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The Possibilistic Mean-Variance Model with Uncertain Possibility Distributions

Furkan GÖKTAŞ[1]

1. Asst. Prof. Dr., Karabük University, furkangoktas@karabuk.edu.tr, https://orcid.org/000-0001-9291-3912

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Abstract

The possibilistic mean–variance (MV) model is the counterpart of Markowitz's MV model in the possibility theory. This study aims to examine the possibilistic MV model when the possibility distributions of stock returns are uncertain triangular fuzzy numbers. We define an uncertainty vector and use its ellipsoidal uncertainty set in a minimax optimization problem to model this uncertainty. We also show that this minimax optimization problem reduces to a strictly convex minimization problem. Thus, unlike the possibilistic MV model, we get diversified optimal portfolios uniquely with our approach. After laying down the theoretical points of our approach, we illustrate it with a real-world example in the literature by using a software package for convex optimization. To the best of our knowledge, this is the first paper that considers uncertain possibility distributions in the possibilistic MV model.

Keywords: Convex Optimization, Fuzzy Set, Normal Distribution, Portfolio Selection, Possibility Theory, Triangular Fuzzy Number.

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1. INTRODUCTION

Although Markowitz's MV model introduced by Markowitz (1952) has deeply influenced the portfolio theory, it is not preferred in practice due to estimation errors (Goldfarb & Iyengar, 2003). Estimating the mean vector is a more significant problem (DeMiguel et al., 2009; Garlappi et al., 2007). Hence, the robust MV model is proposed where this vector belongs to an uncertainty set (Garlappi et al., 2007). When the ellipsoidal uncertainty set is used, this model mainly provides that an efficient portfolio approaches the minimum variance portfolio (Garlappi et al., 2007). In addition, the box-type uncertainty set ignores the correlation matrix of variables. A Bayesian approach introduced by Jorion (1986) can be used to minimize the impact of estimation errors by using a common value. This model mainly provides that the efficient portfolio approaches the minimum variance portfolio if the common value is chosen as the mean of the minimum variance portfolio (Garlappi et al., 2007). Ding's minimax portfolio selection model depends on minimizing the maximum loss when the stock returns vector's joint probability distribution is an elliptical distribution as in the models mentioned above (Ding, 2006). However, the literature questions this assumption (Chicheportiche & Bouchaud, 2012).

Young's minimax portfolio selection model is another alternative to Markowitz's MV model (Young, 1998). It is related to the game theory. It depends on past data without the assumption of elliptical distribution. However, the past data may not contain necessary information for the future. The possibilistic MV model is a notable alternative to Markowitz's MV model (Carlsson et al., 2002). This model enables us to model the imprecise probability and use expert knowledge. Here, possibility distributions are used to model the uncertainty of the stock returns. This model and its solution are examined in many papers, such as Taş et al. (2016), Zhang et al. (2009), Corazza and Nardelli (2019), Göktaş and Duran (2020a). There are also many studies in which possibilistic models are used, such as Zhang (2007), Li et al. (2015), Göktaş & Duran (2019), Göktaş & Duran (2020b), Sui et al. (2020a), Sui et al. (2020b), Hu et al. (2021), Huang et al. (2022), Deng & Lin (2022), Göktaş (2023), Saki et al. (2023), Guillaume at al. (2024). Some of these models are reviewed by Zhang et al. (2018), Mandal & Thakur (2024). On the other hand, to the best of our knowledge, no study considers uncertain possibility distributions in this model. Though it is assumed that the possibility distributions are exactly known in the literature of possibilistic portfolio optimization, this assumption is generally invalid (Hu et al., 2021). Thus, the primary motivation of our paper is to consider not only the uncertainty of stock returns but also the uncertainty of possibility distributions in the portfolio selection.

To fill this gap in the literature, we examine the possibilistic MV model when the possibility distributions are uncertain triangular fuzzy numbers. We define an uncertainty vector and use its ellipsoidal uncertainty set in a minimax optimization problem to model the uncertainty. We also show that this minimax optimization problem reduces to a strictly convex minimization problem. Thus, unlike the possibilistic MV model, we get diversified optimal portfolios uniquely. Furthermore, we derive theoretical results about our approach based on the possibilistic portfolio optimization and robust (worst-case) portfolio optimization frameworks. Our paper's originality comes from being the first paper that considers uncertain possibility distributions in the possibilistic MV model.

We organize the rest of the paper as follows. Section 2 gives theoretical points of the possibilistic MV model and our approach under certain assumptions. Section 3 illustrates our approach with the real-world example in the literature using CVX, a MATLAB software for convex optimization. (See Grant and Boyd (2008) for further information.) We conclude the paper with Section 4.

