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SOME INTEGRAL INEQUALITIES THROUGH TEMPERED FRACTIONAL INTEGRAL OPERATOR

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ABSTRACT. In this article, we adopt the tempered fractional integral operators to develop some novel Minkowski and Hermite-Hadamard type integral inequalities. Thus, we give several special cases of the integral inequalities for tempered fractional integrals obtained in the earlier works.

1. INTRODUCTION

The theory of convexity plays a vital role in different fields of pure and applied sciences. Consequently, the classical concepts of convex sets and convex functions have been generalized in different directions. The concept of function is one of the basic structures of mathematics, and many researchers have focused on new function classes and have made efforts to classify the space of functions. One of important class of functions defined as a product of this intense effort is the convex function, which has applications in statistics, inequality theory, convex programming, and numerical analysis. This interesting class of functions is defined as follows (mentioned in ([6]).

Definition 1. *Let $\mathcal{H}$ be an interval in $\mathbb{R}$. Then $f : \mathcal{H} \to \mathbb{R}$ is said to be convex if*

$$f(\xi a + (1-\xi) b) \leq \xi f(a) + (1-\xi) f(b)$$

for all $a, b \in \mathcal{H}$ and $\xi \in [0, 1]$.

For more information, see the papers [1-5] and [22]- [24].
Another aspect due to which the convexity theory has attracted many researchers

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is its close relationship with theory of inequalities. Many famous inequalities can be obtained using the concept of convex functions. For details related to convexity, interested readers are referred to [6,7]. Among these inequalities, Hermite–Hadamard inequality, which provides us a necessary and sufficient condition for a convex function, is one of the most studied results. This result of Hermite and Hadamard reads as follows:

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval I of real numbers and $a,\ b\ \in\ I$, with $a < b$. The following double inequality holds:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f\left(a\right)+f\left(b\right)}{2} \tag{1}$$

This double inequality is known in the literature as the Hermite-Hadamard inequality for convex functions.

Definition 2. *([17-18]) Let $f \in \mathcal{L}^1(a,b)$. The Riemann Liouville integrals $I_{a^+}^{\alpha} f$ and $I_{b^-}^{\alpha} f$ of order $\alpha > 0$ with $a\ \geq\ 0$ are defined by*

$$I_{a^+}^{\alpha} f\left(x\right) = \frac{1}{\Gamma\left(\alpha\right)}\int_a^x f\left(\xi\right)\left(x-\xi\right)^{\alpha-1}d\xi, \qquad x >\ a \tag{2}$$

and

$$I_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma\left(\alpha\right)}\int_x^b f\left(\xi\right)\left(\xi-x\right)^{\alpha-1}d\xi, \qquad b >\ x \tag{3}$$

The tempered fractional integral was first studied by Buschman [8], but Liu et al. [9], Meerschaert et al. [10] and Fernandez et al. [12] have described the associated tempered fractional calculus more explicitly.

Definition 3. *([10]) Let $[a,b]$ be a real interval and $\zeta \geq 0,\ \alpha > 0$. Then, for a function $f \in \mathcal{L}^1[a,b]$, the left and right tempered fractional integral, respectively, defined by*

$${}_{a^+}\mathfrak{J}^{\alpha,\zeta} f\left(x\right) = \frac{1}{\Gamma(\alpha)}\int_a^x \left(x-\xi\right)^{\alpha-1} e^{-\zeta(x-\xi)} f\left(\xi\right)d\xi \tag{4}$$

and

$${}_{b^-}\mathfrak{J}^{\alpha,\zeta} f\left(x\right) = \frac{1}{\Gamma(\alpha)}\int_x^b \left(\xi-x\right)^{\alpha-1} e^{-\zeta(\xi-x)} f\left(\xi\right)d\xi, \tag{5}$$

where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^\infty t^{z-1}e^{-t}dt$.

For any $\zeta > 0$, the positive one-sided tempered fractional operator of a suitable function $f(x)$ can be given by;

$${}^{\tau}\mathfrak{J}_x^{\alpha,\zeta} f\left(x\right) = \frac{1}{\Gamma(\alpha)}\int_0^x \left(x-\xi\right)^{\alpha-1} e^{-\zeta(x-\xi)} f\left(\xi\right)d\xi.$$

Remark 1. *If we take $\zeta = 0$ in the equations (4) and (5), then we have the left and right R-L operators (2) and (3) respectively.*

First of all, we define the new incomplete Gamma function following definition as in [11].

