International Electronic Journal of Algebra
Volume 35 (2024) 20-31
DOI: 10.24330/ieja.1388822

ON THE CAPITULATION PROBLEM OF SOME PURE METACYCLIC FIELDS OF DEGREE 20 II

Fouad Elmouhib, Mohamed Talbi and Abdelmalek Azizi

Received: 23 May 2022; Revised: 30 September 2022; Accepted: 27 October 2022
Communicated by Tuğçe Pekacar Çalcı

Abstract. Let n be a 5^{th} power-free natural number and $k_0 = \mathbb{Q}(\zeta_5)$ be the cyclotomic field generated by a primitive 5^{th} root of unity ζ_5. Then $k = \mathbb{Q}(\sqrt[5]{n}, \zeta_5)$ is a pure metacyclic field of absolute degree 20. In the case that k possesses a 5-class group $C_{k,5}$ of type $(5,5)$ and all the classes are ambiguous under the action of $Gal(k/k_0)$, the capitulation of 5-ideal classes of k in its unramified cyclic quintic extensions is determined.

Mathematics Subject Classification (2020): 11R04, 11R18, 11R29, 11R37
Keywords: Pure metacyclic field, 5-class group, Hilbert 5-class field, capitulation

1. Introduction

Let k be a number field, and L be an unramified abelian extension of k. We say that an ideal $\mathcal{I}$ of k or its class capitulates in L if $\mathcal{I}$ becomes principal in L.

Let $\Gamma = \mathbb{Q}(\sqrt[5]{n})$ be a pure quintic field, where n is a 5^{th} power free natural number, and $k_0 = \mathbb{Q}(\zeta_5)$ be the cyclotomic field generated by a primitive 5^{th} root of unity ζ_5. Then $k = \Gamma(\zeta_5)$ is the normal closure of Γ and a pure metacyclic field of absolute degree 20. Let $k_5^{(1)}$ be the Hilbert 5-class field of k, $C_{k,5}$ be the 5-ideal class group of k and $C_{k,5}^{(\sigma)}$ be the subgroup of ambiguous ideal classes under the action of $Gal(k/k_0) = \langle\sigma\rangle$.

In the case that $C_{k,5}$ is of type $(5,5)$ and rank $C_{k,5}^{(\sigma)} = 1$, the capitulation of the 5-ideal classes of k in the six intermediate extensions of $k_5^{(1)}/k$ is determined in [2].

Let p and q be primes such that $p \equiv 1 \pmod 5$ and $q \equiv \pm 2 \pmod 5$. According to [1, Theorem 1.1], if $C_{k,5}$ is of type $(5,5)$ and rank $C_{k,5}^{(\sigma)} = 2$, we have three forms of the radicand n as follows:

- $n = p^e$ with $e \in \{1,2,3,4\}$ and $p \equiv 1 \pmod{25}$.
- $n = 5^{e_1} p^{e_2}$ with $e_1, e_2 \in \{1,2,3,4\}$ and $p \not\equiv 1 \pmod{25}$.
- $n = p^{e_1} q^{e_2} \equiv \pm 1, \pm 7 \pmod{25}$ with $e_1, e_2 \in \{1,2,3,4\}$, $p \not\equiv 1 \pmod{25}$ and $q \not\equiv \pm 7 \pmod{25}$.

In this paper, we investigate the capitulation of the 5-ideal classes of the pure metacyclic field k in the unramified cyclic quintic extensions of k within the Hilbert 5-class field $k_5^{(1)}$ of k, whenever $C_{k,5}$ is of type $(5,5)$ and rank $C_{k,5}^{(\sigma)} = 2$, which means that all classes are ambiguous.

We will study the capitulation of $C_{k,5}$ in the six intermediate extensions $K_1, \ldots, K_6$ of $k_5^{(1)}/k$ by distinguishing the three cases of the radicand n. Figure 1 illustrates the situation.

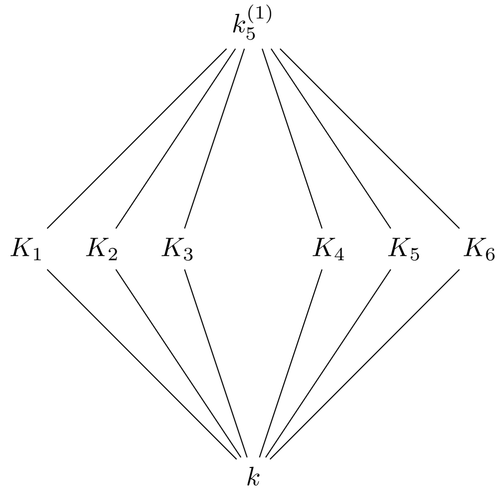

Figure 1: The unramified quintic sub-extensions of $k_5^{(1)}/k$

The theoretical results are underpinned by numerical examples obtained with the computational number theory system PARI/GP [6].

Notations.

Throughout this paper, we use the following notations:

- The lower case letters p and q denote a prime numbers such that, $p \equiv 1 \pmod 5$ and $q \equiv \pm 2 \pmod 5$.
- $\Gamma = \mathbb{Q}(\sqrt[5]{n})$: a pure quintic field, where $n \neq 1$ is a 5^{th} power-free natural number.
- $k_0 = \mathbb{Q}(\zeta_5)$: the cyclotomic field, where $\zeta_5 = e^{2i\pi/5}$ is a primitive 5^{th} root of unity.
- $k = \mathbb{Q}(\sqrt[5]{n}, \zeta_5)$: the normal closure of Γ, a quintic Kummer extension of k_0.
- $\langle\tau\rangle = \mathrm{Gal}(k/\Gamma)$ such that τ is identity on Γ, and sends ζ_5 to its square. Hence τ has order 4.
- $\langle\sigma\rangle = \mathrm{Gal}(k/k_0)$ such that σ is identity on k_0, and sends $\sqrt[5]{n}$ to $\zeta_5\sqrt[5]{n}$. Hence σ has order 5.
- For a number field L, denote by:
  - $\mathcal{O}_L$: the ring of integers of L.
  - C_L, h_L, $C_{L,5}$: the class group, class number, and 5-class group of L.
  - $L_5^{(1)}, L^*$: the Hilbert 5-class field of L, and the absolute genus field of L.
  - $[\mathcal{I}]$: the class of a fractional ideal $\mathcal{I}$ in the class group of L.
- $(\frac{a}{b})_5 = 1 \Leftrightarrow X^5 \equiv a \pmod b$ soluble in $\mathcal{O}_{k_0}$, where a, b are primes in $\mathcal{O}_{k_0}$.

2. Preliminaries

2.1. Decomposition laws in Kummer extension.

Since the pure quintic extensions of the 5^{th} cyclotomic field $k_0 = \mathbb{Q}(\zeta_5)$ and of $k = \mathbb{Q}(\sqrt[5]{n}, \zeta_5)$ are all Kummer's extensions, we recall the decomposition laws of ideals in these extensions.

Proposition 2.1. *Let L be a number field containing the l^{th} roots of unity, where l is prime, and θ be an element of L, such that $\theta \neq \mu^l$, for all $\mu \in L$. Therefore $L(\sqrt[l]{\theta})$ is a cyclic extension of* *degree l over L. We denote by ζ a primitive l^{th} root of unity.*

(1) *We assume that a prime ideal $\mathcal{P}$ of L, divides θ exactly to the power $\mathcal{P}^a$.*

- *If $a = 0$ and $\mathcal{P}$ does not divide l, then $\mathcal{P}$ splits completely in $L(\sqrt[l]{\theta})$ when the congruence $\theta \equiv X^l \pmod{\mathcal{P}}$ has a solution in L.*
- *If $a = 0$ and $\mathcal{P}$ does not divide l, then $\mathcal{P}$ is inert in $L(\sqrt[l]{\theta})$ when the congruence $\theta \equiv X^l \pmod{\mathcal{P}}$ has no solution in L.*
- *If $l \nmid a$, then $\mathcal{P}$ is totally ramified in $L(\sqrt[l]{\theta})$.*

(2) *Let $\mathcal{B}$ be a prime factor of $1-\zeta$ that divides $1-\zeta$ exactly to the a^{th} power. Suppose that $\mathcal{B} \nmid \theta$, then $\mathcal{B}$ splits completely in $L(\sqrt[l]{\theta})$ if the congruence*

$$\theta \equiv X^l \pmod{\mathcal{B}^{al+1}} \qquad (*)$$

has a solution in L. The ideal $\mathcal{B}$ is inert in $L(\sqrt[l]{\theta})$ if the congruence

$$\theta \equiv X^l \pmod{\mathcal{B}^{al}} \qquad (**)$$

has a solution in L, but $(*)$ has no solution. The ideal $\mathcal{B}$ is totally ramified in L if the congruence $(**)$ has no solution.

