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Research article

Statistical Structures with Ricci and Hessian Metrics and Gradient Solitons

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(Dedicated to Professor Bang-Yen CHEN on the occasion of his 80th birthday)

Abstract

We put into light some properties of statistical structures with Ricci and Hessian metrics and provide some examples, relating them to Miao–Tam and Fischer–Marsden equations, and to gradient solitons.

Keywords: *statistical structure, gradient soliton, affine connection, curvature.*
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1. Introduction

It is known that statistical structures constitute a link between information geometry and the general theory of affine connections. Originating from probability theory, their study from the differential geometry point of view has increased. A statistical manifold was originally defined as a differentiable manifold whose points are probability distributions [1, 2, 23, 25]. The geometry of statistical manifolds has been lately applied to different fields of information science, information theory, neural networks, machine learning, image processing, statistical mechanics etc. We recall that a statistical model in information geometry admits as Riemannian metric the Fisher–Rao metric and is equipped with an affine connection without torsion built from the expectation of the probability distribution [2].

In the last two decades, special attention has been paid to the study of statistical manifolds endowed with remarkable geometric structures, as well as their statistical submersions (see, e.g., [14, 18, 19, 21, 22, 31, 32, 36, 37, 39, 40]). In particular, the study of statistical submanifolds in such manifolds is a topic of high interest [4, 5, 6, 11, 12, 18, 24, 27, 28, 29, 30, 33, 41]. Moreover, statistical solitons were recently investigated in [8, 9, 34].

We recall that a *statistical structure* [1] on a smooth manifold M is a pair (g, ∇) of a pseudo-Riemannian metric g and a torsion-free affine connection ∇ such that the tensor field ∇g is totally symmetric, i.e.,

$$(\nabla_X g)(Y, Z) = (\nabla_Y g)(X, Z)\Big(= (\nabla_Z g)(X, Y)\Big)$$

for any vector fields $X, Y, Z \in \Gamma^\infty(TM)$.

In this setting, we shall consider in this article statistical structures having two types of metrics, more precisely, the Ricci tensor field and the Hessian operator. We are going to derive some basic properties and provide several examples. Since these tensor fields and the pseudo-Riemannian metric are connected by the gradient Ricci soliton equation, we will also bring that into discussion.

In all the rest of the paper, we shall consider (M, g) a pseudo-Riemannian manifold (unless something else is explicitly stated) and we will denote by ∇^g its Levi-Civita connection.

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2. Statistical structures with Ricci and Hessian metrics

For an affine connection ∇ and a pseudo-Riemannian metric g on a smooth manifold M, we shall characterize the statistical pairs with the metric either the Ricci tensor or the Hessian operator.

If the Ricci tensor field Ric^{∇} of (g, ∇) is symmetric and non-degenerate, it becomes a pseudo-Riemannian metric and we shall further provide the necessary and sufficient condition for $(\mathrm{Ric}^{\nabla}, \nabla)$ to be a statistical structure whenever (g, ∇) is a statistical structure. If we denote by $Q =: Q^{\nabla}$ the Ricci operator defined by $g(QX, Y) := \mathrm{Ric}^{\nabla}(X, Y)$ for $X, Y \in \Gamma^{\infty}(TM)$, from a direct computation, we obtain

$$(\nabla_X \mathrm{Ric}^{\nabla})(Y, Z) = (\nabla_X g)(QY, Z) + g((\nabla_X Q)Y, Z)$$

which implies, by the symmetry of Ric^{∇} and ∇g, that

$$\begin{aligned}(\nabla_X \mathrm{Ric}^{\nabla})(Y, Z) - (\nabla_Y \mathrm{Ric}^{\nabla})(X, Z) &= (\nabla_X g)(QY, Z) - (\nabla_Y g)(QX, Z) + g((\nabla_X Q)Y, Z) - g((\nabla_Y Q)X, Z) \\ &= (\nabla_X g)(Y, QZ) - (\nabla_Y g)(X, QZ) + g\big((\nabla_X Q)Y - (\nabla_Y Q)X, Z\big),\end{aligned}$$

and we can state

Theorem 2.1. *Let ∇ be a torsion-free affine connection on (M, g) such that the Ricci tensor field Ric^{∇} of (g, ∇) is non-degenerate and symmetric and the Ricci operator Q of (g, ∇) is a ∇-Codazzi tensor field $\big($i.e., $(\nabla_X Q)Y = (\nabla_Y Q)X$ for any $X, Y \in \Gamma^{\infty}(TM)\big)$. Then $(\mathrm{Ric}^{\nabla}, \nabla)$ is a statistical structure if and only if (g, ∇) is a statistical structure.*

Example 2.1. If ∇ is a torsion-free affine connection on an Einstein manifold (M, g), then $(\mathrm{Ric}^{\nabla^g}, \nabla)$ is a statistical structure if and only if (g, ∇) is a statistical structure.