Definition 4. *For the real numbers , $\alpha > 0$ and , $x, \zeta \geq 0$, we define the ζ-incomplete Gamma function by*

$$I_\alpha(\alpha, b) = \frac{1}{\Gamma(\alpha)} \int_0^b x^{\alpha-1} e^{-\zeta t} dx$$

If $\zeta = 1$, it reduces to the incomplete Gamma function

$$I_\alpha(\alpha, b) = \frac{1}{\Gamma(\alpha)} \int_0^b x^{\alpha-1} e^{-x} dx.$$

Remark 2. *For the real numbers $\alpha > 0$ and $x, \zeta \geq 0$, we have*

a. $I_{\zeta(b-a)}(\alpha, 1) = \int_0^1 x^{\alpha-1} e^{-\zeta(b-a)x} dx = \frac{1}{(b-a)^\alpha} \mathrm{I}_\alpha(\alpha, \mathrm{b-a})$
b. $\int_0^1 \mathrm{I}_{\alpha(\mathrm{b-a})}(\alpha, x)\, \mathrm{dx} = \frac{\mathrm{I}_\alpha(\alpha\ ,\mathrm{b-a})}{(b-a)^\alpha} - \frac{\mathrm{I}_\alpha(\alpha+1\ ,\mathrm{b-a})}{(b-a)^{\alpha+1}}$

Recently, Nisar et al. [13] established some inequalities via fractional conformable integral operators. In [14,15], various researchers established Minkowski inequalities involving fractional calculus with general analytic kernels and some novel estimations of Hadamard type inequalities for different kinds of convex functions via tempered fractional integral operator, the Hermite–Hadamard type inequalities for k-fractional conformable integrals are found in [16].
This paper is organized in the following way: In Section 2, the main results, the reverse Minkowski and related Hermite-Hadamard integral inequalities, are established using tempered fractional integral operators. The concluding remarks are given in Section 3.

2. Main Results

In this section, the reverse Minkowski and Hermite-Hadamard type integral inequalities are developed using the tempered integral operator.

Theorem 1. *Let $\zeta \geq 0$, $\alpha > 0$, $p \geq 1$ and let there be two positive functions f_1 and f_2 on $[0,\infty)$ such that for all $x >$ a, ${}^\tau\mathfrak{J}_x^{\alpha,\zeta} {f_1}^p(x) < \infty$, ${}^\tau\mathfrak{J}_x^{\alpha,\zeta} {f_2}^p(x) < \infty$. If $0 < \tau_1 \leq \frac{f_1(\xi)}{f_2(\xi)} \leq \tau_2$, holds for $\tau_1, \tau_2 \in \mathbb{R}^+$ and $\xi \in [0,x]$, then we have:*

$$\left({}^\tau\mathfrak{J}_x^{\alpha,\zeta} {f_1}^p(x)\right)^{\frac{1}{p}} + \left({}^\tau\mathfrak{J}_x^{\alpha,\zeta} {f_2}^p(x)\right)^{\frac{1}{p}} \leq \frac{1+\tau_2(\tau_1+2)}{(\tau_1+1)(\tau_2+1)} \left({}^\tau\mathfrak{J}_x^{\alpha,\zeta} (f_1+f_2)^p(x)\right)^{\frac{1}{p}}. \quad (6)$$

Proof. Under the given condition $\frac{f_1(\xi)}{f_2(\xi)}\leq\tau_2$, $\xi\in[0,x]$, it can be written as

$$(\tau_2+1)^p f_1{}^p(\xi) \leq \tau_2{}^p(f_1+f_2)^p(\xi). \tag{7}$$

Multiplying both sides of (7) by $\frac{(x-\xi)^{\alpha-1}}{\Gamma(\alpha)e^{\zeta(x-\xi)}}$, then integrating the resulting inequality with respect to ξ over $[0,x]$, we obtain,

$$\begin{aligned}(\tau_2+1)^p\frac{1}{\Gamma(\alpha)}\int_0^x(x-\xi)^{\alpha-1}e^{-\zeta(x-\xi)}f_1{}^p(\xi)\,d\xi\\ \leq \tau_2^p\frac{1}{\Gamma(\alpha)}\int_0^x(x-\xi)^{\alpha-1}e^{-\zeta(x-\xi)}(f_1+f_2)^p(\xi)d\xi.\end{aligned} \tag{8}$$

Consequently, we obtain

$$(\tau_2+1)^{p\,\tau}\mathfrak{J}_x^{\alpha,\zeta}f_1{}^p(x) \leq \tau_2{}^{p\,\tau}\mathfrak{J}_x^{\alpha,\zeta}(f_1+f_2)^p(x). \tag{9}$$