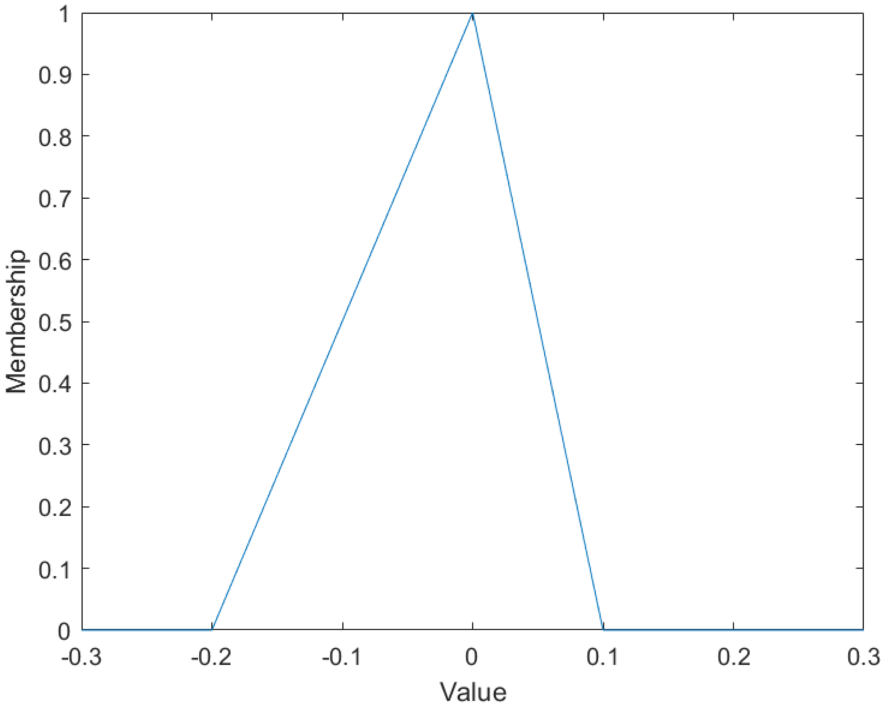
2. METHODS

In this section, we consider the portfolio selection problem where x is the weight vector of the portfolio and its feasible set (F) is as below, like Carlsson et al. (2002). There are at most n stocks in the portfolio and x_i is the weight of the i^{th} stock.

$$F = \left\{ x : \sum_{i=1}^{n} x_i = 1 \wedge x_i \geq 0, \forall i \right\} \tag{1}$$

The possibility theory is related to the fuzzy sets and probability theories. It gives the lower and upper bounds for the imprecise probability. Possibility distributions are usually given with fuzzy numbers (Dubois, 2006; Souliotis et al., 2022). We assume that the possibility distributions of stock returns are given with triangular fuzzy numbers. The graph of the membership function of triangular fuzzy number (-0.2, 0, 0.1) is given in Figure 1.

Figure 1. The graph of the membership function of the triangular fuzzy number



When the i^{th} stock return's possibility distribution is (a_i, b_i, c_i), the possibilistic mean and standard deviation of portfolio return are found as below, respectively (Carlsson et al., 2002). a_i, b_i and

c_i can be taken as equal to the sample minimum, mean and maximum statistics, respectively (Taş et al., 2016). The second expression in (2) implies that the possibilistic correlation matrix consists of only ones under the triangular fuzzy numbers assumption (Corazza & Nardelli, 2019).

$$PM(x) = \sum_{i=1}^{n} x_i \left(\frac{a_i + 4b_i + c_i}{6} \right)$$
$$PSD(x) = \sum_{i=1}^{n} x_i \left(\frac{c_i - a_i}{2\sqrt{6}} \right) \tag{2}$$

For several reasons, we prefer to use triangular fuzzy numbers rather than trapezoidal fuzzy numbers. Firstly, it is more intuitive to use triangular fuzzy numbers since their parameters correspond to the worst-case, base-case and best-case predictions for the stock returns, respectively. Secondly, it is more practical since we can use linear or convex optimization tools in this case. We note that the weighted sum method gives the exact efficient frontier in this case. Thirdly, the possibilistic correlation depends on the cores and supports of trapezoidal fuzzy numbers (Corazza & Nardelli, 2019). However, this may cause misleading results since these are not related to the correlation structure of the stock returns vector. On the other hand, the use of triangular fuzzy numbers corresponds to the uncertain correlation structure of the stock returns vector. Thus, the possibilistic MV model considers the worst-case scenario.