Taking into account that (g, ∇^g) is a statistical structure for any pseudo-Riemannian metric g, we illustrate the previous result by the following example.

Example 2.2. If $(M, \nabla_g f, -\kappa f, g)$ is a gradient almost Ricci soliton (we refer to [38] for its definition) with the Riemannian metric of constant sectional curvature κ and non-degenerate Ricci tensor field Ric^{∇^g}, then $(\mathrm{Ric}^{\nabla^g}, \nabla^g)$ is a statistical structure on M. Indeed, it follows from the fact that, in this case, the tensor field $\mathrm{Hess}^{(g,\nabla^g)}(f) + \kappa f \cdot g (= -\mathrm{Ric}^{\nabla^g})$ is a ∇^g-Codazzi tensor field.

Example 2.3. Another example of statistical structure $(\mathrm{Ric}^{\nabla^g}, \nabla^g)$ arises on any Ricci symmetric manifold (M, g) (i.e., satisfying $\nabla^g \mathrm{Ric}^{\nabla^g} = 0$) with non-degenerate Ricci tensor, in particular on any Einstein manifold. And more general, if $\nabla^g \mathrm{Ric}^{\nabla^g} = \Omega \otimes \eta$, with Ω a symmetric $(0, 2)$-tensor field and η a 1-form, then $(\mathrm{Ric}^{\nabla^g}, \nabla^g)$ is a statistical structure.

We recall that a Riemannian manifold of dimension > 2 is called a *quasi-Einstein manifold* [10] if there exist $\lambda, \mu \in \mathbb{R}$ and a 1-form η such that

$$\mathrm{Ric}^{\nabla^g} = \lambda g + \mu \eta \otimes \eta, \tag{2.1}$$

respectively, an *Einstein manifold* [7] if the Ricci curvature tensor field is a multiple of the metric, $\mathrm{Ric}^{\nabla^g} = \lambda g$ with $\lambda \in \mathbb{R}$.

Proposition 2.1. *Let (M, g) be a quasi-Einstein manifold satisfying (2.1). If η is a ∇^g-Codazzi tensor field, then $(\mathrm{Ric}^{\nabla^g}, \nabla^g)$ is a statistical structure on M if and only if M is an Einstein manifold, or the g-dual vector field of η is ∇^g-parallel.*

Proof. For any $X, Y, Z \in \Gamma^{\infty}(TM)$, we have

$$\begin{aligned}(\nabla^g_X \mathrm{Ric}^{\nabla^g})(Y, Z) - (\nabla^g_Y \mathrm{Ric}^{\nabla^g})(X, Z) &= \mu\big((\nabla^g_X(\eta \otimes \eta))(Y, Z) - (\nabla^g_Y(\eta \otimes \eta))(X, Z)\big) \\ &= \mu\big(\eta(Y)(\nabla_X \eta)Z + \eta(Z)(\nabla_X \eta)Y - \eta(X)(\nabla_Y \eta)Z - \eta(Z)(\nabla_Y \eta)X\big) \\ &= \mu\big(\eta(Y)(\nabla_X \eta)Z - \eta(X)(\nabla_Y \eta)Z\big),\end{aligned}$$

hence, $(\nabla^g_X \mathrm{Ric}^{\nabla^g})(Y, Z) - (\nabla^g_Y \mathrm{Ric}^{\nabla^g})(X, Z) = 0$ for any $X, Y, Z \in \Gamma^{\infty}(TM)$ if and only if $\mu = 0$ or

$$\eta(Y)(\nabla_X \eta)Z - \eta(X)(\nabla_Y \eta)Z = 0$$

for any $X, Y, Z \in \Gamma^\infty(TM)$. Let ξ be the g-dual vector field of η. Then