Proof. See [3, Theorems 118, 119]. □

2.2. Relative genus field $(k/k_0)^*$ of k over k_0.

Let $\Gamma = \mathbb{Q}(\sqrt[5]{n})$ be a pure quintic field, $k_0 = \mathbb{Q}(\zeta_5)$ the 5^{th}-cyclotomic field and $k = \Gamma(\zeta_5)$ be the normal closure of Γ. The relative genus field $(k/k_0)^*$ of k over k_0 is the maximal abelian extension of k_0 which is contained in the Hilbert 5-class field $k_5^{(1)}$ of k.

Let q^* be the exponent defined by $[N_{k/k_0}(k-\{0\}) \cap E_{k_0} : N_{k/k_0}(E_{k_0})] = 5^{q^*}$. Here N_{k/k_0} is the relative norm from k to k_0, and E_{k_0} the group of units of k_0. We note that $N_{k/k_0}(E_{k_0}) = E_{k_0}^5$ and $[E_{k_0} : E_{k_0}^5] = 5^2$, so we get that $q^* \in \{0, 1, 2\}$.

The group E_{k_0} is generated by ζ_5 and $\zeta_5 + 1$, then according to the definition of q^*, we see that:

$$q^* = \begin{cases} 2 & \text{if } \zeta, \zeta+1 \in N_{k/k_0}(k-\{0\}), \\ 1 & \text{if } \zeta^i(\zeta+1)^j \in N_{k/k_0}(k-\{0\}) \text{ for some i and j }, \\ 0 & \text{if } \zeta^i(\zeta+1)^j \notin N_{k/k_0}(k-\{0\}) \text{ for } 0 \le i,j \le 4 \text{ and } i+j \ne 0. \end{cases}$$

The relative genus field $(k/k_0)^*$ is given explicitly by the following proposition by means of the decomposition of n in k_0 and the value of q^*.

Proposition 2.2. *Let $k = k_0(\sqrt[5]{n})$ such that $n = \mu\lambda^{e_\lambda}\pi_1^{e_1} \dots \pi_f^{e_f}\pi_{f+1}^{e_{f+1}} \dots \pi_g^{e_g}$ in k_0, where μ is unity of $\mathcal{O}_{k_0}$, $\lambda = 1-\zeta_5$ the unique prime above 5 in k_0 and each prime $\pi_i \equiv \pm1, \pm7 \pmod{\lambda^5}$ for $1 \le i \le f$ and $\pi_j \not\equiv \pm1, \pm7 \pmod{\lambda^5}$ for $f+1 \le j \le g$. Then we have:*

(i) *There exists $h_i \in \{1, \dots, 4\}$ such that $\pi_{f+1}\pi_i^{h_i} \equiv \pm1, \pm7 \pmod{\lambda^5}$ for $f+2 \le i \le g$.*

(ii) *If $n \not\equiv \pm1, \pm7 \pmod{\lambda^5}$ and $q^* = 1$, then the genus field $(k/k_0)^*$ is given as:*

$$(k/k_0)^* = k\left(\sqrt[5]{\pi_1}, \dots, \sqrt[5]{\pi_f}, \sqrt[5]{\pi_{f+1}\pi_{f+2}^{h_{f+2}}}, \dots, \sqrt[5]{\pi_{f+1}\pi_g^{h_g}}\right)$$

where h_i is chosen as in (i).

(iii) *In the other cases of q^* and the congruence of n, the genus field $(k/k_0)^*$ is given by deleting an appropriate number of 5^{th} root from the right side of (ii).*

Proof. See [4, Proposition 5.8]. □

3. Study of capitulation

Let Γ, k_0 and k as above. If $C_{k,5}$ is of type $(5,5)$ and the subgroup of ambiguous classes $C_{k,5}^{(\sigma)}$ under the action of $Gal(k/k_0) = \langle\sigma\rangle$ has rank 2, we have $C_{k,5} = C_{k,5}^{(\sigma)}$.

By class field theory, the principal genus $C_{k,5}^{1-\sigma}$ corresponds to $(k/k_0)^*$, and since $C_{k,5} = C_{k,5}^{(\sigma)}$ we get that $C_{k,5}^{1-\sigma} = \{1\}$, whence $(k/k_0)^*$ coincides with the Hilbert 5-class field $k_5^{(1)}$ of k.

When $C_{k,5}$ is of type $(5,5)$, it has 6 subgroups of order 5, denoted H_i, $1 \leq i \leq 6$. Let K_i be the intermediate extension of $k_5^{(1)}/k$ which corresponds by class field theory to H_i.
As each K_i is cyclic of order 5 over k, by Hilbert's theorem 94, there is at least one subgroup of order 5 of $C_{k,5}$, i.e. at least one H_l for some $l \in \{1,2,3,4,5,6\}$, which capitulates in K_i.

Definition 3.1. Let $\mathcal{S}_j$ be a generator of H_j $(1 \leq j \leq 6)$ which corresponds to K_j. For $1 \leq j \leq 6$, let $i_j \in \{0,1,2,3,4,5,6\}$. We say that the capitulation is of type $(i_1,i_2,i_3,i_4,i_5,i_6)$ to mean the following:

(1) when $i_j \in \{1,2,3,4,5,6\}$, then only the class $\mathcal{S}_{i_j}$ and its powers capitulate in K_j;
(2) when $i_j = 0$, then all the 5-classes capitulate in K_j.

We find ourselves in front of $7^6 = 117649$ possible types which need to be reduced.

Its easy to see that $C_{k,5} \simeq C_{k,5}^{+} \times C_{k,5}^{-}$ such that $C_{k,5}^{+} = \{\mathcal{A} \in C_{k,5} \mid \mathcal{A}^{\tau^2} = \mathcal{A}\}$ and $C_{k,5}^{-} = \{\mathcal{X} \in C_{k,5} \mid \mathcal{X}^{\tau^2} = \mathcal{X}^{-1}\}$, with $Gal(k/\Gamma) = \langle\tau\rangle$. We order the subgroups H_i of $C_{k,5}$ as follows:
$H_1 = C_{k,5}^{+} = \langle\mathcal{A}\rangle$, $H_6 = C_{k,5}^{-} = \langle\mathcal{X}\rangle$, $H_2 = \langle\mathcal{A}\mathcal{X}\rangle$, $H_3 = \langle\mathcal{A}\mathcal{X}^2\rangle$, $H_4 = \langle\mathcal{A}\mathcal{X}^3\rangle$ and $H_5 = \langle\mathcal{A}\mathcal{X}^4\rangle$.

By the action of $Gal(k/\mathbb{Q})$ on $C_{k,5}$, we can give the following proposition:

Proposition 3.1. *For all continuations of the automorphisms σ and τ we have:*

(1) $K_i^{\sigma} = K_i$ $(i = 1,2,3,4,5,6)$, *i.e σ sets all K_i.*
(2) $K_1^{\tau^2} = K_1$, $K_6^{\tau^2} = K_6$, $K_2^{\tau^2} = K_5$ *and* $K_3^{\tau^2} = K_4$. *i.e τ^2 sets K_1, K_6 and permutes K_2 with K_5 and K_3 with K_4.*

Proof. We will agree that for all $1 \leq i \leq 6$ and for all $w \in Gal(k/\mathbb{Q})$ we have $H_i^w = \{\mathcal{C}^w \mid \mathcal{C} \in H_i\}$.
(1) Since all classes are ambiguous because $C_{k,5} = C_{k,5}^{(\sigma)}$, σ sets all H_i.
(2)We have $H_1 = C_{k,5}^{+} = \langle\mathcal{A}\rangle$ and $H_6 = C_{k,5}^{-} = \langle\mathcal{X}\rangle$, then $H_1^{\tau^2} = H_1$ and $H_6^{\tau^2} = H_6$.
- Since $(\mathcal{A}\mathcal{X})^{\tau^2} = \mathcal{A}^{\tau^2}\mathcal{X}^{\tau^2} = \mathcal{A}\mathcal{X}^{-1} = \mathcal{A}\mathcal{X}^4 \in H_5$, $H_2^{\tau^2} = H_5$.
- Since $(\mathcal{A}\mathcal{X}^2)^{\tau^2} = \mathcal{A}^{\tau^2}(\mathcal{X}^2)^{\tau^2} = \mathcal{A}\mathcal{X}^{-2} = \mathcal{A}\mathcal{X}^3 \in H_4$, $H_3^{\tau^2} = H_4$.
- Since $\tau^4 = 1$, we get that $H_5^{\tau^2} = H_2$ and $H_4^{\tau^2} = H_3$.

The relations between the fields K_i in (1) and (2) are nothing else than the translations of the corresponding relations for the subgroups H_i via class field theory. □

To study the capitulation problem of k whenever $C_{k,5}$ is of type $(5,5)$ and $C_{k,5} = C_{k,5}^{(\sigma)}$, we will investigate the three forms of the radicand n proved in [1, Theorem 1.1], and mentioned above.