Based on (2) and the weighted sum method, the possibilistic MV model can be given with the following linear minimization problem as in this study where β is in [0,1] (Göktaş & Duran, 2019; Göktaş & Duran, 2020a). In the portfolio selection problem, β (1-β) is the weight of the first (second) objective, which is to maximize (minimize) the possibilistic mean (standard deviation). When β is equal to 1, the investor is risk-neutral. When β decreases, the investor's degree of risk aversion increases. (If the investor were a risk lover, the second objective would be maximizing the possibilistic standard deviation instead of minimizing it.)

$$\min_{x \in F} (-\beta) PM(x) + (1-\beta) PSD(x) \tag{3}$$

The maximum performance portfolio (MPP) can be defined as below, where P(x) is the performance function. It is also an optimal solution of (3) for the unknown β value.

$$\max_{x \in F} \left(P(x) := \frac{PM(x)}{PSD(x)} \right) \tag{4}$$

The optimal solution of (3) is found analytically (Göktaş & Duran, 2019; Göktaş & Duran, 2020a). If the objective function is minimum for only one stock, then the optimal portfolio (OP) consists of only this stock. If the objective function is minimum for multiple stocks, then OPs consist of their convex combinations. In this case, alternative optimal solutions exist (Göktaş & Duran, 2019; Göktaş & Duran, 2020a). Similar results are also valid for MPP. (When f(x)=0 in (14), MPP is found. Due to the Fundamental Theorem of Linear Programming, either the optimal solution is a corner point of the feasible set, i.e. a unique stock, or alternative optimal solutions exist.)

In this study, we assume that the i^{th} stock return's possibility distribution is equal to ($a_i+\Theta_i$, $b_i+\Theta_i$, $c_i+\Theta_i$) where a_i, b_i and c_i are determined based on the past data like in Taş et al. (2016). Θ_i is the uncertainty parameter for the i^{th} stock return and the uncertainty vector's (Θ) joint probability distribution is normal. We also assume that its mean vector (μ_Θ) and positive definite covariance matrix ($\sum_\Theta$) are determined based on expert knowledge. Its α-confidence region is found as below, where k_α is the square root of the α-quantile of the chi-square distribution with n degrees of freedom (Ding, 2006; Studer, 1999). (5) is an ellipsoidal uncertainty set with the center μ_Θ. When an element of μ_Θ is positive (negative), the expert is optimistic (pessimistic) about the future return of the corresponding stock with respect to its past returns. $\sum_\Theta$ is related to the variability of Θ with respect to these beliefs (μ_Θ) since $\sum_\Theta$ determines the Mahalanobis distance between Θ and μ_Θ (Breuer, 2006; Breuer & Csiszar, 2013). Here, Θ^T is the transpose of the Θ vector and $\sum^{-1}$ is the inverse of the $\sum$ matrix.

$$S=\left\{\theta:\left(\theta-\mu_\theta\right)^T \Sigma_\theta^{-1}\left(\theta-\mu_\theta\right)\le k_\alpha^2\right\} \tag{5}$$

Based on (2) and our assumption that $X_i \sim (a_i+\Theta_i,\ b_i+\Theta_i,\ c_i+\Theta_i)$, we find the possibilistic mean and standard deviation of portfolio return as below. That is, the possibilistic standard deviation remains the same due to the translation invariance property (Carlsson & Fuller, 2001).

$$\begin{aligned}
PM\left(x,\theta\right)&=\sum_{i=1}^{n}x_i\left(\frac{\left(a_i+\theta_i\right)+4\left(b_i+\theta_i\right)+\left(c_i+\theta_i\right)}{6}\right)\\
&=\sum_{i=1}^{n}x_i\left(\frac{a_i+4b_i+c_i}{6}\right)+\sum_{i=1}^{n}x_i\frac{6\theta_i}{6}=PM\left(x\right)+\sum_{i=1}^{n}x_i\theta_i\\
PSD\left(x,\theta\right)&=\sum_{i=1}^{n}x_i\left(\frac{\left(c_i+\theta_i\right)-\left(a_i+\theta_i\right)}{2\sqrt{6}}\right)\\
&=\sum_{i=1}^{n}x_i\left(\frac{c_i-a_i}{2\sqrt{6}}\right)=PSD\left(x\right)
\end{aligned} \tag{6}$$

Based on (6) and the weighted sum method, we define the possibilistic MV model with uncertain possibility distributions as below, where β is again in [0,1], F is defined in (1) and S is defined in (5).

$$\min_{x\in F}\max_{\theta\in S}\left(-\beta\right)\left[PM\left(x\right)+\sum_{i=1}^{n}x_i\theta_i\right]+\left(1-\beta\right)PSD\left(x\right) \tag{7}$$

This minimax optimization problem is equivalent to the following convex minimization problem. (8) is strictly convex when β is not equal to 0. (8) reduces to (3) when β is equal to 0.