Hence, we can write

$$\left[{}^\tau\mathfrak{J}_x^{\alpha,\zeta}f_1{}^p(x)\right]^{\frac{1}{p}} \leq \frac{\tau_2}{(\tau_2+1)}\left[{}^\tau\mathfrak{J}_x^{\alpha,\zeta}(f_1+f_2)^p(x)\right]^{\frac{1}{p}}. \tag{10}$$

In contrast, as $\tau_1 f_2(\xi)\leq f_1(\xi)$, it follows that

$$\left(1+\frac{1}{\tau_1}\right)^p f_2{}^p(\xi) \leq \frac{1}{\tau_1{}^p}[f_1(\xi)+f_2(\xi)]^p. \tag{11}$$

Again, if we multiplying both sides of (11) by $\frac{(x-\xi)^{\alpha-1}}{\Gamma(\alpha)e^{\zeta\ (x-\xi)}}$, then integrating the resulting inequality with respect to ξ over $[0,x]$, we obtain,

$$\left[{}^\tau\mathfrak{J}_x^{\alpha,\zeta}f_2{}^p(x)\right]^{\frac{1}{p}} \leq \frac{1}{(\tau_1+1)}\left[{}^\tau\mathfrak{J}_x^{\alpha,\zeta}(f_1+f_2)^p(x)\right]^{\frac{1}{p}}. \tag{12}$$

Adding the inequalities (10) and (12) yields the desired inequality. □

Remark 3. *By setting Theorem 1 for $\alpha=1$, $\zeta=0$ and for an arbitrary choice of function, we obtain Theorem 1.2 in [20].*

Remark 4. *In Theorem 1, if we choose $\zeta=0$, we obtain Theorem 2.1 in [19].*

Inequality (6) is referred to as the reverse Minkowski inequality for the tempered fractional integral operator.

Theorem 2. *Let $\zeta \geq 0$, $\alpha > 0$, $p\geq1$ and let there be two positive functions f_1 and f_2 on $[0,\infty)$ such that for all $x>$ a, ${}^\tau\mathfrak{J}_x^{\alpha,\zeta}f_1{}^p(x)<\infty$, ${}^\tau\mathfrak{J}_x^{\alpha,\zeta}f_2{}^p(x)<\infty$. If $0<\tau_1\leq\frac{f_1(\xi)}{f_2(\xi)}\leq\tau_2$, holds for $\tau_1,\tau_2\in\mathbb{R}^+$ and $\xi\in[0,x]$, then we have:*

$$\begin{aligned}\left({}^\tau\mathfrak{J}_x^{\alpha,\zeta}f_1{}^p(x)\right)^{\frac{2}{p}}+\left({}^\tau\mathfrak{J}_x^{\alpha,\zeta}f_2{}^p(x)\right)^{\frac{2}{p}}\\ \geq\left(\frac{(1+\tau_2)(\tau_1+1)}{\tau_2}-2\right)\left[{}^\tau\mathfrak{J}_x^{\alpha,\zeta}f_1{}^p(x)\right]^{\frac{1}{p}}\left[{}^\tau\mathfrak{J}_x^{\alpha,\zeta}f_2{}^p(x)\right]^{\frac{1}{p}}.\end{aligned} \tag{13}$$

Proof. The product of inequalities (10) and (12) yields

$$\frac{(1+\tau_2)(\tau_1+1)}{\tau_2}\left[{}^{\tau}\mathfrak{J}_x^{\alpha,\zeta} f_1{}^p(x)\right]^{\frac{1}{p}}\left[{}^{\tau}\mathfrak{J}_x^{\alpha,\zeta} f_2{}^p(x)\right]^{\frac{1}{p}} \le \left[{}^{\tau}\mathfrak{J}_x^{\alpha,\zeta}(f_1+f_2)^p(x)\right]^{\frac{2}{p}}. \quad (14)$$

Now, utilizing the Minkowski inequality to the right hand side of (14), one obtains

$$\left({}^{\tau}\mathfrak{J}_x^{\alpha,\zeta}(f_1+f_2)^p(x)\right)^{\frac{2}{p}} \le \left(\left[{}^{\tau}\mathfrak{J}_x^{\alpha,\zeta} f_1{}^p(x)\right]^{\frac{1}{p}} + \left[{}^{\tau}\mathfrak{J}_x^{\alpha,\zeta} f_2{}^p(x)\right]^{\frac{1}{p}}\right)^2.$$

