\frac{1}{4}\Bigg[\frac{1}{(b-a)}\int_a^b\big[g(x,c,e)+g(x,c,f)+g(x,d,e)+g(x,d,f)\big]dx \\
&+\frac{1}{(d-c)}\int_c^d\big[g(a,y,e)+g(a,y,f)+g(b,y,e)+g(b,y,f)\big]dy \\
&+\frac{1}{(f-e)}\int_e^f\big[g(a,c,z)+g(a,d,z)+g(b,c,z)+g(b,d,z)\big]dz\Bigg] \\
=\;&\frac{(b-a)(d-c)(f-e)}{8}\int_0^1\int_0^1\int_0^1(1-2r)(1-2s)(1-2t) \\
&\times\frac{\partial^3 g}{\partial t\partial s\partial r}(ta+(1-t)b,sc+(1-s)d,re+(1-r)f)dtdsdr,
\end{aligned}
$$

where $\Delta_1=[a,b]\times[c,d]$, $\Delta_2=[a,b]\times[e,f]$ *and* $\Delta_3=[c,d]\times[e,f]$.

Proof. By integration by parts, we have

$$
\begin{aligned}
&\int_0^1\int_0^1\int_0^1 (1-2r)(1-2s)(1-2t)\\
&\times \frac{\partial^3 g}{\partial t\partial s\partial r}(ta+(1-t)b, sc+(1-s)d, re+(1-r)f)dtdsdr\\
=&\int_0^1\int_0^1 (1-2r)(1-2s)\\
&\times\left\{\frac{(1-2t)}{a-b}\frac{\partial^2 g}{\partial s\partial r}(ta+(1-t)b, sc+(1-s)d, re+(1-r)f)\Big|_0^1\right.\\
&\left.+\frac{2}{a-b}\frac{\partial^2 g}{\partial s\partial r}(ta+(1-t)b, sc+(1-s)d, re+(1-r)f)dt\right\}dsdr\\
=&\int_0^1\int_0^1 (1-2r)(1-2s)\left\{\frac{-1}{a-b}\frac{\partial^2 g}{\partial s\partial r}(a, sc+(1-s)d, re+(1-r)f)\right.\\
&-\frac{1}{a-b}\frac{\partial^2 g}{\partial s\partial r}(b, sc+(1-s)d, re+(1-r)f)\\
&\left.+\frac{2}{a-b}\frac{\partial^2 g}{\partial s\partial r}(ta+(1-t)b, sc+(1-s)d, re+(1-r)f)dt\right\}dsdr
\end{aligned}
\tag{2.2}
$$

$$
\begin{aligned}
=&\frac{1}{b-a}\int_0^1(1-2r)\left\{\int_0^1(1-2s)\left(\frac{\partial^2 g}{\partial s\partial r}(a, sc+(1-s)d, re+(1-r)f)\right.\right.\\
&\left.+\frac{\partial^2 g}{\partial s\partial r}(b, sc+(1-s)d, re+(1-r)f)\right)ds-2\int_0^1\int_0^1(1-2s)\\
&\left.\times\left(\frac{\partial^2 g}{\partial s\partial r}(ta+(1-t)b, sc+(1-s)d, re+(1-r)f)dtds\right)\right\}dr.
\end{aligned}
$$

If we denote the right hand side of (2.2) by I_1 and again by integrating by parts, we have

$$\begin{aligned}
I_1 = & \int_0^1 (1-2r)\Bigg\{ \frac{(1-2s)}{c-d}\left(\frac{\partial g}{\partial r}(a, sc+(1-s)d, re+(1-r)f)\right. \\
& \left.\left. + \frac{\partial g}{\partial r}(b, sc+(1-s)d, re+(1-r)f)\right)\right|_0^1 \\
& + \frac{2}{c-d}\int_0^1 \left[\frac{\partial g}{\partial r}(a, sc+(1-s)d, re+(1-r)f)\right. \\
& \left. + \frac{\partial g}{\partial r}(b, sc+(1-s)d, re+(1-r)f)\right] ds \\
& - 2\int_0^1 \left[\frac{(1-2s)}{c-d}\frac{\partial g}{\partial r}(ta+(1-t)b, sc+(1-s)d, re+(1-r)f)\Big|_0^1\right. \\
& \left. + \frac{2}{c-d}\int_0^1 \frac{\partial g}{\partial r}(ta+(1-t)b, sc+(1-s)d, re+(1-r)f) ds\right] dt \Bigg\} dr
\end{aligned}$$