3.1. The case $n = p^e$ where $p \equiv 1 \pmod{25}$.

Let $k = \Gamma(\zeta_5)$ be the normal closure of $\Gamma = \mathbb{Q}(\sqrt[5]{n})$, where $n = p^e$ such that $p \equiv 1 \pmod{25}$ and $e \in \{1, 2, 3, 4\}$. By [5, Theorem 2.13], since $p \equiv 1 \pmod 5$ we have that p splits completely in $k_0 = \mathbb{Q}(\zeta_5)$ as $p = \pi_1\pi_2\pi_3\pi_4$, with π_i are primes in k_0. As the discriminant of $\Gamma/\mathbb{Q}$ is 5^3p^4, we get that p is ramified in Γ, then the primes π_i are ramified in k.

If $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ and $\mathcal{P}_4$ are respectively the prime ideals of k above π_1, π_2, π_3 and π_4, then $\mathcal{P}_i^5 = \pi_i\mathcal{O}_k$ $(i = 1, 2, 3, 4)$. Since τ acts transitively on π_i, we have that τ^2 permutes π_1 with π_3, hence τ^2 permutes $\mathcal{P}_1$ with $\mathcal{P}_3$. Since $\pi_i^\sigma = \pi_i$, we have $\mathcal{P}_i^\sigma = \mathcal{P}_i$. In fact $[\mathcal{P}_i]$ $(i = 1, 2, 3, 4)$ generate the subgroup of strong ambiguous ideal classes denoted $C_{k,s}^{(\sigma)}$ and defined by $C_{k,s}^{(\sigma)} = \{[\mathcal{P}] \in C_{k,5} \mid \mathcal{P}^\sigma = \mathcal{P}\}$.

The next theorem allow us to determine explicitly the intermediate extensions of $k_5^{(1)}/k$.

Theorem 3.1. *Let k and n as above. Let π_1, π_2, π_3 and π_4 be primes of k_0 congruent to 1 modulo λ^5 such that $p = \pi_1\pi_2\pi_3\pi_4$, then:*

(1) $k_5^{(1)} = k\left(\sqrt[5]{\pi_1}, \sqrt[5]{\pi_3}\right)$.

(2) *The six intermediate extensions of $k_5^{(1)}/k$ are: $k\left(\sqrt[5]{\pi_1}\right)$, $k\left(\sqrt[5]{\pi_3}\right)$, $k\left(\sqrt[5]{\pi_1\pi_3}\right)$, $k\left(\sqrt[5]{\pi_1\pi_3^2}\right)$, $k\left(\sqrt[5]{\pi_1\pi_3^3}\right)$ and $k\left(\sqrt[5]{\pi_1\pi_3^4}\right)$.*
Furthermore τ^2 permutes $k\left(\sqrt[5]{\pi_1}\right)$ with $k\left(\sqrt[5]{\pi_3}\right)$ and $k\left(\sqrt[5]{\pi_1\pi_3^2}\right)$ with $k\left(\sqrt[5]{\pi_1\pi_3^3}\right)$, and sets $k\left(\sqrt[5]{\pi_1\pi_3}\right)$, $k\left(\sqrt[5]{\pi_1\pi_3^4}\right)$.

Proof. (1) We have that $k_5^{(1)} = (k/k_0)^*$. Since $k = k_0(\sqrt[5]{n})$ with $n = p = \pi_1\pi_2\pi_3\pi_4$ in k_0 and $\pi_i \equiv 1 \pmod{\lambda^5}$ $(i = 1, 2, 3, 4)$, by Proposition 2.2 we have $(k/k_0)^* = k\left(\sqrt[5]{\pi_1}, \sqrt[5]{\pi_3}\right)$.
(2) If $k_5^{(1)} = k\left(\sqrt[5]{\pi_1}, \sqrt[5]{\pi_3}\right)$, then the six intermediate extensions are: $k\left(\sqrt[5]{\pi_1}\right)$, $k\left(\sqrt[5]{\pi_3}\right)$, $k\left(\sqrt[5]{\pi_1\pi_3}\right)$, $k\left(\sqrt[5]{\pi_1\pi_3^2}\right)$, $k\left(\sqrt[5]{\pi_1\pi_3^3}\right)$ and $k\left(\sqrt[5]{\pi_1\pi_3^4}\right)$. We have $\tau^2(\pi_1) = \pi_3$, so it is easy to see that τ^2 sets the fields $k\left(\sqrt[5]{\pi_1\pi_3}\right)$, $k\left(\sqrt[5]{\pi_1\pi_3^4}\right)$.
Since $\tau^2(\pi_1) = \tau^2(\sqrt[5]{\pi_1^5}) = (\tau^2(\sqrt[5]{\pi_1}))^5 = \pi_3$, $\tau^2(\sqrt[5]{\pi_1})$ is 5^{th} root of π_3. Thus $k\left(\sqrt[5]{\pi_3}\right) = k\left(\tau^2(\sqrt[5]{\pi_1})\right)$, i.e. $k\left(\sqrt[5]{\pi_3}\right) = k\left(\sqrt[5]{\pi_1}\right)^{\tau^2}$. By the same reasoning we prove that $k\left(\sqrt[5]{\pi_1}\right) = k\left(\sqrt[5]{\pi_3}\right)^{\tau^2}$. Hence τ^2 permutes $k\left(\sqrt[5]{\pi_1}\right)$ with $k\left(\sqrt[5]{\pi_3}\right)$.
We have $\tau^2(\pi_1\pi_3^2) = \pi_1^2\pi_3$ then $\tau^2(\pi_1\pi_3^2) = \tau^2(\sqrt[5]{(\pi_1\pi_3^2)^5}) = (\tau^2(\sqrt[5]{\pi_1\pi_3^2}))^5 = \pi_1^2\pi_3$, then $\tau^2(\sqrt[5]{\pi_1\pi_3^2})$ is 5^{th} root of $\pi_1^2\pi_3$. Thus $k\left(\sqrt[5]{\pi_1^2\pi_3}\right) = k\left(\tau^2(\sqrt[5]{\pi_1\pi_3^2})\right)$ i.e $k\left(\sqrt[5]{\pi_1^2\pi_3}\right) = k\left(\sqrt[5]{\pi_1\pi_3^3}\right) = k\left(\sqrt[5]{\pi_1\pi_3^2}\right)^{\tau^2}$. By the same reasoning we prove that $k\left(\sqrt[5]{\pi_1\pi_3^2}\right) = k\left(\sqrt[5]{\pi_1\pi_3^3}\right)^{\tau^2}$. Hence τ^2 permutes $k\left(\sqrt[5]{\pi_1\pi_3^2}\right)$ with $k\left(\sqrt[5]{\pi_1\pi_3^3}\right)$. □

The generators of $C_{k,5}$ when it is of type $(5, 5)$ and the radicand n is as above are determined as follows:

Theorem 3.2. *Let k and n as above. Let π_1, π_2, π_3 and π_4 be primes of k_0 congruent to 1 $\pmod{\lambda^5}$ such that $n = p = \pi_1\pi_2\pi_3\pi_4$. Let $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ and $\mathcal{P}_4$ be prime ideals of k such that $\mathcal{P}_i^5 = \pi_i\mathcal{O}_{k_0}$ $(i = 1, 2, 3, 4)$. Then:*

$$C_{k,5} = \langle [\mathcal{P}_1\mathcal{P}_3], [\mathcal{P}_1\mathcal{P}_3^4] \rangle.$$

Proof. According to the proof of [1, Theorem 1.1], , for this case of the radicand n, we have that $\zeta_5^i(1+\zeta_5)^j$ is norm of element in $k-\{0\}$ for some exponents i and j. By [4, Section 5.3], if ζ_5 is not norm of unit of k we have $C_{k,5} = C_{k,5}^{(\sigma)} \neq C_{k,s}^{(\sigma)}$, so $C_{k,s}^{(\sigma)}$ contained in $C_{k,5}^{(\sigma)}$. Hence we discuss two cases:

• 1^{st} case: $C_{k,5} = C_{k,5}^{(\sigma)} \neq C_{k,s}^{(\sigma)}$:

We have that $C_{k,s}^{(\sigma)}$ is contained in $C_{k,5} = C_{k,5}^{(\sigma)}$, and by [4, Section 5.3] we have $C_{k,5}^{(\sigma)}/C_{k,s}^{(\sigma)} = C_{k,5}/C_{k,s}^{(\sigma)}$ is cyclic group of order 5. Since $C_{k,5}$ has order 25, $C_{k,s}^{(\sigma)}$ is cyclic of order 5.

We have that $C_{k,s}^{(\sigma)} = \langle[\mathcal{P}_1],[\mathcal{P}_2],[\mathcal{P}_3],[\mathcal{P}_4]\rangle$, $\mathcal{P}_1^{\tau^2} = \mathcal{P}_3$ and $\mathcal{P}_2^{\tau^2} = \mathcal{P}_4$, so $\mathcal{P}_1$ and $\mathcal{P}_2$ can not be both principals in k, otherwise $\mathcal{P}_3 = \mathcal{P}_1^{\tau^2}$ and $\mathcal{P}_4 = \mathcal{P}_2^{\tau^2}$ will be principals too, Thus $C_{k,s}^{(\sigma)} = \{1\}$, which is impossible. By the same reasoning we have that $\mathcal{P}_3$ and $\mathcal{P}_4$ can not be both principals in k.