$$\min_{x\in F}\left[\left(-\beta\right)PM\left(x\right)+\left(1-\beta\right)PSD\left(x\right)-\beta\min_{\theta\in S}\sum_{i=1}^{n}x_i\theta_i\right] \tag{8}$$

We define the following minimization problem over the ellipsoidal uncertainty set. We know that (9) is related to the risk measure called Maximum Loss (ML) (Studer, 1999). Due to the scenario analysis approach, we can say that ML considers the most severe scenario from the plausible scenario set (S) (Breuer, 2006).

$$\min_{\theta \in S} \sum_{i=1}^{n} x_i \theta_i \tag{9}$$

The optimal solution of (9) is found as below by using the Karush-Kuhn-Tucker (KKT) conditions (Ding, 2006; Studer, 1999). As in this study, it is the most severe scenario in S and can be called the worst-case scenario.

$$\theta^*(x) = \mu_\theta - \frac{k_\alpha}{\sqrt{x^T \Sigma_\theta x}} \Sigma_\theta x \tag{10}$$

Then, the optimal value of an objective function of (9) is found as below, where E() and var() are the expectation and variance operators in the probability theory, respectively (Ding, 2006; Studer, 1999).

$$f(x) := x^T \mu_\theta - k_\alpha \sqrt{x^T \Sigma_\theta x} = E\left(x^T \theta\right) - k_\alpha \sqrt{\mathrm{var}\left(x^T \theta\right)} \tag{11}$$

Furthermore, we find that (7) is equivalent to the following convex minimization problem. We call its optimal solution as the worst-case optimal portfolio (WOP).

$$\min_{x \in F} (-\beta)\left[PM(x) + f(x)\right] + (1-\beta) PSD(x) \tag{12}$$

We define the maximum worst-case performance portfolio (MWPP) as below where WCP(x) is the worst-case performance function. It is also an optimal solution of (12) for the unknown β value.

$$\max_{x \in F} \left(WCP(x) := \frac{PM(x) + f(x)}{PSD(x)} \right) \tag{13}$$

When at least one portfolio’s worst-case performance (WCP) is positive, we can find MWPP by standardizing the optimal solution of (14) since the objective function of (13) is a homogeneous function of x (Goldfarb & Iyengar, 2003; Tütüncü & Koenig, 2004). Clearly, (14) is a strictly convex minimization problem. To solve (12) and (14), the CVX/MOSEK solver can be used in MATLAB as in this study (Grant & Boyd, 2008).

$$\begin{aligned} &\min \left(-PM(x) - f(x)\right) \\ &s.t.\ PSD(x) = 1 \\ &\quad x_i \geq 0, \forall i \end{aligned} \tag{14}$$

Remark: In this study, we assume that the joint probability distribution of Θ is normal. Anyone can generalize the theoretical results derived in this section for the other elliptical distributions with finite moments. When another elliptical distribution is used, k_α is replaced by another positive constant c_α. Everything remains the same except for this (Breuer, 2006). See Embrechts et al. (2002) for further information about the elliptical distributions.

Remark: The possibilistic MV model (with uncertain possibility distributions) considers the asymmetry in the past data like Young's minimax model, unlike the other models mentioned in this study. The elliptical distribution of the stock returns vector (uncertainty vector) is determined based on the expert knowledge in Ding's minimax model (in our approach). We believe that modeling the uncertainty vector is simpler. That is, the ease of use of our approach is higher.

When there are no restrictions that x_i is nonnegative in (1), the PM(x) and PSD(x) functions are found as below (Carlsson & Fuller, 2001; Carlsson et al., 2002). In this case, the possibilistic MV model does not give a bounded optimal solution (Corazza & Nardelli, 2019). On the other hand, our approach may give a bounded optimal solution due to f(x) given in (11). It may be applicable, unlike this model, even if there are no restrictions on the short positions. We note that (12) and (14) remain convex minimization problems because the absolute value and sum operations preserve the convexity. (The first constraint of (14) changes to PSD(x)≤1, but it is always tight at an optimal solution due to the risk-return trade-off.)

$$
\begin{aligned}
PM(x) &= \sum_{i=1}^{n} x_i \left(\frac{a_i + 4b_i + c_i}{6} \right) \\
PSD(x) &= \sum_{i=1}^{n} |x_i| \left(\frac{c_i - a_i}{2\sqrt{6}} \right)
\end{aligned}
\tag{15}
$$

It is obvious that when the joint probability distribution of Θ is another elliptical distribution with the same first two moments, we have the same WOP for a different confidence level than α. Using Proposition 1 of Breuer and Csiszar (2013), we can show that a similar result is also valid for μ_Θ. Let the reference distribution be $N(\mu_\Theta,\sum_\Theta)$. Then, using the relative entropy constraint, the worst-case joint probability distribution of Θ is found as below for $E(x^T\Theta)$, where d is the upper bound for the distance to the reference distribution.