$$\begin{aligned}
(2.3) \qquad = & \frac{1}{d-c}\int_0^1 (1-2r)\Bigg\{ \frac{\partial g}{\partial r}(a, c, re+(1-r)f) + \frac{\partial g}{\partial r}(b, c, re+(1-r)f) \\
& + \frac{\partial g}{\partial r}(a, d, re+(1-r)f) + \frac{\partial g}{\partial r}(b, d, re+(1-r)f) \\
& + 4\int_0^1\int_0^1 \frac{\partial g}{\partial r}(ta+(1-t)b, sc+(1-s)d, re+(1-r)f) ds dt \\
& - 2\Bigg[\int_0^1 \left(\frac{\partial g}{\partial r}(a, sc+(1-s)d, re+(1-r)f)\right. \\
& \quad \left. + \frac{\partial g}{\partial r}(b, sc+(1-s)d, re+(1-r)f)\right) ds \\
& \quad + \int_0^1 \left(\frac{\partial g}{\partial r}(ta+(1-t)b, c, re+(1-r)f)\right. \\
& \quad \left. + \frac{\partial g}{\partial r}(ta+(1-t)b, d, re+(1-r)f)\right) dt \Bigg]\Bigg\} dr.
\end{aligned}$$

Similarly, we denote the right hand side of (2.3) by I_2, it follows that

(2.4)

$$
\begin{aligned}
I_2 = \frac{(1-2r)}{e-f}\Big(& g(a,c,re+(1-r)f) + g(b,c,re+(1-r)f) \\
&+ g(a,d,re+(1-r)f) + g(b,d,re+(1-r)f)\Big)\Big|_0^1 \\
&+ \frac{2}{e-f}\int_0^1 \Big[g(a,c,re+(1-r)f) + g(b,c,re+(1-r)f) \\
&+ g(a,d,re+(1-r)f) + g(b,d,re+(1-r)f)\Big]dr \\
&+ \frac{4}{e-f}\int_0^1\int_0^1 \Big[(1-2r)g(ta+(1-t)b, sc+(1-s)d, re+(1-r)f)\Big|_0^1 \\
&+ 2\int_0^1 g(ta+(1-t)b, sc+(1-s)d, re+(1-r)f)dr\Big]dsdt \\
&- \frac{2}{e-f}\int_0^1 \Big[(1-2r)\Big(g(a, sc+(1-s)d, re+(1-r)f) \\
&+ g(b, sc+(1-s)d, re+(1-r)f)\Big)\Big|_0^1 \\
&+ 2\int_0^1 \Big(g(a, sc+(1-s)d, re+(1-r)f) \\
&+ g(b, sc+(1-s)d, re+(1-r)f)\Big)dr\Big]ds \\
&- \frac{2}{e-f}\int_0^1 \Big[(1-2r)\Big(g(ta+(1-t)b, c, re+(1-r)f) \\
&+ g(ta+(1-t)b, d, re+(1-r)f)\Big)\Big|_0^1 \\
&+ 2\int_0^1 \Big((g(ta+(1-t)b, c, re+(1-r)f) \\
&+ g(ta+(1-t)b, d, re+(1-r)f)\Big)dr\Big]dt,
\end{aligned}
$$

writing (2.3) and (2.4) in (2.2), we have

$$
\begin{aligned}
I_3 = & \frac{1}{(b-a)(d-c)(f-e)} \\
& \times\Big\{g(a,c,e) + g(a,c,f) + g(b,c,e) + g(b,c,f) \\
& + g(a,d,e) + g(a,d,f) + g(b,d,e) + g(b,d,f)
\end{aligned}
\tag{2.5}
$$

$$
\begin{aligned}
&-2\int_0^1\Big[g(a,c,re+(1-r)f)+g(b,c,re+(1-r)f)\\
&\qquad+g(a,d,re+(1-r)f)+g(b,d,re+(1-r)f)\Big]dr\\
&+4\int_0^1\int_0^1\Big[g(ta+(1-t)b,sc+(1-s)d,e)\\
&\qquad+g(ta+(1-t)b,sc+(1-s)d,f)\Big]dsdt\\
&-8\int_0^1\int_0^1\int_0^1 g[ta+(1-t)b,sc+(1-s)d,re+(1-r)f)dtdsdr\\
&-2\int_0^1\Big[g(a,sc+(1-s)d,e)+g(a,sc+(1-s)d,f)\\
&\qquad+g(b,sc+(1-s)d,e)+g(b,sc+(1-s)d,f)\Big]ds\\
&+4\int_0^1\int_0^1\Big[g(a,sc+(1-s)d,re+(1-r)f)\\
&\qquad+g(b,sc+(1-s)d,re+(1-r)f)\Big]dsdr\\
&-2\int_0^1\Big[g(ta+(1-t)b,c,e)+g(ta+(1-t)b,c,f)\\
&\qquad+g(ta+(1-t)b,d,e)+g(ta+(1-t)b,d,f)\Big]dt\\
&+4\int_0^1\int_0^1\Big[g(ta+(1-t)b,c,re+(1-r)f)\\
&\qquad+g(ta+(1-t)b,d,re+(1-r)f)\Big]dtdr\Bigg\}.
\end{aligned}
$$

Using the change of variable $x = ta+(1-t)b$, $y = sc+(1-s)d$ and $z = re+(1-r)f$ for $t, s, r \in [0, 1]$, and multiplying the both side by $\frac{(b-a)(d-c)(f-e)}{8}$, we obtain (2.1), which completes the proof. □