Since $C_{k,s}^{(\sigma)}$ is cyclic of order 5 and without loosing generality, we get that $C_{k,s}^{(\sigma)} = \langle[\mathcal{P}_1]\rangle$, so $\mathcal{P}_1$ and $\mathcal{P}_3 = \mathcal{P}_1^{\tau^2}$ are not principals. Since $C_{k,5} \simeq C_{k,5}^{+} \times C_{k,5}^{-}$, it is sufficient to find generators of $C_{k,5}^{+}$ and $C_{k,5}^{-}$. As $[\mathcal{P}_1\mathcal{P}_3]^{\tau^2} = [(\mathcal{P}_1\mathcal{P}_3)^{\tau^2}] = [\mathcal{P}_1\mathcal{P}_3]$, then $C_{k,5}^{+} = \langle[\mathcal{P}_1\mathcal{P}_3]\rangle$ and $[\mathcal{P}_1\mathcal{P}_3^4]^{\tau^2} = [(\mathcal{P}_1\mathcal{P}_3^4)^{\tau^2}] = [\mathcal{P}_1^4\mathcal{P}_3] = [\mathcal{P}_1\mathcal{P}_3^4]^{-1}$, then $C_{k,5}^{-} = \langle[\mathcal{P}_1\mathcal{P}_3^4]\rangle$. Hence $C_{k,5} = \langle[\mathcal{P}_1\mathcal{P}_3],[\mathcal{P}_1\mathcal{P}_3^4]\rangle$.

• 2^{nd} case: $C_{k,5} = C_{k,5}^{(\sigma)} = C_{k,s}^{(\sigma)}$:

We apply the same reasoning as in the 1^{st} case, because none of $\mathcal{P}_i\,(i=1,2,3,4)$ is principal, otherwise $C_{k,5} = C_{k,s}^{(\sigma)} = \{1\}$, which is impossible. Hence $C_{k,5} = \langle[\mathcal{P}_1\mathcal{P}_3],[\mathcal{P}_1\mathcal{P}_3^4]\rangle$. □

Now we are able to state the main theorem of capitulation in this case.

Theorem 3.3. *We keep the same assumptions as in Theorem 3.2. Then:*

(1) *If* $(\frac{\pi_1}{\pi_3})_5 = 1$ *we have* $K_1 = k\left(\sqrt[5]{\pi_1\pi_3}\right)$ *or* $k\left(\sqrt[5]{\pi_1\pi_3^4}\right)$, $K_2 = k\left(\sqrt[5]{\pi_3}\right)$, $K_3 = k\left(\sqrt[5]{\pi_1\pi_3^2}\right)$ *or* $k\left(\sqrt[5]{\pi_1\pi_3^3}\right)$, $K_4 = k\left(\sqrt[5]{\pi_1\pi_3^3}\right)$ *or* $k\left(\sqrt[5]{\pi_1\pi_3^2}\right)$, $K_5 = k\left(\sqrt[5]{\pi_1}\right)$ *and* $K_6 = k\left(\sqrt[5]{\pi_1\pi_3^4}\right)$ *or* $k\left(\sqrt[5]{\pi_1\pi_3}\right)$. *Otherwise we just permute* K_2 *and* K_5 *in equalities.*

(2) $[\mathcal{P}_1\mathcal{P}_3]$ *capitulates in* $k\left(\sqrt[5]{\pi_1\pi_3}\right)$, $[\mathcal{P}_i]$ *capitulates in* $k\left(\sqrt[5]{\pi_i}\right)$ $(i=1,3)$, $[\mathcal{P}_1\mathcal{P}_3^2]$ *capitulates in* $k\left(\sqrt[5]{\pi_1\pi_3^2}\right)$, $[\mathcal{P}_1\mathcal{P}_3^3]$ *capitulates in* $k\left(\sqrt[5]{\pi_1\pi_3^3}\right)$ *and* $[\mathcal{P}_1\mathcal{P}_3^4]$ *capitulates in* $k\left(\sqrt[5]{\pi_1\pi_3^4}\right)$.

(3) (i) *If* $(\frac{\pi_1}{\pi_3})_5 = 1$ *and* $K_6 = k\left(\sqrt[5]{\pi_1\pi_3^4}\right)$, *then the possible types of capitulation are:* $(0,0,0,0,0,0)$, $(1,0,0,0,0,0)$, $(0,2,0,0,5,0)$, $(1,2,0,0,5,0)$, $\{(0,0,3,4,0,0)$ *or* $(0,0,4,3,0,0)\}$, $\{(1,0,3,4,0,0)$ *or* $(1,0,4,3,0,0)\}$, $\{(0,2,3,4,5,0)$ *or* $(0,2,4,3,5,0)\}$, $\{(1,2,3,4,5,0)$ *or* $(1,2,4,3,5,0)\}$.

(ii) *If* $(\frac{\pi_1}{\pi_3})_5 = 1$ *and* $K_6 = k\left(\sqrt[5]{\pi_1\pi_3}\right)$ *then the same possible types of capitulation occur as in* (*i*) *with* $i_6 = 0$ *or* 1 *and* $i_1 = 0$ *or* 6.

(iii) *If* $(\frac{\pi_1}{\pi_3})_5 \neq 1$ *then the same possible types of capitulation occur as* (*i*) *and* (*ii*) *by permuting 2 and 5 in the given types of capitulation.*

Proof. (1) According to Theorem 3.1, we have that τ^2 permutes $k\left(\sqrt[5]{\pi_1}\right)$ with $k\left(\sqrt[5]{\pi_3}\right)$ and $k\left(\sqrt[5]{\pi_1\pi_3^2}\right)$ with $k\left(\sqrt[5]{\pi_1\pi_3^3}\right)$, moreover τ^2 sets $k\left(\sqrt[5]{\pi_1\pi_3}\right)$, $k\left(\sqrt[5]{\pi_1\pi_3^4}\right)$.

By class field theory K_i corresponds to $H_i\,(i=1,2,3,4,5,6)$. We determine explicitly the six subgroups H_i of $C_{k,5}$ as follows:

We have that $C_{k,5} = \langle\mathcal{A},\mathcal{X}\rangle$, where $H_1 = C_{k,5}^{+} = \langle\mathcal{A}\rangle$ and $H_6 = C_{k,5}^{-} = \langle\mathcal{X}\rangle$. By Theorem 3.2 we have $\mathcal{A} = [\mathcal{P}_1\mathcal{P}_3]$ and $\mathcal{X} = [\mathcal{P}_1\mathcal{P}_3^4]$, then $\mathcal{A}\mathcal{X} = [\mathcal{P}_1]^2$, $\mathcal{A}\mathcal{X}^2 = [\mathcal{P}_1\mathcal{P}_3^3]^3$, $\mathcal{A}\mathcal{X}^3 = [\mathcal{P}_1\mathcal{P}_3^2]^4$ and

$\mathcal{AX}^4 = [\mathcal{P}_3]^4$. Thus $H_2 = \langle[\mathcal{P}_1]\rangle$, $H_3 = \langle[\mathcal{P}_1\mathcal{P}_3^3]\rangle$, $H_4 = \langle[\mathcal{P}_1\mathcal{P}_3^2]\rangle$ and $H_5 = \langle[\mathcal{P}_3]\rangle$. Since τ^2 sets $k\left(\sqrt[5]{\pi_1\pi_3}\right)$ and $k\left(\sqrt[5]{\pi_1\pi_3^4}\right)$, if $K_1 = k\left(\sqrt[5]{\pi_1\pi_3}\right)$, then $K_6 = k\left(\sqrt[5]{\pi_1\pi_3^4}\right)$ and vice versa.

If $(\frac{\pi_1}{\pi_3})_5 = 1$ then $X^5 \equiv \pi_1 \pmod{\pi_3}$ is resolved in $\mathcal{O}_{k_0}$ and by Proposition 2.1, we have that π_1 splits completely in $k_0(\sqrt[5]{\pi_3})$, which equivalent to say that $\mathcal{P}_1$ splits completely in $k\left(\sqrt[5]{\pi_3}\right)$, so $K_2 = k\left(\sqrt[5]{\pi_3}\right)$ and we get that $K_5 = k\left(\sqrt[5]{\pi_1}\right)$. If $K_3 = k\left(\sqrt[5]{\pi_1\pi_3^2}\right)$, then $K_4 = k\left(\sqrt[5]{\pi_1\pi_3^3}\right)$ and vice versa. Since π_1 and π_3 divide $\pi_1\pi_3$, $\pi_1\pi_3^2$, $\pi_1\pi_3^3$ and $\pi_1\pi_3^4$, if $(\frac{\pi_1}{\pi_3})_5 \neq 1$, then $K_2 = k\left(\sqrt[5]{\pi_1}\right)$ and $K_5 = k\left(\sqrt[5]{\pi_3}\right)$.

