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Laguerre Collocation Approach of Caputo Fractional Fredholm-Volterra Integro-Differential Equations

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Abstract

This paper discusses the linear fractional Fredholm-Volterra integro-differential equations (IDEs) considered in the Caputo sense. For this purpose, Laguerre polynomials have been used to construct an approximation method to obtain the solutions of the linear fractional Fredholm-Volterra IDEs. By this approximation method, the IDE has been transformed into a linear algebraic equation system using appropriate collocation points. In addition, a novel and exact matrix expression for the Caputo fractional derivatives of Laguerre polynomials and an associated explicit matrix formulation has been established for the first time in the literature. Furthermore, a comparison between the results of the proposed method and those of methods in the literature has been provided by implementing the method in numerous examples.

1. Introduction

The integro-differential equations (IDEs) of the fractional order are used by mathematicians and other scientists to model different physical and biological processes just as the heat conduction problem, radiative equilibrium, fracture mechanics, elasticity, signal processing, control and robotics, population dynamics, and health issues [1]- [12]. Hence, solving these types of equations and investigating the exact and approximate solutions has gained importance in recent years. When these investigations are reviewed it can be obviously seen that the methods handled to solve the fractional Fredholm-Volterra integro-differential equations (FVIDEs) are presented as reliable modified Laplace Adomian decomposition method [13], generalized hat functions [14], Nyström and Newton-Kantorovitch [15], Chebyshev wavelet [16]- [18], wavelet-based methods [19], Chebyshev Neural Network [20], Taylor expansion [21], sinccollocation [22], Legendre wavelet [23], Lucas wavelets with Legendre–Gauss quadrature [24], Bessel polynomials [25], fractional differential transform [26], Bernstein polynomials [27], Genocchi polynomials [28], spectral Jacobi-collocation [29], Block pulse functions [30], fractional-order Bernoulli functions [31], hybrid functions [32], Bernoulli wavelets [33], hybrid orthonormal Bernstein and block-pulse functions wavelet method [34].

Additionally, Laguerre polynomials have been used to solve the IDEs of integer order. Obviously, these integer-order equations can be specialized as 2-evolution equation [35], Altarelli-Parisi equation [36], Dokshitzer-Gribov-Lipatov-Altarelli-Parisi equation [37], linear Fredholm IDE [38], [39], Volterra IDE of pantograph-type [40], delay partial functional differential equation [41], Volterra partial IDE of parabolic-type [42], [43], and nonlinear partial IDE [44]. In other respects, Laguerre polynomials have been applied to attain the solutions of the fractional IDE of the Fredholm type [45].

Moreover, in our research articles, approximation methods based on Laguerre polynomials have been developed. Daşcıoğlu et al. [46] have used a collocation method based upon the Laguerre polynomials to attain the solutions of the linear fractional FVIDEs in conformable sense. The method described in [46] is an improvement of the method that used for the solutions of the linear fractional IDEs of the Fredholm type in the Caputo sense [47] and Caputo fractional linear IDEs of the Volterra type [48].

However, for the linear fractional IDEs of the Fredholm-Volterra type in the Caputo sense with mixed conditions there is no method in the sense of Laguerre polynomials. In this work, a method based on these polynomials is proposed to obtain the solutions of the fractional

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linear IDE of the Fredholm-Volterra type in the following general form:

$$\sum_{i=1}^{m} p_i(x)D^{\alpha_i}y(x)+\sum_{i=1}^{l} q_i(x)y^i(x)=g(x)+\lambda_1\int_a^b F(x,t)y(t)\,dt+\lambda_2\int_a^x V(x,t)y(t)\,dt, \qquad a\le x\le b, \tag{1.1}$$

with the conditions

$$\sum_{k=0}^{v-1} B_{jk}y^{(k)}(\beta_{jk})=\mu_j, \qquad v_i-1<\alpha_i<v_i, \qquad j=0,1,...,v-1, \tag{1.2}$$

where $m,l\in\mathbb{N}$, $v_i\in\mathbb{Z}^+$; $\mu_j,\beta_{jk},B_{jk},\lambda_1,\lambda_2\in\mathbb{R}$, $v=max\left(\left(\underbrace{max}_{0\le i\le m}\ v_i\right),l\right)$. Here $p_i(x),q_i(x),F(x,t),V(x,t)$, and $g(x)$ are known functions, $y(x)$ is the unknown function that has to be determined, $y^i(x)$, shows the ordinary derivatives of the unknown function $y(x)$, $D^{\alpha_i}y(x)$ stands for the Caputo fractional derivative of $y(x)$ whose definition has been given below:

Definition 1.1. *[49] The Caputo fractional differentiation operator D^{α} of order α is defined as:*

$$D^{\alpha}f(x)=\frac{1}{\Gamma(n-\alpha)}\int_0^x\frac{f^{(n)}(t)}{(x-n)^{\alpha+1-n}}dt, \qquad \alpha>0,$$

where $-1<\alpha<n,n\in\mathbb{Z}^+$ and Γ is the well-known Gamma function.