Theorem 2.1. *Let $g : \Gamma \to \mathbb{R}$ be a mapping on Γ. Suppose that g is third partial differentiable on Ω. If $\left|\frac{\partial^3 g}{\partial t\partial s\partial r}\right|$ is a convex function on the co-ordinates on Ω, then the following inequality holds:*

$$
\begin{aligned}
&\left|\frac{1}{(b-a)(d-c)(f-e)}\iiint_\Omega g(x,y,z)dxdydz - A + B - C\right|\\
&\le \frac{(b-a)(d-c)(f-e)}{128}\Bigg(\frac{1}{4}\Bigg\{\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,e)\right| + \left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,f)\right|\\
&\quad+\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,e)\right| + \left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,f)\right| + \left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,e)\right|
\end{aligned}
\tag{2.6}
$$

$$+\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,f)\right|+\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,e)\right|+\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,f)\right|\Bigg\}\Bigg),$$

where A and B and C are defined, respectively with

$$A=\frac{1}{2}\Bigg[\frac{1}{(b-a)(d-c)}\iint_{\Delta_1}\big[g(x,y,e)+g(x,y,f)\big]dxdy$$
$$+\frac{1}{(b-a)(f-e)}\iint_{\Delta_2}\big[g(x,c,z)+g(x,d,z)\big]dxdz$$
$$+\frac{1}{(d-c)(f-e)}\iint_{\Delta_3}\big[g(a,y,z)+g(b,y,z)\big]dydz\Bigg],$$

$$B=\frac{1}{4}\Bigg[\frac{1}{(b-a)}\int_a^b\big[g(x,c,e)+g(x,c,f)+g(x,d,e)+g(x,d,f)\big]dx$$
$$+\frac{1}{(d-c)}\int_c^d\big[g(a,y,e)+g(a,y,f)+g(b,y,e)+g(b,y,f)\big]dy$$
$$+\frac{1}{(f-e)}\int_e^f\big[g(a,c,z)+g(a,d,z)+g(b,c,z)+g(b,d,z)\big]dz\Bigg],$$

and

$$C=\frac{1}{8}\Bigg(g(a,c,e)+g(a,d,e)+g(b,c,e)+g(b,d,e)$$
$$+g(a,c,f)+g(a,d,f)+g(b,c,f)+g(b,d,f)\Bigg).$$

Proof. From Lemma 2.1, we have

$$\left|\frac{1}{(b-a)(d-c)(f-e)}\iiint_\Omega g(x,y,z)dxdydz-A+B-C\right|$$
$$\le\frac{(a-b)(d-c)(f-e)}{8}\int_0^1\int_0^1\int_0^1\big|(1-2r)(1-2s)(1-2t)\big|$$
$$\times\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(ta+(1-t)b,sc+(1-s)d,re+(1-r)f)\right|dtdsdr.$$

Since $\left|\frac{\partial^3 g}{\partial t\partial s\partial r}\right|$ is co-ordinated convex on Ω, then one has:

$$\left|\frac{1}{(b-a)(d-c)(f-e)}\iiint_\Omega g(x,y,z)dxdydz-A+B-C\right|$$
$$\le\frac{(a-b)(d-c)(f-e)}{8}\int_0^1\int_0^1\Bigg[\int_0^1\big|(1-2r)(1-2s)(1-2t)\big|$$
$$\times\Bigg\{t\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,sc+(1-s)d,re+(1-r)f)\right|$$
$$+(1-t)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,sc+(1-s)d,re+(1-r)f)\right|\Bigg\}dt\Bigg]dsdr.$$

By calculating the integral in above inequality we have

$$\begin{aligned}
&\int_0^1 \big|1-2t\big|\bigg\{t\bigg|\frac{\partial^3 g}{\partial t\partial s\partial r}(a, sc+(1-s)d, re+(1-r)f)\bigg| \\
&+(1-t)\bigg|\frac{\partial^3 g}{\partial t\partial s\partial r}(b, sc+(1-s)d, re+(1-r)f)\bigg|\bigg\}dt \\
=&\int_0^{\frac{1}{2}}(1-2t)\bigg\{t\bigg|\frac{\partial^3 g}{\partial t\partial s\partial r}(a, sc+(1-s)d, re+(1-r)f)\bigg| \\
&+(1-t)\bigg|\frac{\partial^3 g}{\partial t\partial s\partial r}(b, sc+(1-s)d, re+(1-r)f)\bigg|\bigg\}dt \\
&+\int_{\frac{1}{2}}^1(2t-1)\bigg\{t\bigg|\frac{\partial^3 g}{\partial t\partial s\partial r}(a, sc+(1-s)d, re+(1-r)f)\bigg| \\
&+(1-t)\bigg|\frac{\partial^3 g}{\partial t\partial s\partial r}(b, sc+(1-s)d, re+(1-r)f)\bigg|\bigg\}dt \\
=&\frac{1}{4}\bigg(\bigg|\frac{\partial^3 g}{\partial t\partial s\partial r}(a, sc+(1-s)d, re+(1-r)f)\bigg| \\
&+\bigg|\frac{\partial^3 g}{\partial t\partial s\partial r}(b, sc+(1-s)d, re+(1-r)f)\bigg|\bigg).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\bigg|\frac{1}{(b-a)(d-c)(f-e)}\iiint_\Omega g(x,y,z)dxdydz - A + B - C\bigg| \\
\leq\ &\frac{(a-b)(d-c)(f-e)}{32} \\
&\times\int_0^1\int_0^1\big|(1-2r)(1-2s)\big|\bigg\{\bigg|\frac{\partial^3 g}{\partial t\partial s\partial r}(a, sc+(1-s)d, re+(1-r)f)\bigg| \\
&+\bigg|\frac{\partial^3 g}{\partial t\partial s\partial r}(b, sc+(1-s)d, re+(1-r)f)\bigg|\bigg\}dsdr.
\end{aligned}$$