(2) Since $\mathcal{P}_i^5 = \pi_i\mathcal{O}_k$ $(i = 1, 3)$, we have $(\mathcal{P}_1\mathcal{P}_3)^5 = \pi_1\pi_3\mathcal{O}_k$, then $(\mathcal{P}_1\mathcal{P}_3)^5 = \pi_1\pi_3\mathcal{O}_{k(\sqrt[5]{\pi_1\pi_3})}$ in $k\left(\sqrt[5]{\pi_1\pi_3}\right)$ and $\pi_1\pi_3\mathcal{O}_{k(\sqrt[5]{\pi_1\pi_3})} = \left(\sqrt[5]{\pi_1\pi_3}\mathcal{O}_{k(\sqrt[5]{\pi_1\pi_3})}\right)^5$, whence $\mathcal{P}_1\mathcal{P}_3\mathcal{O}_{k(\sqrt[5]{\pi_1\pi_3})} = \sqrt[5]{\pi_1\pi_3}\mathcal{O}_{k(\sqrt[5]{\pi_1\pi_3})}$. Thus $\mathcal{P}_1\mathcal{P}_3$ seen in $\mathcal{O}_{k(\sqrt[5]{\pi_1\pi_3})}$ becomes principal, i.e $[\mathcal{P}_1\mathcal{P}_3]$ capitulates in $k\left(\sqrt[5]{\pi_1\pi_3}\right)$.
Since $(\mathcal{P}_1\mathcal{P}_3^2)^5 = \pi_1\pi_3^2\mathcal{O}_k$, we have $(\mathcal{P}_1\mathcal{P}_3^2)^5 = \pi_1\pi_3^2\mathcal{O}_{k\left(\sqrt[5]{\pi_1\pi_3^2}\right)}$ in $k\left(\sqrt[5]{\pi_1\pi_3^2}\right)$ and $\pi_1\pi_3^2\mathcal{O}_{k\left(\sqrt[5]{\pi_1\pi_3^2}\right)} = \left(\sqrt[5]{\pi_1\pi_3^2}\mathcal{O}_{k\left(\sqrt[5]{\pi_1\pi_3^2}\right)}\right)^5$, hence $\mathcal{P}_1\mathcal{P}_3^2\mathcal{O}_{k\left(\sqrt[5]{\pi_1\pi_3^2}\right)} = \sqrt[5]{\pi_1\pi_3^2}\mathcal{O}_{k\left(\sqrt[5]{\pi_1\pi_3^2}\right)}$. Thus $\mathcal{P}_1\mathcal{P}_3^2$ seen in $\mathcal{O}_{k\left(\sqrt[5]{\pi_1\pi_3^2}\right)}$ becomes principal, i.e $[\mathcal{P}_1\mathcal{P}_3^2]$ capitulates in $k\left(\sqrt[5]{\pi_1\pi_3^2}\right)$.

By the same reasoning, we have $[\mathcal{P}_1\mathcal{P}_3^3]$ capitulates in $k\left(\sqrt[5]{\pi_1\pi_3^3}\right)$ and $[\mathcal{P}_1\mathcal{P}_3^4]$ capitulates in $k\left(\sqrt[5]{\pi_1\pi_3^4}\right)$.

We have $\mathcal{P}_1^5 = \pi_1\mathcal{O}_k$, then $\mathcal{P}_1\mathcal{O}_{k(\sqrt[5]{\pi_1})} = \sqrt[5]{\pi_1}\mathcal{O}_{k(\sqrt[5]{\pi_1})}$. Hence $[\mathcal{P}_1]$ capitulates in $k\left(\sqrt[5]{\pi_1}\right)$. By the same reasoning, we have $[\mathcal{P}_3]$ capitulates in $k\left(\sqrt[5]{\pi_3}\right)$.

(3) (i) If $(\frac{\pi_1}{\pi_3})_5 = 1$ and $K_6 = k\left(\sqrt[5]{\pi_1\pi_3^4}\right)$ we have $[\mathcal{P}_1\mathcal{P}_3^4]$ capitulates in K_6. According to [[4], Lemma 6.2], we have that $C_{k,5}^+ \simeq C_{\Gamma,5}$ and by class field theory $C_{\Gamma,5} \simeq Gal(\Gamma_5^{(1)}/\Gamma)$, then we obtain $C_{k,5}/C_{k,5}^- \simeq Gal(\Gamma_5^{(1)}/\Gamma) \simeq Gal(k\Gamma_5^{(1)}/k)$. Thus $k\Gamma_5^{(1)}$ is an unramified cyclic extension of k corresponds to $C_{k,5}^-$. We denote by $j_{k/\Gamma} : C_{\Gamma,5} \longrightarrow C_{k,5}$ the homomorphism induced by extension of ideals of Γ in k. Since $C_{k,5}^+ = \langle[\mathcal{P}_1\mathcal{P}_3]\rangle$ and $\mathcal{P}_1\mathcal{P}_3 = j_{k/\Gamma}(\mathcal{J})$ such that $C_{\Gamma,5} = \langle\mathcal{J}\rangle$, $[\mathcal{P}_1\mathcal{P}_3]$ capitulates in $K_6 = k\Gamma_5^{(1)}$. As $C_{k,5} = \langle[\mathcal{P}_1\mathcal{P}_3], [\mathcal{P}_1\mathcal{P}_3^4]\rangle$, then all classes capitulate in K_6.

We determine possible types of capitulation $(i_1, i_2, i_3, i_4, i_5, i_6)$. We have that $i_6 = 0$, $K_2 = K_5^{\tau^2}$, $K_3 = K_4^{\tau^2}$, then the same number of classes capitulate in K_2, K_5 and similarly for K_3, K_4.

If $i_1 \neq 0$ we have $i_1 = 1$, if $i_2 \neq 0$ we have $i_2 = 2$ and if $i_5 \neq 0$ we have $i_5 = 5$. i_3 and i_4 are both nulls or non nulls, so if i_3 and $i_4 \neq 0$, then $(i_3, i_4) = (3, 4)$ or $(4, 3)$. Thus the possible types of capitulation are:
$(0,0,0,0,0,0)$, $(1,0,0,0,0,0)$, $(0,2,0,0,5,0)$, $(1,2,0,0,5,0)$, $\{(0,0,3,4,0,0)$ or $(0,0,4,3,0,0)\}$, $\{(1,0,3,4,0,0)$ or $(1,0,4,3,0,0)\}$, $\{(0,2,3,4,5,0)$ or $(0,2,4,3,5,0)\}$, $\{(1,2,3,4,5,0)$ or $(1,2,4,3,5,0)\}$.

(ii) If $(\frac{\pi_1}{\pi_3})_5 = 1$ and $K_6 = k\left(\sqrt[5]{\pi_1\pi_3}\right)$ we have $[\mathcal{P}_1\mathcal{P}_3]$ capitulates in K_6, then if $i_6 \neq 0$ we have $i_6 = 1$. $[\mathcal{P}_1\mathcal{P}_3^4]$ capitulates in K_1, then if $i_1 \neq 0$ we have $i_1 = 6$, so the same possible types of capitulation occur as in (i) with $i_6 = 0$ or 1 and $i_1 = 0$ or 6.

(iii) If $(\frac{\pi_1}{\pi_3})_5 \neq 1$, by (1) we have $K_2 = k\left(\sqrt[5]{\pi_3}\right)$ and $K_5 = k\left(\sqrt[5]{\pi_1}\right)$ then the same possible types of capitulation occur as (i) and (ii) by permuting 2 and 5 in the given types of capitulation. □

3.2. The case $n = p^{e_1}q^{e_2} \equiv \pm1, \pm7 \pmod{25}$ where $p \not\equiv 1 \pmod{25}$, $q \not\equiv \pm7 \pmod{25}$.

Let $k = \Gamma(\zeta_5)$ be the normal closure of $\Gamma = \mathbb{Q}(\sqrt[5]{n})$, where $n = p^{e_1}q^{e_1} \equiv \pm1, \pm7 \pmod{25}$ such that $p \not\equiv 1, \pmod{25}$, $q \not\equiv \pm7 \pmod{25}$ and $e_1, e_2 \in \{1, 2, 3, 4\}$. By [5, Theorem 2.13], since

$q \equiv \pm 2 \pmod 5$ we have that q is inert in $k_0 = \mathbb{Q}(\zeta_5)$, so we set in the squel $q = \pi_5$ as prime in k_0.