Then, we have

$$\begin{aligned}\left({}^{\tau}\mathfrak{J}_x^{\alpha,\zeta}(f_1+f_2)^p(x)\right)^{\frac{2}{p}} \le &\left[{}^{\tau}\mathfrak{J}_x^{\alpha,\zeta} f_1{}^p(x)\right]^{\frac{2}{p}} + \left[{}^{\tau}\mathfrak{J}_x^{\alpha,\zeta} f_2{}^p(x)\right]^{\frac{2}{p}} \\ &+ 2\left[{}^{\tau}\mathfrak{J}_x^{\alpha,\zeta} f_1{}^p(x)\right]\left[{}^{\tau}\mathfrak{J}_x^{\alpha,\zeta} f_2{}^p(x)\right].\end{aligned} \quad (15)$$

Thus, from the above inequalities, we obtain the inequality (13).

□

Remark 5. *By setting Theorem 2 for $\alpha = 1$, $\zeta = 0$ and for an arbitrary choice of function, we obtain Theorem 2.2 in [21].*

Remark 6. *In Theorem 2, if we choose $\zeta = 0$, we obtain Theorem 2.3 in [19].*

Lemma 1. *([19]) Let $\mathcal{G}$ be a concave function on $[a,\ b]$. Then the following double inequality holds:*

$$\mathcal{G}(a) + \mathcal{G}(b) \le \mathcal{G}(b+a-x) + \mathcal{G}(x) \le 2\mathcal{G}\left(\frac{a+b}{2}\right). \quad (16)$$

Theorem 3. *Let $\zeta \ge 0, \alpha > 0, p \ge 1$ and let there be two positive functions $\hbar$ and $\mathcal{L}$ on $[0,\infty)$. If $\hbar^p$ and $\mathcal{L}^q$ are two concave functions on $[0,\infty)$, then we have:*

$$\begin{aligned}&2^{-p-q}\big(\hbar(0)+\hbar(x)\big)^p\big(\mathcal{L}(0)+\mathcal{L}(x)\big)^q\left[{}^{\tau}\mathfrak{J}_x^{\alpha,\zeta}(x^{\alpha-1})\right]^2 \\ &\qquad\le {}^{\tau}\mathfrak{J}_x^{\alpha,\zeta}\left(x^{\alpha-1}\hbar^p(x)\right){}^{\tau}\mathfrak{J}_x^{\alpha,\zeta}\left(x^{\alpha-1}\mathcal{L}^q(x)\right).\end{aligned} \quad (17)$$

Proof. Since the $\hbar^p$ and $\mathcal{L}^q$ are two concave functions on [0,∞), then by Lemma 1, for any $\xi > 0$ we obtain,

$$\hbar^p(0) + \hbar^p(x) \le \hbar^p(x-\xi) + \hbar^p(\xi) \le 2\hbar^p\left(\frac{x}{2}\right), \quad (18)$$

and

$$\mathcal{L}^q(0) + \mathcal{L}^q(x) \le \mathcal{L}^q(x-\xi) + \mathcal{L}^q(\xi) \le 2\mathcal{L}^q\left(\frac{x}{2}\right). \quad (19)$$

Multiplying both sides of (18) and (19) by $\frac{(x-\xi)^{\alpha-1}\xi^{\alpha-1}}{\Gamma(\alpha)e^{\zeta\ (x-\xi)}}$, then integrating the resulting inequality with respect to ξ over $[0,x]$, we obtain,

$$\begin{aligned}\frac{\hbar^p(0)+\hbar^p(x)}{\Gamma(\alpha)}\int_0^x \frac{(x-\xi)^{\alpha-1}e^{-\zeta(x-\xi)}}{\xi^{1-\alpha}}d\xi \\ \leq \frac{1}{\Gamma(\alpha)}\int_0^x \frac{(x-\xi)^{\alpha-1}e^{-\zeta(x-\xi)}}{\xi^{1-\alpha}}\hbar^p(x-\xi)\,d\xi \\ +\frac{1}{\Gamma(\alpha)}\int_0^x \frac{(x-\xi)^{\alpha-1}e^{-\zeta(x-\xi)}}{\xi^{1-\alpha}}\hbar^p(\xi)\,d\xi \\ \leq \frac{2\hbar^p\left(\frac{x}{2}\right)}{\Gamma(\alpha)}\int_0^x \frac{(x-\xi)^{\alpha-1}e^{-\zeta(x-\xi)}}{\xi^{1-\alpha}}d\xi,\end{aligned} \tag{20}$$