A similar way for other integral, since $\big|\frac{\partial^3 g}{\partial t\partial s\partial r}\big|$ is co-ordinated convex on Ω, we have

$$\begin{aligned}
&\int_0^1 \big|1-2s\big|\bigg\{\bigg|\frac{\partial^3 g}{\partial t\partial s\partial r}(a, sc+(1-s)d, re+(1-r)f)\bigg| \\
&+\bigg|\frac{\partial^3 g}{\partial t\partial s\partial r}(b, sc+(1-s)d, re+(1-r)f)\bigg|\bigg\}ds
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{\frac{1}{2}} (1-2s)\Bigg\{ s\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,re+(1-r)f)\right| \\
&+ (1-s)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,re+(1-r)f)\right| \\
&+ s\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,re+(1-r)f)\right| + (1-s)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,re+(1-r)f)\right|\Bigg\}ds \\
&+ \int_{\frac{1}{2}}^1 (2s-1)\Bigg\{ s\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,re+(1-r)f)\right| \\
&+ (1-s)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,re+(1-r)f)\right| \\
&+ s\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,re+(1-r)f)\right| + (1-s)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,re+(1-r)f)\right|\Bigg\}ds \\
&= \frac{1}{4}\Bigg\{ \left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,re+(1-r)f)\right| + \left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,re+(1-r)f)\right| \\
&+ \left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,re+(1-r)f)\right| + \left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,re+(1-r)f)\right|\Bigg\},
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|\frac{1}{(b-a)(d-c)(f-e)}\iiint_\Omega g(x,y,z)dxdydz - A + B - C\right| \\
&\leq \frac{(a-b)(d-c)(f-e)}{128}\int_0^1 |1-2r| \\
&\times \Bigg\{ \left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,re+(1-r)f)\right| + \left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,re+(1-r)f)\right| \\
&+ \left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,re+(1-r)f)\right| + \left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,re+(1-r)f)\right|\Bigg\}dr.
\end{aligned} \tag{2.7}
$$

Thus,

$$
\begin{aligned}
&\int_0^1 |1-2r|\Bigg\{ \left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,re+(1-r)f)\right| + \left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,re+(1-r)f)\right| \\
&+ \left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,re+(1-r)f)\right| + \left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,re+(1-r)f)\right|\Bigg\}dr
\end{aligned} \tag{2.8}
$$

$$= \int_0^{\frac{1}{2}} (1-2r)\Bigg\{ r\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,e)\right| + (1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,f)\right|$$
$$+ r\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,e)\right| + (1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,f)\right|$$
$$+ r\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,e)\right| + (1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,f)\right|$$
$$+ r\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,e)\right| + (1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,f)\right|\Bigg\} dr$$
$$+ \int_{\frac{1}{2}}^{1} (2r-1)\Bigg\{ r\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,e)\right| + (1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,f)\right|$$
$$+ r\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,e)\right| + (1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,f)\right|$$
$$+ r\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,e)\right| + (1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,f)\right|$$
$$+ r\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,e)\right| + (1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,f)\right|\Bigg\} dr$$
$$= \frac{1}{4}\Bigg\{ \left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,e)\right| + \left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,f)\right| + \left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,e)\right| + \left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,f)\right|$$
$$+ \left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,e)\right| + \left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,f)\right| + \left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,e)\right| + \left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,f)\right|\Bigg\}.$$

By the (2.7) and (2.8), we get the inequality (2.6). □

Another similar result may be extended in the following theorem.