$$
\theta \sim N\left(\mu_\theta - \sqrt{2d}\,\frac{\sum_\theta x}{\sqrt{x^T \sum_\theta x}}, \sum_\theta \right) \tag{16}
$$

By using (16) instead of $N(\mu_\Theta,\sum_\Theta)$, we find f(x) as below for the given confidence level c. By comparing (11) and (17), we see that the same WOPs are derived when the scalars before the square root expressions are equal. Based on this information, we remark that it is more essential to determine Σ_Θ. Thus, the expert may focus on its determination rather than the others.

$$f(x) = x^T \left(\mu_\theta - \sqrt{2d} \frac{\Sigma_\theta x}{\sqrt{x^T \Sigma_\theta x}} \right) - k_c \sqrt{x^T \Sigma_\theta x} = x^T \mu_\theta - \left(k_c + \sqrt{2d} \right) \sqrt{x^T \Sigma_\theta x} \tag{17}$$

3. RESULTS AND DISCUSSION

In this section, we illustrate our approach with the real-world example given by Zhang (2007) and Huang et al. (2022), where there are five stocks (X1, X2, X3, X4, X5) and the possibility distributions are as below in the possibilistic MV model. Clearly, in our approach, they are shifted by Θ_i for all i due to the uncertainty in the possibilistic means where the uncertainty vector's (Θ) joint probability distribution is normal with mean vector μ_Θ and positive definite covariance matrix $\sum_\Theta$.

$$\begin{aligned} X_1 &\sim (0.019, 0.073, 0.16) \\ X_2 &\sim (0.03, 0.105, 0.207) \\ X_3 &\sim (0.042, 0.138, 0.261) \\ X_4 &\sim (0.042, 0.168, 0.33) \\ X_5 &\sim (0.04, 0.208, 0.421) \end{aligned} \tag{18}$$

Then, we calculate the possibilistic means and standard deviations with the performances as in Table 1 using the formulas given in (2) or (4). We show the best values in bold. We get X3 when x equals the column vector $(0, 0, 1, 0, 0)^T$.

Table 1. The possibilistic MV model's metrics

Stocks	P. Mean	P. Std. Dev.	Performance
X1	0.079	**0.029**	2.727
X2	0.110	0.036	3.031
X3	0.143	0.045	**3.188**
X4	0.174	0.059	2.960
X5	**0.216**	0.078	2.771

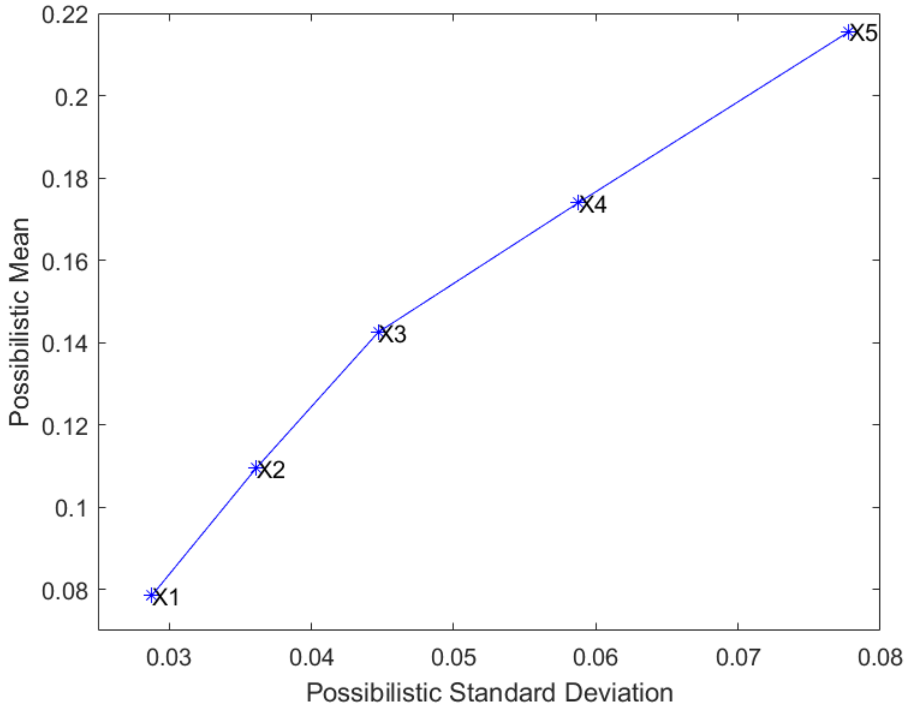
Based on Table 1 and the information given in Section 2, we uniquely calculate some OPs as in Table 2. As we expect, they are not diversified.