$$\begin{aligned}\eta(Y)(\nabla_X\eta)Z - \eta(X)(\nabla_Y\eta)Z &= \eta(Y)(\nabla_Z\eta)X - \eta(X)(\nabla_Z\eta)Y \\ &= \eta(Y)g(\nabla^g_Z\xi, X) - \eta(X)g(\nabla^g_Z\xi, Y) \\ &= g(\nabla^g_Z\xi, \eta(Y)X - \eta(X)Y)\end{aligned}$$

for any $X, Y, Z \in \Gamma^\infty(TM)$, and we get the conclusion. □

Example 2.4. If (M, g) is a quasi-Einstein Vaisman manifold (we refer to [16] for its definition and to [35] for explicit examples) satisfying $\mathrm{Ric}^{\nabla^g} = \lambda g + \mu\eta\otimes\eta$ with η the Lee 1-form, then $(\mathrm{Ric}^{\nabla^g}, \nabla^g)$ is a statistical structure on M.

For a connection $\nabla := \nabla^g + S$, with S a $(0, 2)$-tensor field on M, we shall further provide conditions on S such that the couples (g, ∇), $(\mathrm{Ric}^{\nabla^g}, \nabla)$, and $(\mathrm{Ric}^{\nabla}, \nabla)$ to be statistical structures, giving, in each case, concrete examples.

By direct computations we get

Proposition 2.2. *Let $\nabla := \nabla^g + S$ be an affine connection on (M, g), where S is a $(1, 2)$-tensor field on M. Then*

$$\begin{aligned}T^{\nabla}(X, Y) &= S(X, Y) - S(Y, X),\\ (\nabla_X g)(Y, Z) &= -g(S(X, Y), Z) - g(S(X, Z), Y)\end{aligned}$$

for any $X, Y, Z \in \Gamma^\infty(TM)$, therefore, (g, ∇) is a statistical structure if and only if

$$\begin{aligned}S(X, Y) &= S(Y, X),\\ g(S(X, Y), Z) &= g(S(X, Z), Y)\end{aligned}$$

for any $X, Y, Z \in \Gamma^\infty(TM)$.

Since

$$\nabla_X\nabla_Y Z = \nabla^g_X\nabla^g_Y Z + \nabla^g_X S(Y, Z) + S(X, \nabla^g_Y Z) + S(X, S(Y, Z))$$

we find that

Proposition 2.3. *The Riemann curvature of the statistical structure $(g, \nabla := \nabla^g + S)$ defined in Proposition 2.2 is given by*

$$R^{\nabla}(X, Y)Z = R^{\nabla^g}(X, Y)Z + (\nabla^g_X S)(Y, Z) - (\nabla^g_Y S)(X, Z) + S(X, S(Y, Z)) - S(Y, S(X, Z))$$

for any $X, Y, Z \in \Gamma^\infty(TM)$.

We can further deduce

Corollary 2.1. *Let $(g, \nabla := \nabla^g + S)$ be the statistical structure defined in Proposition 2.2. If $S(S(X, Y), Z) = S(S(Z, Y), X)$ for any $X, Y, Z \in \Gamma^\infty(TM)$, then S is a ∇^g-Codazzi tensor field if and only if $R^{\nabla} = R^{\nabla^g}$.*

Taking into account that the Ricci tensor field of a Riemannian metric with harmonic curvature is a Codazzi tensor field, and considering the above properties, we give the following examples of statistical structures with Ricci metric.

Example 2.5. If (M, g) is a Riemannian manifold with harmonic curvature tensor and the Ricci tensor field Ric^{∇^g} is non-degenerate, then $(\mathrm{Ric}^{\nabla^g}, \nabla^g)$ is a statistical structure.

Moreover

$$(\nabla_X \mathrm{Ric}^{\nabla^g})(Y, Z) = (\nabla^g_X \mathrm{Ric}^{\nabla^g})(Y, Z) - g(S(X, Y), QZ) - g(QY, S(X, Z)),$$

where $g(QX, Y) := \mathrm{Ric}^{\nabla^g}(X, Y)$, and we can state

Proposition 2.4. *If (M, g) is a Riemannian manifold with harmonic curvature tensor and the Ricci tensor field Ric^{∇^g} is non-degenerate, then $(\mathrm{Ric}^{\nabla^g}, \nabla := \nabla^g + S)$ is a statistical structure, where S is a symmetric $(1,2)$-tensor field, satisfying*

$$g(S(X,Y),Z) = g(S(X,Z),Y),$$
$$S(X,QY) = S(QX,Y)$$

for any $X, Y, Z \in \Gamma^\infty(TM)$, and $g(QX,Y) := \mathrm{Ric}^{\nabla^g}(X,Y)$.