Theorem 2.2. *Let $g : \Gamma \to \mathbb{R}$ be a mapping on Γ. Suppose that g is third partial differentiable on Ω. If $\left|\frac{\partial^3 g}{\partial t\partial s\partial r}\right|^q$, $q > 1$, is a convex function on the co-ordinates on Ω, then the following inequality holds:*

$$\left|\frac{1}{(b-a)(d-c)(f-e)}\iiint_\Omega g(x,y,z)dxdydz - A + B - C\right|$$
$$\leq \frac{(b-a)(d-c)(f-e)}{8(p+1)^{\frac{3}{p}}}\Bigg(\frac{1}{4}\Bigg\{\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,e)\right|^q + \left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,f)\right|^q$$
$$(2.9)\qquad + \left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,e)\right|^q + \left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,f)\right|^q + \left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,e)\right|^q$$
$$+ \left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,f)\right|^q + \left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,e)\right|^q + \left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,f)\right|^q\Bigg\}\Bigg)^{\frac{1}{q}},$$

where A, B and C is defined in theorem 2.1, and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. From Lemma 2.1, we have

$$\begin{aligned}&\left|\frac{1}{(b-a)(d-c)(f-e)}\iiint_{\Omega} g(x,y,z)dxdydz - A + B - C\right|\\ &\leq \frac{(a-b)(d-c)(f-e)}{8}\int_0^1\int_0^1\int_0^1 \big|(1-2r)(1-2s)(1-2t)\big|\\ &\quad\times\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(ta+(1-t)b, sc+(1-s)d, re+(1-r)f)\right|dtdsdr.\end{aligned}$$

By using the well known Hölder inequality for triple integrals, then one has:

$$\begin{aligned}&\left|\frac{1}{(b-a)(d-c)(f-e)}\iiint_{\Omega} g(x,y,z)dxdydz - A + B - C\right|\\ &\leq \frac{(a-b)(d-c)(f-e)}{8}\left(\int_0^1\int_0^1\int_0^1 \big|1-2r\big|^p\big|1-2s\big|^p\big|1-2t\big|^p dtdsdr\right)^{\frac{1}{p}}\\ &\times\left(\int_0^1\int_0^1\int_0^1 \left|\frac{\partial^3 g}{\partial t\partial s\partial r}(ta+(1-t)b, sc+(1-s)d, re+(1-r)f)\right|^q dtdsdr\right)^{\frac{1}{q}}.\end{aligned}$$

Since $\left|\frac{\partial^3 g}{\partial t\partial s\partial r}\right|^q$, $q>1$ is convex function on the co-ordinates on Ω, for $t\in[0,1]$ we have

$$\begin{aligned}&\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(ta+(1-t)b, sc+(1-s)d, re+(1-r)f)\right|^q\\ &\leq t\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a, sc+(1-s)d, re+(1-r)f)\right|^q\\ &\quad+(1-t)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b, sc+(1-s)d, re+(1-r)f)\right|^q.\end{aligned}$$

Similarly

$$\begin{aligned}&\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(ta+(1-t)b, sc+(1-s)d, re+(1-r)f)\right|^q\\ &\leq ts\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a, c, re+(1-r)f)\right|^q + t(1-s)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a, d, re+(1-r)f)\right|^q\\ &\quad+(1-t)s\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b, c, re+(1-r)f)\right|^q\\ &\quad+(1-t)(1-s)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b, d, re+(1-r)f)\right|^q,\end{aligned}$$

and

$$\begin{aligned}
&\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(ta+(1-t)b, sc+(1-s)d, re+(1-r)f)\right|^q \\
&\le tsr\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,e)\right|^q + ts(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,f)\right|^q \\
&\quad + tr(1-s)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,e)\right|^q + t(1-s)(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,f)\right|^q \\
&\quad + (1-t)sr\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,e)\right|^q + (1-t)(1-r)s\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,f)\right|^q \\
&\quad + (1-t)(1-s)r\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,e)\right|^q \\
&\quad + (1-t)(1-s)(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,f)\right|^q.
\end{aligned}$$

Hence, it follows that

$$\begin{aligned}
&\left|\frac{1}{(b-a)(d-c)(f-e)}\iiint_\Omega g(x,y,z)dxdydz - A + B - C\right| \\
&\le \frac{(b-a)(d-c)(f-e)}{8(p+1)^3}\Bigg(\int_0^1\int_0^1\int_0^1 \Big\{tsr\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,e)\right|^q \\
&\quad + ts(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,f)\right|^q + tr(1-s)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,e)\right|^q \\
&\quad + t(1-s)(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,f)\right|^q + (1-t)sr\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,e)\right|^q \\
&\quad + (1-t)(1-r)s\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,f)\right|^q + (1-t)(1-s)r\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,e)\right|^q \\
&\quad + (1-t)(1-s)(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,f)\right|^q\Big\}dtdsdr\Bigg)^{\frac{1}{q}} \\
&= \frac{(b-a)(d-c)(f-e)}{8(p+1)^3} \times \Bigg(\frac{1}{4}\Bigg\{\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,e)\right|^q + \left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,f)\right|^q \\
&\quad + \left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,e)\right| + \left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,f)\right|^q + \left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,e)\right|^q \\
&\quad + \left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,f)\right|^q + \left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,e)\right|^q + \left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,f)\right|^q\Bigg\}\Bigg)^{\frac{1}{q}}.
\end{aligned}$$