Table 2. Some optimal portfolios

Stocks	OP for β=1	OP for β=0	MPP
X1	0	1	0
X2	0	0	0
X3	0	0	1
X4	0	0	0
X5	1	0	0

We get the possibilistic MV model's efficient frontier as in Figure 2. We find that all stocks are an OP with some β values. The possibilistic mean (standard deviation) increases with the increase in β. The efficient frontier consists of consecutive line segments due to the linearity in (2).

Figure 2. The possibilistic MV model's efficient frontier



We determine that μ_Θ is equal to the column vector $(0.02, 0.01, 0, -0.01, -0.02)^T$, whereas $\sum_\Theta$ is as in Table 3. We set the confidence level (α) as 0.9 in this study. Then, we find k_α as 3.039 based on the normal distribution assumption. Due to μ_Θ, we can say that the expert is optimistic (pessimistic) about the future returns of X1 and X2 (X4 and X5) with respect to the possibility distributions given in (18). We note that the probabilistic standard deviation vector of Θ is equal to the column vector $(0.03, 0.02, 0.02, 0.02, 0.03)^T$ while $\Theta1$ and $\Theta2$ ($\Theta4$ and $\Theta5$) are negatively correlated each other.

Table 3. The covariance matrix of the uncertainty vector (Θ)

Stocks	X1	X2	X3	X4	X5
X1	0.0009	-0.0003	0	0	0
X2	-0.0003	0.0004	0	0	0
X3	0	0	0.0004	0	0
X4	0	0	0	0.0004	-0.0003
X5	0	0	0	-0.0003	0.0009

WOP1 (WOP2) is an abbreviation of WOP for β=1 (for β=0.5). We uniquely find WOP1, WOP2 and MWPP as in Table 4 using CVX/MOSEK solver in MATLAB for (12) or (14). The WOPs are diversified, unlike the OPs. Even if there are no restrictions that x_i is nonnegative in (1), we get the same WOPs due to the effect of f(x) given in (11). We remark that the possibilistic MV model does not give a bounded optimal solution in this case.

Table 4. Some worst-case optimal portfolios

Stocks	WOP1	WOP2	MWPP
X1	0	0.090	0.201
X2	0	0.180	0.355
X3	0.020	0.161	0.216
X4	0.553	0.350	0.167
X5	0.427	0.218	0.061

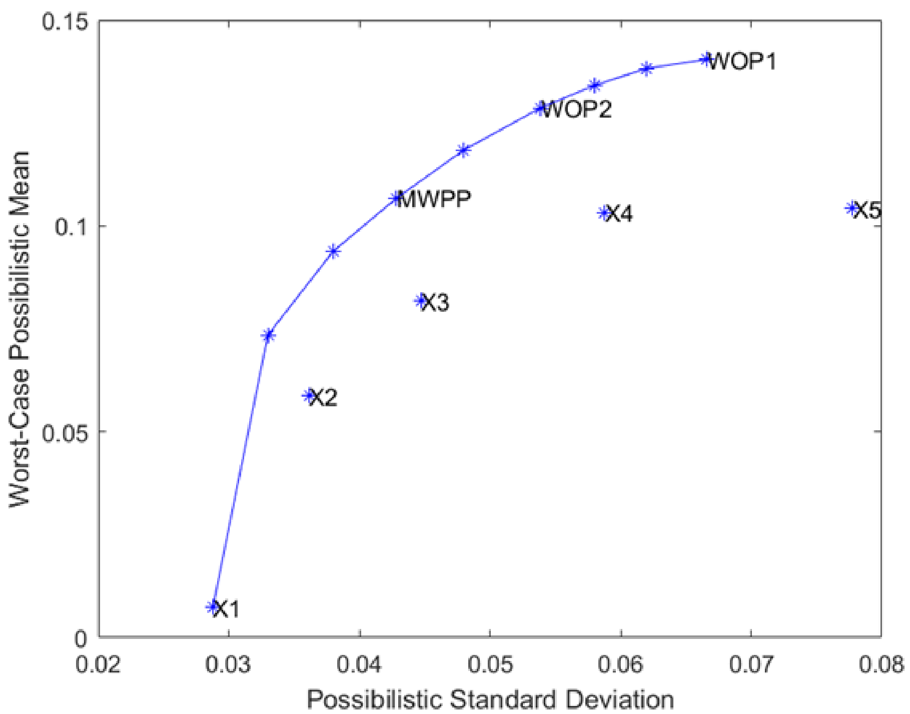
In Table 5, we give the $\Theta^*(x)$ vectors for the WOPs by using (10). When this information and (18) are combined, the worst-case possibility distributions of stock returns are derived for the WOPs. These vary from portfolio (x) to portfolio (x).