and

$$\begin{aligned}\frac{\mathcal{L}^q(0)+\mathcal{L}^q(x)}{\Gamma(\alpha)}\int_0^x \frac{(x-\xi)^{\alpha-1}e^{-\zeta(x-\xi)}}{\xi^{1-\alpha}}d\xi \\ \leq \frac{1}{\Gamma(\alpha)}\int_0^x \frac{(x-\xi)^{\alpha-1}e^{-\zeta(x-\xi)}}{\xi^{1-\alpha}}\mathcal{L}^q(x-\xi)\,d\xi \\ +\frac{1}{\Gamma(\alpha)}\int_0^x \frac{(x-\xi)^{\alpha-1}e^{-\zeta(x-\xi)}}{\xi^{1-\alpha}}\mathcal{L}^q(\xi)\,d\xi \\ \leq \frac{2\mathcal{L}^q\left(\frac{x}{2}\right)}{\Gamma(\alpha)}\int_0^x \frac{(x-\xi)^{\alpha-1}e^{-\zeta(x-\xi)}}{\xi^{1-\alpha}}d\xi.\end{aligned} \tag{21}$$

Using the change of variables $x-\xi=y$, we have

$$\frac{1}{\Gamma(\alpha)}\int_0^x \frac{(x-\xi)^{\alpha-1}e^{-\zeta(x-\xi)}}{\xi^{1-\alpha}}\hbar^p(x-\xi)\,d\xi = {}^{\tau}\mathfrak{J}_x^{\alpha,\zeta}\left(x^{\alpha-1}\hbar^p(x)\right), \tag{22}$$

and

$$\frac{1}{\Gamma(\alpha)}\int_0^x \frac{(x-\xi)^{\alpha-1}e^{-\zeta(x-\xi)}}{\xi^{1-\alpha}}\mathcal{L}^q(x-\xi)\,d\xi = {}^{\tau}\mathfrak{J}_x^{\alpha,\zeta}\left(x^{\alpha-1}\mathcal{L}^q(x)\right). \tag{23}$$

Thus, by using (20) and (22) yields,

$$\begin{aligned}\hbar^p(0)+\hbar^p(x)\left({}^{\tau}\mathfrak{J}_x^{\alpha,\zeta}\left(x^{\alpha-1}\right)\right) &\leq 2\,{}^{\tau}\mathfrak{J}_x^{\alpha,\zeta}\left(x^{\alpha-1}\hbar^p(x)\right) \\ &\leq \hbar^p\left(\frac{x}{2}\right)\left({}^{\tau}\mathfrak{J}_x^{\alpha,\zeta}\left(x^{\alpha-1}\right)\right),\end{aligned} \tag{24}$$

Similarly, the use of (21) and (23) yields,

$$\begin{aligned}\hbar^q(0)+\mathcal{L}^q(x)\left({}^{\tau}\mathfrak{J}_x^{\alpha,\zeta}\left(x^{\alpha-1}\right)\right) &\leq 2\,{}^{\tau}\mathfrak{J}_x^{\alpha,\zeta}\left(x^{\alpha-1}\mathcal{L}^q(x)\right) \\ &\leq \mathcal{L}^q\left(\frac{x}{2}\right)\left({}^{\tau}\mathfrak{J}_x^{\alpha,\zeta}\left(x^{\alpha-1}\right)\right).\end{aligned} \tag{25}$$

The inequalities (24) and (25) imply that

$$\begin{aligned}(\hbar^p(0)+\hbar^p(x))(\mathcal{L}^q(0)+\mathcal{L}^q(x))\left({}^{\tau}\mathfrak{J}_x^{\alpha,\zeta}\left(x^{\alpha-1}\right)\right)^2 \\ \leq 4\,{}^{\tau}\mathfrak{J}_x^{\alpha,\zeta}\left(x^{\alpha-1}\hbar^p(x)\right){}^{\tau}\mathfrak{J}_x^{\alpha,\zeta}\left(x^{\alpha-1}\mathcal{L}^q(x)\right).\end{aligned} \tag{26}$$

Since $\hbar$ and $\mathcal{L}$ are positive functions, therefore for any $x > 0, p \geq 1$, and $q \geq 1$, we have

$$\left(\frac{\hbar^p(0)+\hbar^p(x)}{2}\right)^{\frac{1}{p}} \geq 2^{-1}(\hbar(0)+\hbar(x)),$$

and

$$\left(\frac{\mathcal{L}^q(0)+\hbar^p(x)}{2}\right)^{\frac{1}{p}} \geq 2^{-1}(\mathcal{L}(0)+\mathcal{L}(x)).$$