□

Theorem 2.3. *Let $g : \Gamma \to \mathbb{R}$ be a mapping on Γ. Suppose that g is third partial differentiable on Ω. If $\left|\frac{\partial^3 g}{\partial t\partial s\partial r}\right|^q$, $q \ge 1$, is a convex function on the co-ordinates on*

Ω, then the following inequality holds:

$$\left|\frac{1}{(b-a)(d-c)(f-e)}\iiint_{\Omega} g(x,y,z)dxdydz - A + B - C\right|$$
$$\le \frac{(b-a)(d-c)(f-e)}{128}\left(\frac{1}{4}\left\{\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,e)\right|^q + \left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,f)\right|^q\right.\right.$$
$$(2.10)\qquad + \left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,e)\right| + \left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,f)\right|^q + \left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,e)\right|^q$$
$$\left.\left. + \left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,f)\right|^q + \left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,e)\right|^q + \left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,f)\right|^q\right\}\right)^{\frac{1}{q}},$$

where A, B and C is defined in theorem 2.1.

Proof. From Lemma 2.1, we have

$$\left|\frac{1}{(b-a)(d-c)(f-e)}\iiint_{\Omega} g(x,y,z)dxdydz - A + B - C\right|$$
$$\le \frac{(a-b)(d-c)(f-e)}{8}\int_0^1\int_0^1\int_0^1 \big|(1-2r)(1-2s)(1-2t)\big|$$
$$\times \left|\frac{\partial^3 g}{\partial t\partial s\partial r}(ta+(1-t)b, sc+(1-s)d, re+(1-r)f)\right| dtdsdr.$$

By using the well known power mean for triple integrals, then one has:

$$\left|\frac{1}{(b-a)(d-c)(f-e)}\iiint_{\Omega} g(x,y,z)dxdydz - A + B - C\right|$$
$$\le \frac{(b-a)(d-c)(f-e)}{8}\left(\int_0^1\int_0^1\int_0^1 \big|(1-2r)(1-2s)(1-2t)\big|\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\int_0^1\int_0^1 \big|(1-2r)(1-2s)(1-2t)\big|\right.$$
$$\left.\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(ta+(1-t)b, sc+(1-s)d, re+(1-r)f)\right|^q dtdsdr\right)^{\frac{1}{q}}.$$

Since $\left|\frac{\partial^3 g}{\partial t\partial s\partial r}\right|^q$ is convex function on the co-ordinates on Ω, for $t\in[0,1]$ we have

$$\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(ta+(1-t)b, sc+(1-s)d, re+(1-r)f)\right|^q$$
$$\le t\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a, sc+(1-s)d, re+(1-r)f)\right|^q$$
$$+ (1-t)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b, sc+(1-s)d, re+(1-r)f)\right|^q,$$

and

$$\begin{aligned}
&\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(ta+(1-t)b, sc+(1-s)d, re+(1-r)f)\right|^q \\
&\leq ts\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,re+(1-r)f)\right|^q + t(1-s)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,re+(1-r)f)\right|^q \\
&\quad + (1-t)s\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,re+(1-r)f)\right|^q \\
&\quad + (1-t)(1-s)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,re+(1-r)f)\right|^q,
\end{aligned}$$

and

$$\begin{aligned}
&\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(ta+(1-t)b, sc+(1-s)d, re+(1-r)f)\right|^q \\
&\leq tsr\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,e)\right|^q + ts(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,f)\right|^q \\
&\quad + tr(1-s)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,e)\right|^q + t(1-s)(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,f)\right|^q \\
&\quad + (1-t)sr\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,e)\right|^q + (1-t)(1-r)s\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,f)\right|^q \\
&\quad + (1-t)(1-s)r\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,e)\right|^q \\
&\quad + (1-t)(1-s)(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,f)\right|^q,
\end{aligned}$$

hence, it follows that

$$\begin{aligned}
&\left|\frac{1}{(b-a)(d-c)(f-e)}\iiint_\Omega g(x,y,z)dxdydz - A + B - C\right| \\
&\leq \frac{(a-b)(d-c)(f-e)}{8}\left(\frac{1}{8}\right)^{1-\frac{1}{q}}\Bigg(\int_0^1\int_0^1\int_0^1 \left|(1-2r)(1-2s)(1-2t)\right| \\
&\quad \times\Bigg\{tsr\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,e)\right|^q + ts(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,f)\right|^q \\
&\quad + tr(1-s)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,e)\right|^q + t(1-s)(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,f)\right|^q \\
&\quad + (1-t)sr\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,e)\right|^q + (1-t)(1-r)s\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,f)\right|^q \\
&\quad + (1-t)(1-s)r\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,e)\right|^q \\
&\quad + (1-t)(1-s)(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,f)\right|^q\Bigg\}dtdsdr\Bigg)^{\frac{1}{q}}.
\end{aligned}$$