The main purpose of this work is to obtain an approximate solution of given problem (1.1)-(1.2) in the form

$$y(x)\cong y_N(x)=\sum_{n=0}^{N}a_nL_n(x), \tag{1.3}$$

where N is any taken positive integer such that $N\ge v$, the unknown coefficients a_n's must be discovered,and $L_n(x)$ stand for the Laguerre polynomials of the order n stated by Bell [50] as:

$$L_n(x)=\sum_{k=0}^{n}(-1)^k\frac{n!}{(n-k)!(k!)^2}x^k.$$

The rest of the paper is arranged as follows: In section 2, the fundamental matrix relations for each term in fractional IDE (1.1) are constituted. In section 3, a functional collocation method based on the Laguerre polynomials is introduced. In section 4, numerical examples are resolved, their results are presented, and these solutions are compared with the existing results in the literature to affirm the precision and effectiveness of the proposed method. The last section of the paper presents the conclusions.

2. Elementary Matrix Formulas

In this section, we attempt to transform Eq. (1.1) by formulating the matrix forms of the unknown function and its fractional derivatives in the Caputo sense.

First, we can formulate the approximate solution (1.3) as the product of $\mathbf{L}(x)$ which can be called as the Laguerre matrix and the coefficient matrix $\mathbf{A}$ by

$$y_N(x)=\mathbf{L}(x)\mathbf{A}, \tag{2.1}$$

where the matrices are given as

$$\mathbf{A}=\begin{bmatrix}a_0 & a_1 & \cdots & a_N\end{bmatrix}^T \text{ and } \mathbf{L}(x)=\begin{bmatrix}L_0(x) & L_1(x) & \cdots & L_N(x)\end{bmatrix}.$$

Then, the following theorem has been given which demonstrates the connection between the Laguerre polynomials and the fractional derivative of Laguerre polynomials in the Caputo sense, which has been given and proved in our previous paper:

Theorem 2.1. *[46] Let $L_n(x)$ be the Laguerre polynomial of order n, then the Caputo fractional derivative of $L_n(x)$ in terms of Laguerre polynomials is found as follows:*

$$D^{\alpha}L_n(x)=0,n<\lceil\alpha\rceil,$$

and otherwise

$$D^{\alpha}L_n(x)=x^{1-\alpha}\sum_{k=\lceil\alpha\rceil}^{n}\sum_{r=0}^{k-1}(-1)^{r+k}\frac{(k-1)!}{\Gamma(k+1-\alpha)}\binom{n}{k}\binom{k-1}{r}L_r(x),$$

where $\lceil\alpha\rceil$ indicates the smallest integer greater than or equal to α which is known as the ceiling function.

Secondly, the matrix relations of the differential side of the Eq. (1.1) are formulated. The relation between the Laguerre matrix $\mathbf{L}(x)$ and its integer order derivatives of the Laguerre matrix $\mathbf{L}(x)$ will be used in the form given in Eq. (2.2) which can be seen in Ref. [40] to present the matrix relation for the derivatives of the integer order of the unknown function $y(x)$,

$$\mathbf{L}^{(i)}(x)=\mathbf{L}(x)\mathbf{M}^{i}, \qquad i=0,1,\ldots,N, \tag{2.2}$$

where the matrix $\mathbf{M}$ is

$$\mathbf{M}=\begin{bmatrix} 0 & -1 & -1 & & -1\\ 0 & 0 & -1 & \cdots & -1\\ 0 & 0 & 0 & & -1\\ & \vdots & & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & -1\\ 0 & 0 & 0 & & 0 \end{bmatrix}.$$

Therefore, the derivatives of integer order of the unknown function $y(x)$ in Eq. (1.1) can be represented as below by using Eq. (2.2),

$$y^{(i)}(x)\cong\mathbf{L}(x)\mathbf{M}^{i}\mathbf{A}. \tag{2.3}$$

Theorem 2.2. *Let $\boldsymbol{L}(x)$ be the Laguerre matrix defined in (2.1) and $D^{\alpha}\boldsymbol{L}(x)$ be the Caputo fractional derivative of $\boldsymbol{L}(x)$ of the α-th order, then the Caputo fractional derivative of Laguerre matrix is given as*

$$D^{\alpha}\boldsymbol{L}(x)=x^{1-\alpha}\boldsymbol{L}(x)\boldsymbol{S}_{\alpha}, \tag{2.4}$$

where $\boldsymbol{S}_{\alpha}$ is an $(N+1)$ dimensional square matrix specified as