Proof. Remark that the conditions upon S imply

$$\begin{aligned} g(QY, S(X,Z)) &= g(Z, S(X,QY)) = g(Z, S(QX,Y)) \\ &= g(Z, S(Y,QX)) = g(QX, S(Y,Z)) \end{aligned}$$

and we obtain the conclusion. □

For a suitable choice of S, we give the following example of statistical structure with Ric^{∇^g} metric.

Example 2.6. Let $S := \Big(g(Q\xi_0,\cdot) \otimes g(\xi_0,\cdot) + g(\xi_0,\cdot) \otimes g(Q\xi_0,\cdot)\Big) \otimes (\xi_0 + Q\xi_0)$ be the $(1,2)$-tensor field from Proposition 2.4, where ξ_0 is a smooth vector field satisfying $Q^2\xi_0 = f\xi_0$, for an $n \in \mathbb{N}$ and a smooth function f on M. If the Riemannian metric g has harmonic curvature tensor and the Ricci tensor field Ric^{∇^g} is non-degenerate, then $(\mathrm{Ric}^{\nabla^g}, \nabla := \nabla^g + S)$ is a statistical structure.

Proposition 2.5. *If (M, g) is a Riemannian manifold with harmonic curvature tensor and the Ricci tensor field Ric^{∇^g} is non-degenerate, then $(\mathrm{Ric}^{\nabla}, \nabla := \nabla^g + S)$ is a statistical structure, where S is a symmetric and ∇^g-Codazzi (in particular, ∇^g-parallel) $(1,2)$-tensor field, satisfying*

$$g(S(X,Y),Z) = g(S(Z,X),Y),$$
$$S(S(X,Y),Z) = S(S(Z,Y),X),$$
$$S(X,QY) = S(QX,Y)$$

for any $X, Y, Z \in \Gamma^\infty(TM)$, and $g(QX,Y) := \mathrm{Ric}^{\nabla^g}(X,Y)$.

Proof. From Proposition 2.3 we get

$$R^\nabla(X,Y)Z = R^{\nabla^g}(X,Y)Z + S(X, S(Y,Z)) - S(Y, S(X,Z)).$$

Also the conditions upon S imply

$$S(Y, S(X,Z)) = S(S(X,Z),Y) = S(S(Y,Z),X) = S(X, S(Y,Z)),$$

hence $\mathrm{Ric}^\nabla = \mathrm{Ric}^{\nabla^g}$, and we obtain the conclusion. □

An example of a statistical structure with Ric^∇ metric is the following.

Example 2.7. Let $S := g(Q^n\xi_0,\cdot) \otimes g(Q^n\xi_0,\cdot) \otimes Q^n\xi_0$ be the $(1,2)$-tensor field from Proposition 2.5, where ξ_0 is a smooth vector field satisfying $Q^{n+1}\xi_0 = fQ^n\xi_0$, for an $n \in \mathbb{N}$ and a smooth function f on M. If the Riemannian metric g has harmonic curvature tensor and the Ricci tensor field Ric^{∇^g} is non-degenerate, then $(\mathrm{Ric}^\nabla, \nabla := \nabla^g + S)$ is a statistical structure.

Also by means of the Levi-Civita connection, now we shall consider other kind of affine connection.

Proposition 2.6. *Let J be a symmetric and invertible $(1,1)$-tensor field on (M, g). Then $\nabla := J^{-1}(\nabla^g \circ J)$ is an affine connection and*

$$T^\nabla(X,Y) = J^{-1}\big((\nabla^g_X J)Y - (\nabla^g_Y J)X\big),$$
$$R^\nabla(X,Y)Z = J^{-1}\big(R^{\nabla^g}(X,Y)JZ\big)$$

for any $X, Y, Z \in \Gamma^\infty(TM)$.

Proof. Remark that from $J(\nabla_X Y) = \nabla^g_X JY$, we get
$$(\nabla_X J)Y = J^{-1}(\nabla^g_X J^2 Y) - \nabla^g_X JY = J^{-1}((\nabla^g_X J)JY).$$
Also
$$\begin{aligned} T^{\nabla}(X,Y) &= J^{-1}(\nabla^g_X JY) - J^{-1}(\nabla^g_Y JX) - [X,Y] \\ &= J^{-1}\Big(\nabla^g_X JY - \nabla^g_Y JX - J(\nabla^g_X Y) + J(\nabla^g_Y X)\Big) \end{aligned}$$
and
$$\nabla^g_X \nabla^g_Y JZ = \nabla^g_X (J(\nabla_Y Z)) = J(\nabla_X \nabla_Y Z),$$
and we obtain the conclusions. □

An example of statistical structure with Ric^{∇^g} metric and this type of connection is the following.