Hence, it follows that

$$\frac{(\hbar^p(0)+\hbar^p(x))}{2}{}^{\tau}\mathfrak{J}_x^{\alpha,\zeta}\left(x^{\alpha-1}\right) \geq 2^{-p}(\hbar(0)+\hbar(x))^{p\,\tau}\mathfrak{J}_x^{\alpha,\zeta}\left(x^{\alpha-1}\right), \tag{27}$$

$$\frac{(\mathcal{L}^q(0)+\mathcal{L}^q(x))}{2}{}^{\tau}\mathfrak{J}_x^{\alpha,\zeta}\left(x^{\alpha-1}\right) \geq 2^{-q}(\mathcal{L}(0)+\mathcal{L}(x))^{q\,\tau}\mathfrak{J}_x^{\alpha,\zeta}\left(x^{\alpha-1}\right). \tag{28}$$

The inequalities (27) and (28) imply

$$\begin{aligned}\frac{1}{4}(\hbar^p(0)+\hbar^p(x))(\mathcal{L}^q(0)+\mathcal{L}^q(x))\left[{}^{\tau}\mathfrak{J}_x^{\alpha,\zeta}\left(x^{\alpha-1}\right)\right]^2 \\ \geq 2^{-p-q}(\hbar(0)+\hbar(x))^p(\mathcal{L}(0)+\mathcal{L}(x))^q\left[{}^{\tau}\mathfrak{J}_x^{\alpha,\zeta}\left(x^{\alpha-1}\right)\right]^2.\end{aligned} \tag{29}$$

Thus, by combining (21) and (24), we get the desired result. □

Remark 7. *By considering Theorem 3, for $\alpha = 1$, $\zeta = 0$ and for an arbitrary choice of function, we obtain Theorem 2.3 in [21].*

Remark 8. *In Theorem 3, if we choose $\zeta = 0$, we obtain Theorem 2.5 in [19].*

Theorem 4. *Let $\zeta \geq 0$, $\alpha, \beta > 0$, $p \geq 1$ and let there be two positive functions $\hbar$ and $\mathcal{L}$ on $[0,\infty)$. If $\hbar^p$ and $\mathcal{L}^q$ are two concave functions on $[0,\infty)$, then we have:*

$$\begin{aligned}2^{2-p-q}(\hbar(0)+\hbar(x))^p(\mathcal{L}(0)+\mathcal{L}(x))^q\left[{}^{\tau}\mathfrak{J}_x^{\alpha,\zeta}\left(x^{\beta-1}\right)\right]^2 \\ \leq \left[\frac{\Gamma(\beta)}{\Gamma(\alpha)}{}^{\beta\tau}\mathfrak{J}_x^{\alpha,\zeta}\left(x^{\alpha-1}\hbar^p(x)\right)+{}^{\tau}\mathfrak{J}_x^{\alpha,\zeta}\left(x^{\beta-1}\hbar^p(x)\right)\right] \\ \times\left[\frac{\Gamma(\beta)}{\Gamma(\alpha)}{}^{\beta\tau}\mathfrak{J}_x^{\alpha,\zeta}\left(x^{\alpha-1}\mathcal{L}^q(x)\right)+{}^{\tau}\mathfrak{J}_x^{\alpha,\zeta}\left(x^{\beta-1}\mathcal{L}^q(x)\right)\right].\end{aligned} \tag{30}$$

Proof. Multiplying both sides of (18) and (19) by $\frac{(x-\xi)^{\alpha-1}\xi^{\beta\alpha-1}}{\Gamma(\alpha)e^{\zeta\ (x-\xi)}}$, then integrating the resulting inequality with respect to ξ over $[0,x]$, we obtain

$$\begin{aligned}\frac{\hbar^{p}\left(0\right)+\hbar^{p}\left(x\right)}{\Gamma(\alpha)}&\int_{0}^{x}\frac{\left(x-\xi\right)^{\alpha-1}e^{-\zeta(x-\xi)}}{\xi^{1-\beta\alpha}}d\xi\\&\leq\frac{1}{\Gamma(\alpha)}\int_{0}^{x}\frac{\left(x-\xi\right)^{\alpha-1}e^{-\zeta(x-\xi)}}{\xi^{1-\beta\alpha}}\hbar^{p}\left(x-\xi\right)d\xi\\&\quad+\frac{1}{\Gamma(\alpha)}\int_{0}^{x}\frac{\left(x-\xi\right)^{\alpha-1}e^{-\zeta(x-\xi)}}{\xi^{1-\beta\alpha}}\hbar^{p}\left(\xi\right)d\xi\\&\leq\frac{2\hbar^{p}\left(\frac{x}{2}\right)}{\Gamma(\alpha)}\int_{0}^{x}\frac{\left(x-\xi\right)^{\alpha-1}e^{-\zeta(x-\xi)}}{\xi^{1-\beta\alpha}}d\xi,\end{aligned}\tag{31}$$