By $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4$ and $\mathcal{P}_5$ we denote respectively the prime ideals of k above $\pi_1, \pi_2, \pi_3, \pi_4$ and π_5 in k_0, such that $\mathcal{P}_i^5 = \pi_i \mathcal{O}_k$ $(i = 1, 2, 3, 4, 5)$. We have that τ^2 permutes π_1 with π_3, then τ^2 permutes $\mathcal{P}_1$ with $\mathcal{P}_3$, moreover τ^2 sets $q = \pi_5$ and also $\mathcal{P}_5$.
The six intermediate extensions of $k_5^{(1)}/k$ are determined as follows:

Theorem 3.4. *Let $k, n, \pi_1, \pi_2, \pi_3, \pi_4$ and π_5 as above. Put $x_1 = \pi_1 \pi_5^{h_1}$ and $x_2 = \pi_1 \pi_3^4$ where $h_1 \in \{1, 2, 3, 4\}$ is chosen such that $x_1 \equiv x_2 \equiv 1 \pmod{\lambda^5}$, where $h_1 \in \{1, 2, 3, 4\}$. Then:*

(1) $k_5^{(1)} = k\left(\sqrt[5]{x_1}, \sqrt[5]{x_2}\right)$.

(2) *The six intermediate extensions of $k_5^{(1)}/k$ are:*
$k\left(\sqrt[5]{x_1}\right)$, $k\left(\sqrt[5]{x_2}\right)$, $k\left(\sqrt[5]{\pi_1\pi_3\pi_5^{2h_1}}\right)$, $k\left(\sqrt[5]{\pi_1^2\pi_3^4\pi_5^{h_1}}\right)$, $k\left(\sqrt[5]{\pi_1^4\pi_3^2\pi_5^{h_1}}\right)$ *and* $k\left(\sqrt[5]{\pi_3\pi_5^{h_1}}\right)$.
Furthermore τ^2 permutes $k\left(\sqrt[5]{\pi_1^2\pi_3^4\pi_5^{h_1}}\right)$ *with* $k\left(\sqrt[5]{\pi_1^4\pi_3^2\pi_5^{h_1}}\right)$ *and* $k\left(\sqrt[5]{x_1}\right)$ *with* $k\left(\sqrt[5]{\pi_3\pi_5^{h_1}}\right)$, *and sets* $k\left(\sqrt[5]{x_2}\right)$, $k\left(\sqrt[5]{\pi_1\pi_3\pi_5^{2h_1}}\right)$.

Proof. Since $k = k_0\left(\sqrt[5]{n}\right)$ we can write n in k_0 as $n = \pi_1^e\pi_2^e\pi_3^e\pi_4^e\pi_5$, with π_i do not all verified $\pi_i \equiv 1 \pmod{\lambda^5}$, because we have $p \not\equiv 1 \pmod{25}$. By Proposition 2.2 there exist $h_1, h_2 \in \{1, .., 4\}$ such that $\pi_1\pi_5^{h_1} \equiv \pm 1, \pm 7 \pmod{\lambda^5}$ and $\pi_1\pi_3^{h_2} \equiv \pm 1, \pm 7 \pmod{\lambda^5}$. To investigate the correspondence between the six intermediate extensions of $k_5^{(1)}/k$ and the six subgroups of $C_{k,5}$, we assume that $h_2 = 4$. Put $x_1 = \pi_1\pi_5^{h_1}$ and $x_2 = \pi_1\pi_3^4$.
(1) The fact that $k_5^{(1)} = k\left(\sqrt[5]{x_1}, \sqrt[5]{x_2}\right)$ follows from Proposition 2.2.
(2) The six intermediate extensions are: $k\left(\sqrt[5]{x_1}\right)$, $k\left(\sqrt[5]{x_2}\right)$, $k\left(\sqrt[5]{x_1x_2}\right)$, $k\left(\sqrt[5]{x_1x_2^2}\right)$, $k\left(\sqrt[5]{x_1x_2^3}\right)$ and $k\left(\sqrt[5]{x_1x_2^4}\right)$. Since $x_1 = \pi_1\pi_5^{h_1}$ and $x_2 = \pi_1\pi_5^4$, we have $k\left(\sqrt[5]{x_1x_2}\right) = k\left(\sqrt[5]{\pi_1^2\pi_3^4\pi_5^{h_1}}\right)$, $k\left(\sqrt[5]{x_1x_2^2}\right) = k\left(\sqrt[5]{\pi_1\pi_3\pi_5^{2h_1}}\right)$, $k\left(\sqrt[5]{x_1x_2^3}\right) = k\left(\sqrt[5]{\pi_1^4\pi_3^2\pi_5^{h_1}}\right)$ and $k\left(\sqrt[5]{x_1x_2^4}\right) = k\left(\sqrt[5]{\pi_3\pi_5^{h_1}}\right)$.
Since $\pi_1^{\tau^2} = \pi_3$, $\pi_3^{\tau^2} = \pi_1$ and $\pi_5^{\tau^2} = \pi_5$, and by the same reasoning as (2) of Theorem 3.1 we prove that τ^2 permutes $k\left(\sqrt[5]{\pi_1^2\pi_3^4\pi_5^{h_1}}\right)$ with $k\left(\sqrt[5]{\pi_1^4\pi_3^2\pi_5^{h_1}}\right)$ and $k\left(\sqrt[5]{x_1}\right)$ with $k\left(\sqrt[5]{\pi_3\pi_5^{h_1}}\right)$, and sets $k\left(\sqrt[5]{x_2}\right)$, $k\left(\sqrt[5]{\pi_1\pi_3\pi_5^{2h_1}}\right)$. □

The generators of $C_{k,5}$ in this case are determined as follows:

Theorem 3.5. *Let $k, n, \pi_1, \pi_2, \pi_3, \pi_4, \pi_5$ and h_1 as above. Let $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4$ and $\mathcal{P}_5$ prime ideals of k such that $\mathcal{P}_i^5 = \pi_i\mathcal{O}_{k_0}$ $(i = 1, 2, 3, 4, 5)$. Then:*

$$C_{k,5} = \langle [\mathcal{P}_1\mathcal{P}_3\mathcal{P}_5^{2h_1}], [\mathcal{P}_1\mathcal{P}_3^4] \rangle$$

Proof. According to [1, Theorem 1.1], for this case of the radicand n, we have that $\zeta_5^i(1+\zeta_5)^j$ is not norm of element in $k - \{0\}$ for any exponents i and j, then by [4, Section 5.3], we have $C_{k,5} = C_{k,5}^{(\sigma)} = C_{k,s}^{(\sigma)} = \langle [\mathcal{P}_1], [\mathcal{P}_2], [\mathcal{P}_3], [\mathcal{P}_4], [\mathcal{P}_5] \rangle$. Since $\mathcal{P}_1^{\tau^2} = \mathcal{P}_3$, $\mathcal{P}_2^{\tau^2} = \mathcal{P}_4$ and $\mathcal{P}_5^{\tau^2} = \mathcal{P}_5$, as the proof of Theorem 3.2 we have that $\mathcal{P}_1, \mathcal{P}_3$ and $\mathcal{P}_5$ are non principals. As $[\mathcal{P}_1\mathcal{P}_3\mathcal{P}_5^{2h_1}]^{\tau^2} = [(\mathcal{P}_1\mathcal{P}_3\mathcal{P}_5^{2h_1})^{\tau^2}] = [\mathcal{P}_3\mathcal{P}_1\mathcal{P}_5^{2h_1}] = [\mathcal{P}_1\mathcal{P}_3\mathcal{P}_5^{2h_1}]$ then $C_{k,5}^{+} = \langle [\mathcal{P}_1\mathcal{P}_3\mathcal{P}_5^{2h_1}] \rangle$, and we have that $C_{k,5}^{-} = \langle [\mathcal{P}_1\mathcal{P}_3^4] \rangle$. Hence $C_{k,5} = \langle [\mathcal{P}_1\mathcal{P}_3\mathcal{P}_5^{2h_1}], [\mathcal{P}_1\mathcal{P}_3^4] \rangle$. □

The main theorem of capitulation in this case is as follows:

Theorem 3.6. *We keep the same assumptions as Theorem 3.5. Then:*
(1) $K_1 = k\left(\sqrt[5]{\pi_1\pi_3^4}\right)$ *or* $k\left(\sqrt[5]{\pi_1\pi_3\pi_5^{2h_1}}\right)$, $K_2 = k\left(\sqrt[5]{\pi_1\pi_5^{h_1}}\right)$ *or* $k\left(\sqrt[5]{\pi_3\pi_5^{h_1}}\right)$, $K_3 = k\left(\sqrt[5]{\pi_1^2\pi_3^4\pi_5^{h_1}}\right)$ *or* $k\left(\sqrt[5]{\pi_1^4\pi_3^2\pi_5^{h_1}}\right)$, $K_4 = k\left(\sqrt[5]{\pi_1^4\pi_3^2\pi_5^{h_1}}\right)$ *or* $k\left(\sqrt[5]{\pi_1^2\pi_3^4\pi_5^{h_1}}\right)$, $K_5 = k\left(\sqrt[5]{\pi_3\pi_5^{h_1}}\right)$ *or* $k\left(\sqrt[5]{\pi_1\pi_5^{h_1}}\right)$ *and* $K_6 = k\left(\sqrt[5]{\pi_1\pi_3\pi_5^{2h_1}}\right)$ *or* $k\left(\sqrt[5]{\pi_1\pi_3^4}\right)$.
(2) $[\mathcal{P}_1\mathcal{P}_3\mathcal{P}_5^{2h_1}]$ *capitulates in* $k\left(\sqrt[5]{\pi_1\pi_3\pi_5^{2h_1}}\right)$, $[\mathcal{P}_1\mathcal{P}_5^{h_1}]$ *capitulates in* $k\left(\sqrt[5]{\pi_1\pi_5^{h_1}}\right)$, $[\mathcal{P}_1^2\mathcal{P}_3^4\mathcal{P}_5^{h_1}]$ *capitulates in* $k\left(\sqrt[5]{\pi_1^2\pi_3^4\pi_5^{h_1}}\right)$, $[\mathcal{P}_1^4\mathcal{P}_3^2\mathcal{P}_5^{h_1}]$ *capitulates in* $k\left(\sqrt[5]{\pi_1^4\pi_3^2\pi_5^{h_1}}\right)$, $[\mathcal{P}_3\mathcal{P}_5^{h_1}]$ *capitulates in* $k\left(\sqrt[5]{\pi_3\pi_5^{h_1}}\right)$ *and* $[\mathcal{P}_1\mathcal{P}_3^4]$ *capitulates in* $k\left(\sqrt[5]{\pi_1\pi_3^4}\right)$.
(3) *If* $K_1 = k\left(\sqrt[5]{\pi_1\pi_3\pi_5^{2h_1}}\right)$, *then the possible types of capitulation are:*
$(0,0,0,0,0,0)$, $(1,0,0,0,0,0)$, $\{(0,5,0,0,2,0)$ *or* $(0,2,0,0,5,0)\}$,
$\{(1,5,0,0,2,0)$ *or* $(1,2,0,0,5,0)\}$, $\{(0,5,4,3,2,0)$ *or* $(0,2,4,3,5,0)\}$,
$\{(1,5,4,3,2,0)$ *or* $(1,2,4,3,5,0)\}$,
$\{(0,5,3,4,2,0)$ *or* $(0,2,3,4,5,0)\}$, $\{(1,5,3,4,2,0)$ *or* $(1,2,3,4,5,0)\}$,
$\{(0,0,3,4,0,0)$ *or* $(0,0,4,3,0,0)\}$, $\{(1,0,3,4,0,0)$ *or* $(1,0,4,3,0,0)\}$.
If $K_1 = k\left(\sqrt[5]{\pi_1\pi_3^4}\right)$, *then the same possible types occur, where* i_6 *takes the value 0 or 1.*

Proof. (1) According to Theorem 3.4, we have that τ^2 permutes $k\left(\sqrt[5]{\pi_1^2\pi_3^4\pi_5^{h_1}}\right)$ with $k\left(\sqrt[5]{\pi_1^4\pi_3^2\pi_5^{h_1}}\right)$ and $k\left(\sqrt[5]{x_1}\right)$ with $k\left(\sqrt[5]{\pi_3\pi_5^{h_1}}\right)$, and sets $k\left(\sqrt[5]{x_2}\right)$, $k\left(\sqrt[5]{\pi_1\pi_3\pi_5^{2h_1}}\right)$. We determine first the six subgroups H_i of $C_{k,5}$. We have that $C_{k,5} = \langle \mathcal{A}, \mathcal{X}\rangle$, where $H_1 = C_{k,5}^+ = \langle\mathcal{A}\rangle$ and $H_6 = C_{k,5}^- = \langle\mathcal{X}\rangle$. By Theorem 3.5 we have $\mathcal{A} = [\mathcal{P}_1\mathcal{P}_3\mathcal{P}_5^{2h_1}]$ and $\mathcal{X} = [\mathcal{P}_1\mathcal{P}_3^4]$, then $\mathcal{A}\mathcal{X} = [\mathcal{P}_1\mathcal{P}_5^{h_1}]^2$, $\mathcal{A}\mathcal{X}^2 = [\mathcal{P}_1^2\mathcal{P}_3^4\mathcal{P}_5^{h_1}]^4$, $\mathcal{A}\mathcal{X}^3 = [\mathcal{P}_1^4\mathcal{P}_3^2\mathcal{P}_5^{h_1}]$ and $\mathcal{A}\mathcal{X}^4 = [\mathcal{P}_3\mathcal{P}_5^{h_1}]^3$. Hence $H_2 = \langle[\mathcal{P}_1\mathcal{P}_5^{h_1}]\rangle$, $H_3 = \langle[\mathcal{P}_1^2\mathcal{P}_3^4\mathcal{P}_5^{h_1}]\rangle$, $H_4 = \langle[\mathcal{P}_1^4\mathcal{P}_3^2\mathcal{P}_5^{h_1}]\rangle$ and $H_5 = \langle[\mathcal{P}_3\mathcal{P}_5^{h_1}]\rangle$.
Since τ^2 sets $k\left(\sqrt[5]{\pi_1\pi_3\pi_5^{2h_1}}\right)$ and $k\left(\sqrt[5]{\pi_1\pi_3^4}\right)$, so if $K_1 = k\left(\sqrt[5]{\pi_1\pi_3\pi_5^{2h_1}}\right)$ then $K_6 = k\left(\sqrt[5]{\pi_1\pi_3^4}\right)$ and inversely.

By class field theory, the fact that $H_i\,(i=2,5)$ corresponds to $K_i\,(i=2,5)$ means that $\mathcal{P}_1\mathcal{P}_5^{h_1}$ splits completely in K_2 and $\mathcal{P}_3\mathcal{P}_5^{h_1}$ splits completely in K_5. As $\pi_1\pi_5^{h_1}$ divides $\pi_1^2\pi_3^4\pi_5^{h_1}$ and $\pi_1^4\pi_3^2\pi_5^{h_1}$, by Proposition 2.1, $\pi_1\pi_5^{h_1}$ can not split in $k_0(\sqrt[5]{\pi_1^2\pi_3^4\pi_5^{h_1}})$ and $k_0(\sqrt[5]{\pi_1^4\pi_3^2\pi_5^{h_1}})$, this equivalent to say that $\mathcal{P}_1\mathcal{P}_5^{h_1}$ can not split completely in $k\left(\sqrt[5]{\pi_1^2\pi_3^4\pi_5^{h_1}}\right)$ and $k\left(\sqrt[5]{\pi_1^4\pi_3^2\pi_5^{h_1}}\right)$. By the same reasoning we have that $\mathcal{P}_3\mathcal{P}_5^{h_1}$ can not split completely in $k\sqrt[5]{\pi_1^2\pi_3^4\pi_5^{h_1}}$ and $k\sqrt[5]{\pi_1^4\pi_3^2\pi_5^{h_1}}$. Thus if $K_2 = k\left(\sqrt[5]{\pi_1\pi_5^{h_1}}\right)$ then $K_5 = k\left(\sqrt[5]{\pi_3\pi_5^{h_1}}\right)$ and inversely, which allow us to deduce that if $K_3 = k\left(\sqrt[5]{\pi_1^2\pi_3^4\pi_5^{h_1}}\right)$ then $K_5 = k\left(\sqrt[5]{\pi_1^4\pi_3^2\pi_5^{h_1}}\right)$ and inversely.
(2) We keep the same reasoning as the proof of (2) Theorem 3.3.
(3) If $K_1 = k\left(\sqrt[5]{\pi_1\pi_3\pi_5^{2h_1}}\right)$, then $K_6 = k\Gamma_5^{(1)} = k\left(\sqrt[5]{\pi_1\pi_3^4}\right)$ and we have that $[\mathcal{P}_1\mathcal{P}_3^4]$ capitulates in K_6, moreover since $C_{k,5}^+ = \langle[\mathcal{P}_1\mathcal{P}_3\mathcal{P}_5^{2h_1}]\rangle \simeq C_{\Gamma,5}$ $\mathcal{P}_1\mathcal{P}_3\mathcal{P}_5^{2h_1} = j_{k/\Gamma}(\mathcal{J})$ such

that $C_{\Gamma,5} = \langle \mathcal{J} \rangle$, then $[\mathcal{P}_1\mathcal{P}_3\mathcal{P}_5^{2h_1}]$ capitulates in K_6. As $C_{k,5} = \langle [\mathcal{P}_1\mathcal{P}_3\mathcal{P}_5^{2h_1}], [\mathcal{P}_1\mathcal{P}_3^4] \rangle$, then all classes capitulate in $K_6 = k\left(\sqrt[5]{\pi_1\pi_3^4}\right)$. We determine the possible types of capitulation $(i_1, i_2, i_3, i_4, i_5, i_6)$.
We have that $i_6 = 0$, $K_2 = K_5^{\tau^2}$, $K_3 = K_4^{\tau^2}$, then the same number of classes capitulate in K_2, K_5 and similarly for K_3, K_4. If $i_1 \neq 0$ we have $i_1 = 1$. i_2 and i_5 are both nulls or non nulls, so if i_2 and $i_5 \neq 0$, then $(i_2, i_5) = (2,5)$ or $(5,2)$ depending on $\mathcal{P}_1\mathcal{P}_5^{h_1}$ splits completely in $k\left(\sqrt[5]{\pi_1\pi_5^{h_1}}\right)$ or in $k\left(\sqrt[5]{\pi_3\pi_5^{h_1}}\right)$. Similarly if i_3 and $i_4 \neq 0$, then $(i_3, i_4) = (3,4)$ or $(4,3)$. Hence the possible types given are proved.