By calculating the integral in the above inequality, we get

$$\begin{aligned}
&\int_0^1 \big|1-2t\big|\Bigg(tsr\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,e)\right|^q + ts(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,f)\right|^q \\
&tr(1-s)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,e)\right|^q + t(1-s)(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,f)\right|^q \\
&+(1-t)sr\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,e)\right|^q + (1-t)(1-r)s\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,f)\right|^q \\
&+(1-t)(1-s)r\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,e)\right|^q \\
&+(1-t)(1-s)(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,f)\right|^q\Bigg)dt \\
=&\int_0^{\frac{1}{2}} (1-2t)\Bigg(tsr\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,e)\right|^q + ts(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,f)\right|^q \\
&+tr(1-s)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,e)\right|^q + t(1-s)(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,f)\right|^q \\
&+(1-t)sr\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,e)\right|^q + (1-t)(1-r)s\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,f)\right|^q \\
&+(1-t)(1-s)r\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,e)\right|^q \\
&+(1-t)(1-s)(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,f)\right|^q\Bigg)dt \\
&+\int_{\frac{1}{2}}^1 (2t-1)\Bigg(tsr\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,e)\right|^q + ts(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,f)\right|^q \\
&+tr(1-s)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,e)\right|^q + t(1-s)(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,f)\right|^q \\
&+(1-t)sr\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,e)\right|^q + (1-t)(1-r)s\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,f)\right|^q \\
&+(1-t)(1-s)r\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,e)\right|^q \\
&+(1-t)(1-s)(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,f)\right|^q\Bigg)dt \\
=&\frac{sr\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,e)\right|^q}{24} + \frac{s(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,f)\right|^q}{24} \\
&+\frac{r(1-s)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,e)\right|^q}{24} + \frac{(1-s)(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,f)\right|^q}{24} \\
&+\frac{5sr\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,e)\right|^q}{24} + \frac{5s(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,f)\right|^q}{24} \\
&+\frac{5r(1-s)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,e)\right|^q}{24} + \frac{5(1-r)(1-s)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,f)\right|^q}{24}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{5sr\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,e)\right|^q}{24}+\frac{5s(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,f)\right|^q}{24}\\
&+\frac{5r(1-s)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,e)\right|^q}{24}+\frac{5(1-s)(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,f)\right|^q}{24}\\
&+\frac{sr\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,e)\right|^q}{24}+\frac{s(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,f)\right|^q}{24}\\
&+\frac{r(1-s)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,e)\right|^q}{24}+\frac{(1-r)(1-s)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,f)\right|^q}{24}\\
=&\frac{sr\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,e)\right|^q}{4}+\frac{s(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,f)\right|^q}{4}\\
&+\frac{r(1-s)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,e)\right|^q}{4}+\frac{(1-s)(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,f)\right|^q}{4}\\
&+\frac{sr\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,e)\right|^q}{4}+\frac{s(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,f)\right|^q}{4}\\
&+\frac{r(1-s)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,e)\right|^q}{4}+\frac{(1-r)(1-s)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,f)\right|^q}{4}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\left|\frac{1}{(b-a)(d-c)(f-e)}\iiint_\Omega g(x,y,z)dxdydz-A+B-C\right|\\
&\leq\frac{(b-a)(d-c)(f-e)}{32}\Bigg[\int_0^1\int_0^1\left|(1-2r)(1-2s)\right|\Bigg(sr\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,e)\right|^q\\
&+s(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,f)\right|^q+r(1-s)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,e)\right|^q\\
&+(1-s)(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,f)\right|^q+sr\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,e)\right|^q\\
&+s(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,f)\right|^q+r(1-s)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,e)\right|^q\\
&+(1-s)(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,f)\right|^q\Bigg)dsdr\Bigg]^{\frac{1}{q}}.
\end{aligned}
\tag{2.11}
$$

By a similar way for other integrals, we deduce that

$$
\begin{aligned}
&\int_0^1 \big|1-2s\big|\Bigg(sr\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,e)\right|^q + s(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,f)\right|^q\\
&r(1-s)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,e)\right|^q + (1-s)(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,f)\right|^q\\
&+sr\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,e)\right|^q + (1-r)s\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,f)\right|^q\\
&+(1-s)r\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,e)\right|^q + (1-s)(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,f)\right|^q\Bigg)ds\\
=&\int_0^{\frac{1}{2}} (1-2s)\Bigg(sr\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,e)\right|^q + s(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,f)\right|^q\\
&+r(1-s)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,e)\right|^q + (1-s)(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,f)\right|^q\\
&+sr\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,e)\right|^q + (1-r)s\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,f)\right|^q\\
&+(1-s)r\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,e)\right|^q + (1-s)(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,f)\right|^q\Bigg)ds\\
&+\int_{\frac{1}{2}}^1 (2s-1)\Bigg(sr\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,e)\right|^q + s(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,f)\right|^q\\
&+r(1-s)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,e)\right|^q + (1-s)(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,f)\right|^q\\
&+sr\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,e)\right|^q + (1-r)s\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,f)\right|^q\\
&+(1-s)r\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,e)\right|^q + (1-s)(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,f)\right|^q\Bigg)ds\\
=&\frac{r\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,e)\right|^q}{24} + \frac{(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,f)\right|^q}{24}\\
&+\frac{5r\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,e)\right|^q}{24} + \frac{5(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,f)\right|^q}{24}\\
&+\frac{r\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,e)\right|^q}{24} + \frac{(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,f)\right|^q}{24}\\
&+\frac{5r\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,e)\right|^q}{24} + \frac{5(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,f)\right|^q}{24}
\end{aligned}
\tag{2.12}
$$