Example 2.8. If (M,g) is a Riemannian manifold with harmonic curvature tensor, the Ricci tensor field Ric^{∇^g} is non-degenerate, and the Ricci operator Q is invertible, then $(\mathrm{Ric}^{\nabla^g}, \nabla := Q^{-1}(\nabla^g \circ Q))$ is a statistical structure. Indeed, it follows from the fact that Q is a symmetric and, in this case, also a ∇^g-Codazzi tensor field, and it satisfies
$$Q((\nabla_X Q)Y) = (\nabla^g_X Q)QY$$
and
$$\begin{aligned} (\nabla_X \mathrm{Ric}^{\nabla^g})(Y,Z) - (\nabla_Y \mathrm{Ric}^{\nabla^g})(X,Z) &= g(X,(\nabla^g_Y Q)Z) - g(Y,(\nabla^g_X Q)Z) \\ &= g(X,(\nabla^g_Z Q)Y) - g(Y,(\nabla^g_Z Q)X) = 0 \end{aligned}$$
for any $X,Y,Z \in \Gamma^\infty(TM)$.

Another kind of metric of special interest is the Hessian metric. In [9] we obtained some properties of statistical structures with Hessian metrics, which, we shall briefly recall, bringing into light their natural connections to gradient solitons.

Let f be a smooth function on (M,g). If the Hessian of f with respect to (g,∇^g), denoted by $\mathrm{Hess}(f) =: \mathrm{Hess}^{(g,\nabla^g)}(f)$, is non-degenerate and of constant signature, then $\mathrm{Hess}(f)$ is a pseudo-Riemannian metric. A nice geometrical interpretation of Hessian metrics has recently appeared in mirror symmetry [20], their importance in ecology being shown in [3].

Denoting by $\nabla f =: \nabla_g f$ the gradient of f with respect to g and assuming that $\mathrm{Hess}(f)$ is a pseudo-Riemannian metric, we proved in [9] the following result.

Theorem 2.2. (i) $(\mathrm{Hess}(f), \nabla^g)$ *is a statistical structure if and only if the radial curvature vanishes, i.e.,* $R^{\nabla^g}(X,Y)\nabla f = 0$ *for any* $X, Y \in \Gamma^\infty(TM)$.

(ii) *If* (g,∇) *is a statistical structure, then* $(\mathrm{Hess}(f),\nabla)$ *is a statistical structure if and only if* $R^{\nabla}(X,Y)\nabla f = 0$ *for any* $X, Y \in \Gamma^\infty(TM)$.

3. Statistical structures and solitons

We shall relate statistical structures with Ricci and Hessian metrics to gradient Ricci solitons (see [8, 9]).

We recall that, for a pseudo-Riemannian metric g on a smooth manifold M and two smooth functions f and λ on M, the data $(M,g,\nabla f,\lambda)$ is called a *gradient almost Ricci soliton* [38] if
$$\mathrm{Hess}(f) + \mathrm{Ric} = \lambda g,$$
where $\nabla f =: \nabla_g f$ and $\mathrm{Hess}(f) =: \mathrm{Hess}^{(g,\nabla^g)}(f)$ is the gradient and the Hessian of f and $\mathrm{Ric} =: \mathrm{Ric}^{\nabla^g}$ is the Ricci tensor field of g. If λ is a constant, then we drop "almost" from the previous definition and we call the soliton *gradient Ricci soliton*.