and

$$\begin{aligned}\frac{\mathcal{L}^{q}\left(0\right)+\mathcal{L}^{q}\left(x\right)}{\Gamma(\alpha)}&\int_{0}^{x}\frac{\left(x-\xi\right)^{\alpha-1}e^{-\zeta(x-\xi)}}{\xi^{1-\beta\alpha}}d\xi\\&\leq\frac{1}{\Gamma(\alpha)}\int_{0}^{x}\frac{\left(x-\xi\right)^{\alpha-1}e^{-\zeta(x-\xi)}}{\xi^{1-\beta\alpha}}\mathcal{L}^{q}\left(x-\xi\right)d\xi\\&\quad+\frac{1}{\Gamma(\alpha)}\int_{0}^{x}\frac{\left(x-\xi\right)^{\alpha-1}e^{-\zeta(x-\xi)}}{\xi^{1-\beta\alpha}}\mathcal{L}^{q}\left(\xi\right)d\xi\\&\leq\frac{2\mathcal{L}^{q}\left(\frac{x}{2}\right)}{\Gamma(\alpha)}\int_{0}^{x}\frac{\left(x-\xi\right)^{\alpha-1}e^{-\zeta(x-\xi)}}{\xi^{1-\beta\alpha}}d\xi.\end{aligned}\tag{32}$$

Using the change of variables $x-\xi=y$, we have

$$\begin{aligned}\frac{\Gamma(\beta)}{\Gamma(\beta)\Gamma(\alpha)}\int_{0}^{x}\frac{\left(x-\xi\right)^{\alpha-1}e^{-\zeta(x-\xi)}}{\xi^{1-\beta\alpha}}&\hbar^{p}\left(x-\xi\right)d\xi\\&=\frac{\Gamma(\beta)}{\Gamma(\alpha)}{}^{\beta\tau}\mathfrak{J}_{x}^{\alpha,\zeta}\left(x^{\alpha-1}\hbar^{p}\left(x\right)\right),\end{aligned}\tag{33}$$

and

$$\begin{aligned}\frac{\Gamma(\beta)}{\Gamma(\beta)\Gamma(\alpha)}\int_{0}^{x}\frac{\left(x-\xi\right)^{\alpha-1}e^{-\zeta(x-\xi)}}{\xi^{1-\beta\alpha}}&\mathcal{L}^{q}\left(x-\xi\right)d\xi\\&=\frac{\Gamma(\beta)}{\Gamma(\alpha)}{}^{\beta\tau}\mathfrak{J}_{x}^{\alpha,\zeta}\left(x^{\alpha-1}\mathcal{L}^{q}\left(x\right)\right).\end{aligned}\tag{34}$$

Thus, from (31) and (33), we write

$$\begin{aligned}\left(\hbar^{p}(0)+\hbar^{p}(x)\right){}^{\tau}\mathfrak{J}_{x}^{\alpha\beta,\zeta}\left(x^{\alpha-1}\right)&\\ &\leq\frac{\Gamma(\beta)}{\Gamma(\alpha)}{}^{\beta\tau}\mathfrak{J}_{x}^{\alpha,\zeta}\left(x^{\alpha-1}\hbar^{p}(x)\right)+{}^{\tau}\mathfrak{J}_{x}^{\alpha,\zeta}\left(x^{\beta-1}\hbar^{p}(x)\right)\\ &\leq 2\hbar^{p}\left(\frac{x}{2}\right){}^{\tau}\mathfrak{J}_{x}^{\alpha,\zeta}\left(x^{\beta-1}\right),\end{aligned}\tag{35}$$

and with (32) and (34), we can write,

$$\begin{aligned}\left(\mathcal{L}^{q}(0)+\mathcal{L}^{q}(x)\right){}^{\tau}\mathfrak{J}_{x}^{\alpha\beta,\zeta}\left(x^{\alpha-1}\right)&\\ &\leq\frac{\Gamma(\beta)}{\Gamma(\alpha)}{}^{\beta\tau}\mathfrak{J}_{x}^{\alpha,\zeta}\left(x^{\alpha-1}\mathcal{L}^{q}(x)\right)+{}^{\tau}\mathfrak{J}_{x}^{\alpha,\zeta}\left(x^{\beta-1}\mathcal{L}^{q}(x)\right)\\ &\leq 2\mathcal{L}^{q}\left(\frac{x}{2}\right){}^{\tau}\mathfrak{J}_{x}^{\alpha,\zeta}\left(x^{\beta-1}\right).\end{aligned}\tag{36}$$