Table 5. The $\Theta^*(x)$ vectors for the WOPs

Parameters	WOP1	WOP2	MWPP
Θ_1	0.02	0.010	-0.007
Θ_2	0.01	-0.007	-0.020
Θ_3	-0.002	-0.024	-0.031
Θ_4	-0.034	-0.038	-0.028
Θ_5	-0.075	-0.054	-0.022

In Figure 3, we roughly give the efficient frontier in our approach. We see that X2, X3, X4 and X5 are not a WOP for any β. (WOP for β=0 consists of only X1.)

Figure 3. The efficient frontier in our approach



We compare the WOPs and the OPs in Table 6 using the formulas (4) or (13). The WOPs, except for X1, have sufficient performance and worst-case performance, unlike the OPs. Spearman's rank correlation coefficient equals Pearson's linear correlation coefficient of the ranks (Akinyi et al., 2022; Li et al., 2022). In the example, we find it to be 0.429, which is not high. These two measures do not give similar results in the example.

Table 6. Comparisons of the OPs and the WOPs

Portfolio	Performance Value	Performance Rank	Worst-Case P. V.	Worst-Case P. R.
MWPP	2.988	3.	2.491	1.
WOP2	2.928	5.	2.388	2.
WOP1	2.869	6.	2.109	3.
X1	2.727	8.	0.255	8.
X2	3.031	2.	1.625	6.
X3	3.188	1.	1.828	4.
X4	2.960	4.	1.756	5.
X5	2.771	7.	1.341	7.

Based on Table 2, Table 4 and Table 6, we remark that the WOPs, except for X1, are more appropriate for conservative investors since they are robust to the worst-case scenario and more diversified by definition. This is like the relationship between Markowitz's MV model and the robust MV model proposed by Garlappi et al. (2007).

In Table 7, we give WOP1 for the different confidence levels. The weight of X5 (X3) decreases (increases) with the increase in the confidence level (α).

Table 7. WOP1 for the different confidence levels

Stocks	α=0	α=0.25	α=0.5	α=0.9	α=0.95	α=0.99	α=0.999
X1	0	0	0	0	0	0	0
X2	0	0	0	0	0	0	0
X3	0	0	0	0.020	0.048	0.086	0.115
X4	0	0.497	0.531	0.553	0.543	0.530	0.519
X5	1	0.503	0.469	0.427	0.409	0.385	0.366

In Table 8, we give WOP2 for the different confidence levels. The weight of X5 (X2) decreases (increases) with the increase in α.

Table 8. WOP2 for the different confidence levels

Stocks	α=0	α=0.25	α=0.5	α=0.9	α=0.95	α=0.99	α=0.999
X1	0	0.011	0.053	0.090	0.096	0.105	0.113
X2	0	0.077	0.131	0.180	0.189	0.201	0.211
X3	0	0.174	0.167	0.162	0.161	0.159	0.158
X4	0	0.447	0.396	0.350	0.342	0.331	0.322
X5	1	0.291	0.253	0.218	0.213	0.204	0.197

In Table 9, we give MWPP for the different confidence levels. The weight of X3 decreases with the increase in α, whereas the weights of X4 and X5 increase with the increase in α.

Table 9. MWPP for the different confidence levels

Stocks	α=0	α=0.25	α=0.5	α=0.9	α=0.95	α=0.99	α=0.999
X1	0	0.277	0.254	0.201	0.190	0.175	0.164
X2	0	0.476	0.442	0.355	0.337	0.312	0.293
X3	1	0.248	0.249	0.216	0.207	0.196	0.187
X4	0	0	0.055	0.167	0.187	0.214	0.235
X5	0	0	0	0.061	0.079	0.103	0.122

Shannon's entropy can be used as a diversification measure. It takes values between 0 and ln(n). Diversification level increases with the increase in Shannon's entropy (Kamaludin et al., 2021; Lam et al., 2021). (Since n equals 5 in the example, ln(n) equals 1.609.) Table 10 gives Shannon's entropy values for the different confidence levels. We note that the diversification levels (Shannon's entropy values) of the WOPs increase with the increase in the confidence (conservativeness) level. This is because, the increase in the confidence level improves the effect of f(x) given in (11).

Table 10. The Shannon's entropy values for the different confidence levels

WOP	α=0	α=0.25	α=0.5	α=0.9	α=0.95	α=0.99	α=0.999
WOP1	0	0.693	0.691	0.770	0.843	0.915	0.956
WOP2	0	1.055	1.214	1.491	1.522	1.552	1.566
MWPP	0	1.271	1.435	1.519	1.529	1.542	1.551

Based on the last four tables, we say that the WOPs are sufficiently diversified even if the confidence (conservativeness) level is not high.