We assume that Ric and $\mathrm{Hess}(f)$ are non-degenerate and $\mathrm{Hess}(f)$ is of constant signature. Taking the covariant derivative in the soliton equation, we infer
$$(\nabla^g_X \mathrm{Hess}(f))(Y,Z) + (\nabla^g_X \mathrm{Ric})(Y,Z) = X(\lambda)g(Y,Z)$$
for any $X, Y, Z \in \Gamma^\infty(TM)$ and we have [9]

Proposition 3.1. *If $(M, g, \nabla f, \lambda)$ is*

(i) *a gradient almost Ricci soliton, then (Ric, ∇^g) is a statistical structure if and only if*

$$R^{\nabla^g}(\cdot,\cdot)\nabla f = d\lambda \otimes I - I \otimes d\lambda;$$

(ii) *a gradient Ricci soliton, then (Ric, ∇^g) is a statistical structure if and only if $(\mathrm{Hess}(f), \nabla^g)$ is a statistical structure.*

We recall that a pseudo-Riemannian metric is said to satisfy the *Miao–Tam equation* [26] if

$$\mathrm{Hess}(f) - f\,\mathrm{Ric} = -\frac{rf+1}{n-1}g, \tag{3.1}$$

respectively, the *Fischer–Marsden equation* [17] if

$$\mathrm{Hess}(f) - f\,\mathrm{Ric} = -\frac{rf}{n-1}g, \tag{3.2}$$

where r denotes the scalar curvature of (M^n, g).

Remark 3.1. (i) By taking the trace in (3.1) or (3.2), and applying then Δ, we infer

$$\Delta(\Delta(f)) = -\frac{1}{n-1}\Delta(f),$$

hence, if f is not a harmonic function, then $\Delta(f)$ is an eigenfunction of Δ corresponding to the eigenvalue $-\frac{1}{n-1}$.

(ii) If f is a harmonic function and g satisfies the Miao–Tam equation, then $rf = -n$, and if g satisfies the Fischer–Marsden equation, then $rf = 0$. In these cases, if f is nowhere zero, then $\nabla r = -r\nabla(\ln|f|)$.

Denoting by λ the coefficient of g from (3.1) or (3.2), for any $X, Y, Z \in \Gamma^\infty(TM)$, we get

$$(\nabla^g_X \mathrm{Hess}(f))(Y,Z) - (\nabla^g_Y \mathrm{Hess}(f))(X,Z) = f\Big((\nabla^g_X \mathrm{Ric})(Y,Z) - (\nabla^g_Y \mathrm{Ric})(X,Z)\Big) + X(\lambda)g(Y,Z) - Y(\lambda)g(X,Z)$$

and we can state

Proposition 3.2. *Let the metric g satisfy either the Miao–Tam equation or the Fischer–Marsden equation with f a smooth function. If (Ric, ∇^g) is a statistical structure, then $(\mathrm{Hess}(f), \nabla^g)$ is a statistical structure if and only if (rf) is a constant.*

From Remark 3.1 and Proposition 3.2, we conclude

Corollary 3.1. *If the metric g satisfy either the Miao–Tam equation or the Fischer–Marsden equation with f a harmonic function, then (Ric, ∇^g) is a statistical structure if and only if $(\mathrm{Hess}(f), \nabla^g)$ is a statistical structure.*

Example 3.1. Proposition 3.2 can be used as a criterion to decide when a solution (g, f) of the Miao–Tam equation (3.1) or the Fischer–Marsden equation (3.2) leads to a statistical structure $(\mathrm{Hess}(f), \nabla^g)$. For instance, it is known from [15, Theorem 3.1] that the n-dimensional Euclidean sphere $S^n(c)$ admits a nontrivial concircular vector field with a potential function ρ satisfying the Fischer–Marsden equation. In this case, it follows immediately that (Ric, ∇^g) is a statistical structure on $S^n(c)$ (where g denotes the standard metric on $S^n(c)$), but $(\mathrm{Hess}(\rho), \nabla^g)$ is not due to the fact that $(r\rho)$ is not a constant (since the scalar curvature r is a non-zero constant and the potential function ρ is not constant).

We extended the notion of a statistical structure as follows. If h is a symmetric $(0, 2)$-tensor field and ∇ is a torsion-free affine connection, we call (h, ∇) a *nearly statistical structure* on M [8] if ∇h is totally symmetric.

We will further consider a more general notion of soliton, not necessarily in the presence of a pseudo-Riemannian metric. More precisely, we call $(M, \nabla, J, \xi, \lambda)$ a (∇, J, ξ)*-soliton* [13] if

$$\nabla\xi + J = \lambda I, \tag{3.3}$$

where ∇ is an affine connection, J is a $(1, 1)$-tensor field, ξ is a vector field and λ is a smooth function on M. Regarding to it, in [9] and [8] we proved the following results.