$$
\begin{aligned}
&+\frac{5r\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,e)\right|^q}{24}+\frac{5(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,f)\right|^q}{24}\\
&+\frac{r\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,e)\right|^q}{24}+\frac{(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,f)\right|^q}{24}\\
&+\frac{5r\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,e)\right|^q}{24}+\frac{5(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,f)\right|^q}{24}\\
&+\frac{r\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,e)\right|^q}{24}+\frac{(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,f)\right|^q}{24}\\
=&\frac{r\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,e)\right|^q}{4}+\frac{(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,f)\right|^q}{4}\\
&+\frac{r\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,e)\right|^q}{4}+\frac{(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,f)\right|^q}{4}\\
&+\frac{r\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,e)\right|^q}{4}+\frac{(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,f)\right|^q}{4}\\
&+\frac{r\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,e)\right|^q}{4}+\frac{(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,f)\right|^q}{4}.
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
&\left|\frac{1}{(b-a)(d-c)(f-e)}\iiint_\Omega g(x,y,z)dxdydz-A+B-C\right|\\
&\leq\frac{(b-a)(d-c)(f-e)}{128}\\
&\times\left[\int_0^1|1-2r|\left(r\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,e)\right|^q+(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,f)\right|^q\right.\right.\\
&+r\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,e)\right|^q+(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,f)\right|^q\\
&+r\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,e)\right|^q+(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,f)\right|^q\\
&\left.\left.+r\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,e)\right|^q+(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,f)\right|^q\right)dr\right]^{\frac{1}{q}}.
\end{aligned}
\tag{2.13}
$$

A similar way for other integrals, implies that

$$\begin{aligned}
&\int_0^1 \big|1-2r\big|\Bigg(r\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,e)\right|^q+(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,f)\right|^q\\
&+r\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,e)\right|^q+(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,f)\right|^q\\
&+r\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,e)\right|^q+(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,f)\right|^q\\
&+r\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,e)\right|^q+(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,f)\right|^q\Bigg)dr\\
=&\int_0^{\frac{1}{2}} \big|1-2r\big|\Bigg(r\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,e)\right|^q+(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,f)\right|^q\\
&+r\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,e)\right|^q+(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,f)\right|^q\\
&+r\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,e)\right|^q+(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,f)\right|^q\\
&+r\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,e)\right|^q+(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,f)\right|^q\Bigg)dr\\
&+\int_{\frac{1}{2}}^1 \big|2r-1\big|\Bigg(r\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,e)\right|^q+(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,f)\right|^q\\
&+r\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,e)\right|^q+(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,f)\right|^q \qquad (2.14)\\
&+r\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,e)\right|^q+(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,f)\right|^q\\
&+r\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,e)\right|^q+(1-r)\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,f)\right|^q\Bigg)dr\\
=&\frac{\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,e)\right|^q}{24}+\frac{5\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,f)\right|^q}{24}\\
&+\frac{\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,e)\right|^q}{24}+\frac{5\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,f)\right|^q}{24}\\
&+\frac{\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,e)\right|^q}{24}+\frac{5\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,f)\right|^q}{24}\\
&+\frac{\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,e)\right|^q}{24}+\frac{5\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,f)\right|^q}{24}\\
&+\frac{5\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,e)\right|^q}{24}+\frac{\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,f)\right|^q}{24}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{5\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,e)\right|^q}{24}+\frac{\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,f)\right|^q}{24}\\
&+\frac{5\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,e)\right|^q}{24}+\frac{\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,f)\right|^q}{24}\\
&+\frac{5\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,e)\right|^q}{24}+\frac{\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,f)\right|^q}{24}\\
=&\frac{\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,e)\right|^q}{4}+\frac{\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,c,f)\right|^q}{4}\\
&+\frac{\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,e)\right|^q}{4}+\frac{\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(a,d,f)\right|^q}{4}\\
&+\frac{\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,e)\right|^q}{4}+\frac{\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,c,f)\right|^q}{4}\\
&+\frac{\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,e)\right|^q}{4}+\frac{\left|\frac{\partial^3 g}{\partial t\partial s\partial r}(b,d,f)\right|^q}{4}.
\end{aligned}
$$

By the (2.11), (2.12), (2.13) and (2.14), we get the inequality (2.10). □

Remark 2.1. Since $\frac{1}{8} < \frac{1}{(p+1)^{\frac{3}{p}}} < 1$, if $p > 1$, the estimation given in Theorem 2.3 is better than the one given in Theorem 2.2.

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