Let the maximum error matrix (V) be positive semi-definite. That is, its eigenvalues are nonnegative. Since $\sum_{\Theta}$ is positive definite, it is a fact that $V+\sum_{\Theta}$ is also positive definite. That is, its eigenvalues are positive. Based on the framework given in Tütüncü and Koenig (2004), we can make the worst-case analysis for $var(x^T\Theta)$ by only using $V+\sum_{\Theta}$ in (10)-(11) instead of $\sum_{\Theta}$. This is because, $var(x^T\Theta)$ is maximized for any portfolio (x) when the positive definite covariance matrix of Θ is equal to $V+\sum_{\Theta}$ due to the restrictions on the short positions. See Proposition 3 of Tütüncü and Koenig (2004). In the example, we take V as $0.0009*A_{5x5}$, where A_{5x5} consists of only ones. (The maximum element of $\sum_{\Theta}$ is equal to 0.0009 as shown in Table 3.) Then, we find the WOPs as in Table 11.

Table 11. The results of the worst-case analysis for $var(x^T\Theta)$

Stocks	WOP1	WOP2	MWPP
X1	0	0	0
X2	0	0	0
X3	0	0.105	0.146
X4	0.550	0.516	0.505
X5	0.450	0.380	0.349

In the previous section, we remark that it is more essential to determine Σ_Θ. By comparing the information given in Table 4 and Table 11, we say that the WOPs may change highly and get close to WOP1 with the increase in $var(x^T\Theta)$. It is consistent with the fact that $var(x^T\Theta)$ affects WOP1 more than the other WOPs. We note that WOP1 is found by maximizing the worst-case possibilistic mean.

It can be useful to analyze the certainly known and uncertain possibility distributions together. In this context, we consider five additional stocks, of which possibility distributions are certainly known, for the portfolio selection problem. We present their information in Table 12.

Table 12. The information for the additional stocks

Stocks	P. Mean	P. Std. Dev.	Performance
X6	0.10	0.06	1.667
X7	0.11	0.07	1.571
X8	0.12	0.08	1.500
X9	0.13	0.09	1.444
X10	0.15	0.15	1.000

Since X4 has a higher possibilistic mean and lower possibilistic standard deviation than these stocks, they are not an optimal portfolio for the possibilistic MV model. That is, the efficient frontier in Figure 2 does not change. On the other hand, the optimal portfolios in our approach may change as shown in Table 13.

Table 13. Some worst-case optimal portfolios for the extended example

Stocks	WOP1	WOP2	MWPP
X1	0	0.090	0.201
X2	0	0.180	0.355
X3	0	0.161	0.216
X4	0	0.350	0.167
X5	0	0.218	0.961
X6	0	0	0
X7	0	0	0
X8	0	0	0
X9	0	0	0
X10	1	0	0

By comparing Table 4 and Table 13, we see that WOP2 and MWPP do not change unlike WOP1. WOP1 is not diversified like OPs. That is, our approach may give undiversified portfolios when the certainly known and uncertain possibility distributions are used together. We also remark that our approach may not give unique solutions in this case. On the other hand, Tikhonov's regularized problem can be used to approximate the unique optimal solution closest to the origin (Beck and Sabach, 2014; Ketabchi et al., 2021).

4. CONCLUSION

This study proposes an approach to cope with uncertain possibility distributions in the possibilistic MV model under certain assumptions. To the best of our knowledge, this is the first paper that considers this problem. After laying down the theoretical points of our approach, we give a real-world example. We show that our approach may be applicable, unlike the possibilistic MV model, even if there are no restrictions on the short positions. The main limitation of our approach is that there is no formal procedure to determine the uncertainty vector’s (Θ) joint probability distribution. In addition, our approach may be unpreferable for the portfolio selection problems including the high number of stocks since the primary determinant of its success is the quality of the expert knowledge. The main strength of our approach is that it captures uncertainty’s two aspects (fuzziness and randomness) since it depends on possibility and probability theories. In addition, we uniquely get diversified optimal portfolios with our approach, unlike the possibilistic MV model. Furthermore, our approach considers various future scenarios based on the determined confidence (conservativeness) level. For these reasons, our approach may be an alternative to the possibilistic MV model. Another limitation of our approach is that it only considers the possibilistic mean vector’s uncertainty. As future research, our approach can be extended by using multiple uncertainty vectors to model the possibilistic mean and standard deviation vectors’ uncertainty together.

The study does not necessitate Ethics Committee permission.

The study has been crafted in adherence to the principles of research and publication ethics.

The author declares that there exists no financial conflict of interest involving any institution, organization, or individual(s) associated with the article.

The entire work was carried out by its only, stated author.

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