Proposition 3.3. *Let (M, g) be a pseudo-Riemannian manifold, let J be a $(1,1)$-tensor field on M and let $\Omega := g(J\cdot, \cdot)$.*

(i) *If $(\nabla, J, \xi, \lambda)$ defines a (∇, J, ξ)-soliton on (M, g), then the 2-form Ω is symmetric if and only if the dual 1-form of ξ is closed, which is equivalent to*

$$g(\nabla_X \xi, Y) = g(X, \nabla_Y \xi)$$

for any $X, Y \in \Gamma^\infty(TM)$.

(ii) *The 2-form Ω is a ∇-Codazzi tensor field, i.e., $(\nabla_X \Omega)(Y, Z) = (\nabla_Y \Omega)(X, Z)$ for any $X, Y, Z \in \Gamma^\infty(TM)$, if and only if*

$$(\nabla_X g)(JY, Z) - (\nabla_Y g)(JX, Z) = g\big((\nabla_Y J)X - (\nabla_X J)Y, Z\big)$$

for any $X, Y, Z \in \Gamma^\infty(TM)$. In particular, Ω is a ∇^g-Codazzi tensor field if and only if J is a ∇^g-Codazzi tensor field, i.e., $(\nabla^g_X J)Y = (\nabla^g_Y J)X$ for any $X, Y \in \Gamma^\infty(TM)$.

As particular cases, we deduce from [8] and [9] the following results.

Proposition 3.4. *Let $(\nabla, J, \xi, \lambda)$ define a (∇, J, ξ)-soliton on (M, g). If $\Omega := g(J\cdot, \cdot)$ is symmetric and ∇ is torsion-free, then (Ω, ∇) is a nearly statistical structure if and only if*

$$g(R^\nabla(X, Y)\xi, Z) = (\nabla_X g)(JY, Z) - (\nabla_Y g)(JX, Z) + g\big(X(\lambda)Y - Y(\lambda)X, Z\big)$$

for any $X, Y, Z \in \Gamma^\infty(TM)$.

Also

Corollary 3.2. *If $(\nabla^g, J, \xi, \lambda)$ defines a (∇^g, J, ξ)-soliton on (M, g) and $\Omega := g(J\cdot, \cdot)$ is symmetric, then (Ω, ∇^g) is a nearly statistical structure if and only if*

$$R^{\nabla^g}(\cdot, \cdot)\xi = d\lambda \otimes I - I \otimes d\lambda.$$

For $\xi = \nabla f$, from the soliton equation (3.3) we find

$$\mathrm{Hess}^{(g,\nabla)}(f) + \Omega = \lambda g,$$

hence, $\nabla_X \mathrm{Hess}^{(g,\nabla)}(f) + \nabla_X \Omega = X(\lambda)g + \lambda \nabla_X g$ for any $X \in \Gamma^\infty(TM)$. In this case, we get

Proposition 3.5. *Let (g, ∇) be a statistical structure and let $(\nabla, J, \nabla f, \lambda)$ define a $(\nabla, J, \nabla f)$-soliton on (M, g) with λ a constant. Then (Ω, ∇) is a nearly statistical structure if and only if $(\mathrm{Hess}^{(g,\nabla)}, \nabla)$ is a nearly statistical structure.*

In particular, we have

Corollary 3.3. *If $(\nabla^g, J, \nabla f, \lambda)$ defines a $(\nabla^g, J, \nabla f)$-soliton on (M, g) with λ a constant, then the following statements are equivalent:*

(i) *(Ω, ∇^g) is a nearly statistical structure;*
(ii) *$R^{\nabla^g}(\cdot, \cdot)\nabla f = 0$;*
(iii) *$(\mathrm{Hess}^{(g,\nabla)}(f), \nabla^g)$ is a nearly statistical structure.*

Remark 3.2. If $(\nabla^g, \phi, \xi, \lambda)$ defines a (∇^g, ϕ, ξ)-soliton on a trans-Sasakian manifold (M, ϕ, ξ, η, g) with α and β real constants, then M is a Sasakian manifold. Indeed, since $\nabla^g_X \xi = \alpha\Big(X - \eta(X)\xi\Big) - \beta\phi X$, then

$$(\beta - 1)\phi X = (\alpha - \lambda)X - \alpha\eta(X)\xi$$

and further, by applying ϕ:

$$(\alpha - \lambda)\phi X = -(\beta - 1)X + (\beta - 1)\eta(X)\xi$$

and we get $\alpha = 0, \beta = 1, \lambda = 0$, hence the conclusion.

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Author's contributions

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