From (30) and (31), it follows that

$$\begin{aligned}\left(\hbar^{p}(0)+\hbar^{p}(x)\right)\left(\mathcal{L}^{q}(0)+\mathcal{L}^{q}(x)\right)\left[{}^{\tau}\mathfrak{J}_{x}^{\alpha,\zeta}\left(x^{\beta-1}\right)\right]^{2}&\\ &\leq\left[\frac{\Gamma(\beta)}{\Gamma(\alpha)}{}^{\beta\tau}\mathfrak{J}_{x}^{\alpha,\zeta}\left(x^{\alpha-1}\hbar^{p}(x)\right)+{}^{\tau}\mathfrak{J}_{x}^{\alpha,\zeta}\left(x^{\beta-1}\hbar^{p}(x)\right)\right]\\ &\times\left[\frac{\Gamma(\beta)}{\Gamma(\alpha)}{}^{\beta\tau}\mathfrak{J}_{x}^{\alpha,\zeta}\left(x^{\alpha-1}\mathcal{L}^{q}(x)\right)+{}^{\tau}\mathfrak{J}_{x}^{\alpha,\zeta}\left(x^{\beta-1}\mathcal{L}^{q}(x)\right)\right].\end{aligned}\tag{37}$$

Since $\hbar$ and $\mathcal{L}$ are positive functions, therefore for any $x>0$, $p\geq 1$, and $q\geq 1$, we have

$$\frac{\left(\hbar^{p}(0)+\hbar^{p}(x)\right)}{2}{}^{\tau}\mathfrak{J}_{x}^{\alpha,\zeta}\left(x^{\beta-1}\right)\geq 2^{-p}(\hbar(0)+\hbar(x))^{p\tau}\mathfrak{J}_{x}^{\alpha,\zeta}\left(x^{\beta-1}\right),\tag{38}$$

and

$$\frac{\left(\mathcal{L}^{q}(0)+\mathcal{L}^{q}(x)\right)}{2}{}^{\tau}\mathfrak{J}_{x}^{\alpha,\zeta}\left(x^{\beta-1}\right)\geq 2^{-q}(\mathcal{L}(0)+\mathcal{L}(x))^{q\tau}\mathfrak{J}_{x}^{\alpha,\zeta}\left(x^{\beta-1}\right).\tag{39}$$

Thus from (38) and (39) it follows that

$$\begin{aligned}\frac{1}{4}\left(\hbar^{p}(0)+\hbar^{p}(x)\right)\left(\mathcal{L}^{q}(0)+\mathcal{L}^{q}(x)\right)\left[{}^{\tau}\mathfrak{J}_{x}^{\alpha,\zeta}\left(x^{\beta-1}\right)\right]^{2}&\\ &\geq 2^{-p-q}(\hbar(0)+\hbar(x))^{p}(\mathcal{L}(0)+\mathcal{L}(x))^{q}\left[{}^{\tau}\mathfrak{J}_{x}^{\alpha,\zeta}\left(x^{\beta-1}\right)\right]^{2}.\end{aligned}\tag{40}$$

Combining inequalities (37) and (40), we get the desired proof.

□

Remark 9. *By considering Theorem 4 for $\alpha=1$, $\zeta=0$ and for an arbitrary choice of function, we obtain Theorem 2.4 in [21].*

Remark 10. *In Theorem 4, if we choose , $\zeta=0$, we obtain Theorem 2.8 in [19].*

Remark 11. *In Theorem 4, if we choose $\alpha=\beta$, we obtain Theorem 2.4.*

3. Conclusion

The Minkowski and Hermite-Hadamard inequalities for the tempered fractional integral operator have been newly established in this paper. Not only do we prove that the results obtained are mathematically more valuable, but similar inequalities can also be constructed, for example with the help of the incomplete Gamma function used in Remark 2. We hope that our results can stimulate further research in various fields of pure and applied science.

Author Contribution Statements Erdal Gül and Abdüllatif Yalçın have contributed to the establishment of the problem and investigation. Also, Erdal Gül has supervised the editing and conceptualization processes. All the authors have read and approved the final form of the manuscript.

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