If $K_1 = k\left(\sqrt[5]{\pi_1\pi_3^4}\right)$ then $K_6 = k\Gamma_5^{(1)} = k\left(\sqrt[5]{\pi_1\pi_3\pi_5^{2h_1}}\right)$ and we have $C_{k,5}^{+} = \langle [\mathcal{P}_1\mathcal{P}_3\mathcal{P}_5^{2h_1}] \rangle$ capitulates in K_6, the possible values of i_2, i_3, i_4, i_5 are as above, $(i_2, i_5) = (2,5)$ or $(5,2)$ if they are non nulls, $(i_3, i_4) = (3,4)$ or $(4,3)$ if they are non nulls. If $i_1 \neq 0$ then $i_1 = 6$ because $H_6 = \langle [\mathcal{P}_1\mathcal{P}_3^4] \rangle$, and if $i_6 \neq 0$ then $i_1 = 1$ because $H_1 = \langle [\mathcal{P}_1\mathcal{P}_3\mathcal{P}_5^{2h_1}] \rangle$. Hence the possible types given are proved. □

3.3. The case $n = 5^{e_1}p^{e_2}$ where $p \not\equiv 1 \pmod{25}$.

Let $k = \Gamma(\zeta_5)$ be the normal closure of $\Gamma = \mathbb{Q}(\sqrt[5]{n})$, where $n = 5^{e_1}p^{e_2}$ such that $p \not\equiv 1, \pmod{25}$ and $e_1, e_2 \in \{1,2,3,4\}$. By [4, Lemma 5.1], since $n = 5^{e_1}p^{e_2} \not\equiv \pm 1, \pm 7, \pmod{25}$ we have $\lambda = 1 - \zeta_5$ is ramified in k/k_0.

Let π_1, π_2, π_3 and π_4 primes of k_0 such that $p = \pi_1\pi_2\pi_3\pi_4$. Let $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4$ and $\mathcal{I}$ prime ideals of k above $\pi_1, \pi_2, \pi_3, \pi_4$ and λ, we have $\mathcal{P}_i^5 = \pi_i\mathcal{O}_k$ and $\mathcal{I}^5 = \lambda\mathcal{O}_k$. According to [1, Theorem 1.1], for this case of the radicand n, we have that $\zeta_5^i(1+\zeta_5)^j$ is not norm of element in $k - \{0\}$ for any exponents i and j, then we have $C_{k,5} = C_{k,5}^{(\sigma)} = C_{k,s}^{(\sigma)}$. Hence the results about the six intermediate extensions of $k_5^{(1)}/k$, the generators of $C_{k,5}$ and the capitulation problem in this case are the same as case 2 by substituting q by 5, π_5 by λ and $\mathcal{P}_5$ by $\mathcal{I}$.

4. Numerical examples

The task to determine the capitulation in a cyclic quintic extension of a base field of degree 20, that is, in a field of absolute degree 100, is definitely far beyond the reach of computational algebra systems like MAGMA and Pari/GP. For this reason we give examples of pure metacyclic fields $k = \mathbb{Q}(\sqrt[5]{n}, \zeta_5)$ such that $C_{k,5}$ is of type $(5,5)$ and $C_{k,5} = C_{k,5}^{(\sigma)}$.

Table 1: $k = \mathbb{Q}(\sqrt[5]{n}, \zeta_5)$ with $C_{k,5}$ of type $(5,5)$ and $C_{k,5} = C_{k,5}^{(\sigma)}$.

No	n	Factorization	$n(\mathrm{mod}\,25)$	Section	$C_{k,5}$	$C_{k,5}^{(\sigma)}$
1	55	$5\,.\,11$	$+5$	3.3	$(5,5)$	2
2	82	$2\,.\,41$	$+7$	3.2	$(5,5)$	2
3	93	$3\,.\,31$	-7	3.2	$(5,5)$	2
4	99	$3^2\,.\,11$	-1	3.2	$(5,5)$	2
5	124	$2^2\,.\,31$	-1	3.2	$(5,5)$	2
6	143	$11\,.\,13$	-7	3.2	$(5,5)$	2
7	151	151	$+1$	3.1	$(5,5)$	2
8	176	$2^4\,.\,11$	$+1$	3.2	$(5,5)$	2
9	205	$5\,.\,41$	$+5$	3.3	$(5,5)$	2
10	251	251	$+1$	3.1	$(5,5)$	2
11	355	$5\,.\,71$	$+5$	3.3	$(5,5)$	2
12	382	$2\,.\,191$	$+7$	3.2	$(5,5)$	2
13	393	$3\,.\,131$	-7	3.2	$(5,5)$	2
14	407	$11\,.\,37$	$+7$	3.2	$(5,5)$	2
15	524	$2^2\,.\,131$	-1	3.2	$(5,5)$	2
16	543	$3\,.\,181$	-7	3.2	$(5,5)$	2
17	568	$2^3\,.\,71$	-7	3.2	$(5,5)$	2
18	601	601	$+1$	3.1	$(5,5)$	2
19	605	$5\,.\,11^2$	$+5$	3.3	$(5,5)$	2
20	655	$5\,.\,131$	$+5$	3.3	$(5,5)$	2
21	724	$2^2\,.\,181$	-1	3.2	$(5,5)$	2
22	905	$5\,.\,181$	$+5$	3.3	$(5,5)$	2
23	943	$23\,.\,41$	-7	3.2	$(5,5)$	2
24	976	$2^4\,.\,61$	$+1$	3.2	$(5,5)$	2
25	982	$2\,.\,491$	$+7$	3.2	$(5,5)$	2
26	993	$3\,.\,331$	-7	3.2	$(5,5)$	2
27	1051	1051	$+1$	3.1	$(5,5)$	2
28	1301	1301	$+1$	3.1	$(5,5)$	2
29	1457	$31\,.\,47$	$+7$	3.2	$(5,5)$	2
30	1555	$5\,.\,311$	$+5$	3.3	$(5,5)$	2
31	1775	$5^2\,.\,71$	0	3.3	$(5,5)$	2
32	1801	1801	$+1$	3.1	$(5,5)$	2
33	1901	1901	$+1$	3.1	$(5,5)$	2
34	2155	$5\,.\,431$	$+5$	3.3	$(5,5)$	2
35	6943	$53\,.\,131$	-7	3.2	$(5,5)$	2
36	8275	$5^2\,.\,331$	0	3.3	$(5,5)$	2
37	8507	$47\,.\,181$	$+7$	3.2	$(5,5)$	2
38	12707	$97\,.\,131$	$+7$	3.2	$(5,5)$	2
39	30125	$5^3\,.\,241$	0	3.3	$(5,5)$	2

Acknowledgement. The authors would like to thank the referee for the valuable suggestions and comments.

References

[1] A. Azizi, F. Elmouhib and M. Talbi, *5-rank of ambiguous class groups of quintic Kummer extensions*, Proc. Indian Acad. Sci. Math. Sci., 132(12) (2022), 14 pp.

[2] F. Elmouhib, M. Talbi and A. Azizi, *On the capitulation problem of some pure metacyclic fields of degree 20.*, Palest. J. Math., 11(1) (2022), 260-267.

[3] E. Hecke, Lectures on the Theory of Algebraic Numbers, Graduate Texts in Mathematics, 77, Springer-Verlag, New York-Berlin, 1981.

[4] M. Kulkarni, D. Majumdar and B. Sury, *l-class groups of cyclic extension of prime degree l*, J. Ramanujan Math. Soc., 30(4) (2015), 413-454.

[5] L. C. Washington, Introduction to Cyclotomic Fields, Graduate Texts in Mathematics, 83, Springer-Verlag, New-York, 1982.

[6] The PARI Group, PARI/GP, Version 2.4.9, Bordeaux, 2017, http://pari.math.u-bordeaux.fr

Fouad Elmouhib (Corresponding Author) and **Abdelmalek Azizi**
Department of Mathematics and Computer Sciences
Faculty of Science
Mohammed First University
Oujda, Morocco
e-mails: fouad.cd@gmail.com (F. Elmouhib)
abdelmalekazizi@yahoo.fr (A. Azizi)

Mohamed Talbi
Regional Center of Professions of Education and Training
Oujda, Morocco
e-mail: ksirat1971@gmail.com