$$\boldsymbol{S}_{\alpha}=\begin{bmatrix} 0 & \binom{0}{0}S_{1,1} & \binom{0}{0}S_{1,2}+\binom{1}{0}S_{2,2} & \cdots & \sum_{k=1}^{N}\binom{k-1}{0}S_{k,N}\\ 0 & 0 & -\binom{1}{1}S_{2,2} & \cdots & -\sum_{k=2}^{N}\binom{k-1}{1}S_{k,N}\\ 0 & 0 & 0 & \cdots & \sum_{k=3}^{N}\binom{k-1}{2}S_{k,N}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & (-1)^{N+1}S_{N,N}\\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

or

$$\boldsymbol{S}_{\alpha}=\left[(-1)^{i}\sum_{k=i+1}^{j}\binom{k-1}{i}S_{k,j}\right], \qquad i,j=0,1,\ldots,N.$$

Here, the $S_{k,j}$ terms in the entries of the matrix S_{α} are defined as

$$\boldsymbol{S}_{k,j}=\begin{cases}(-1)^{k}\frac{(k-1)!}{\Gamma(k+1-\alpha)}\binom{j}{k}, & \text{if} \quad \lceil\alpha\rceil\le k\le j\\ 0, & \text{otherwise}\end{cases}.$$

Proof. First, the Caputo fractional derivative of $\mathbf{L}(x)$ which is denoted by $D^{\alpha}\mathbf{L}(x)$ has been defined by

$$D^{\alpha}\mathbf{L}(x)=\begin{bmatrix}D^{\alpha}L_0(x) & D^{\alpha}L_1(x) & \cdots & D^{\alpha}L_N(x)\end{bmatrix}.$$

By using Theorem 1 above, for $j<\lceil\alpha\rceil, D^{\alpha}L_j(x)=0$, and for $j\ge\lceil\alpha\rceil, k=1,2,\ldots,j$

$$D^{\alpha}L_j(x)=x^{1-\alpha}\sum_{k=\lceil\alpha\rceil}^{j}\sum_{r=0}^{k-1}(-1)^{r+k}\frac{(k-1)!}{\Gamma(k+1-\alpha)}\binom{j}{k}\binom{k-1}{r}L_r(x).$$

At this point, since the term $S_{k,j}, k=1,2,\ldots,j$ is defined as follows:

$$S_{k,j}=\begin{cases}(-1)^{k}\frac{(k-1)!}{\Gamma(k+1-\alpha)}\binom{j}{k}, & \lceil\alpha\rceil\le k\le j\\ 0, & \text{otherwise}\end{cases}.$$

$D^{\alpha}L_0(x)=0$ and for $j=1,2,\ldots,N$

$$D^{\alpha}L_j(x)=x^{1-\alpha}\sum_{k=1}^{j}\sum_{r=0}^{k-1}(-1)^{r}\binom{k-1}{r}S_{k,j}L_r(x).$$

Here, for $j=0, D^{\alpha}L_0(x)=0$ and for $j\in\{1,\ldots,N\}$

$$D^{\alpha}L_j(x)=x^{1-\alpha}\sum_{k=1}^{j}\sum_{r=0}^{k-1}(-1)^{r}\binom{k-1}{r}S_{k,j}L_r(x)$$

$$=x^{1-\alpha}\left\{\sum_{k=1}^{j}\binom{k-1}{0}S_{k,j}L_0(x)-\sum_{k=2}^{j}\binom{k-1}{1}S_{k,j}L_1(x)-\cdots(-1)^{j-1}\binom{j-1}{j-1}S_{j,j}L_{j-1}(x)\right\}$$

Therefore, all the entries in the 0-th column and all the entries in the N-th row of $D^{\alpha}\mathbf{L}(x)$ is zero, and otherwise, the i,j-th element of the matrix $D^{\alpha}\mathbf{L}(x)$ is given as

$$x^{1-\alpha}\sum_{k=i+1}^{j}(-1)^{i}\binom{k-1}{i}S_{k,j}L_{i}(x).$$

Thus, the relation between $D^{\alpha}\mathbf{L}(x)$ and $\mathbf{L}(x)$ as expressed in Eq. (2.4) has been obtained. □

This relation proves the theorem.

Then, using the result of Theorem 2 and using relations (2.1) and (2.4), the Caputo fractional derivative of the unknown function $y(x)$ which is the differential part of Eq. (1.1) can be represented by

$$D^{\alpha}y(x)\cong D^{\alpha}\mathbf{L}(x)\mathbf{A}=x^{1-\alpha}\mathbf{L}(x)\mathbf{S}_{\alpha}\mathbf{A}. \tag{2.5}$$

Now, finally, the corresponding matrix formula for mixed conditions (1.1) could be given in the form

$$\sum_{k=0}^{v-1}B_{jk}\mathbf{L}(\beta_{jk})\mathbf{M}^{k}\mathbf{A}=\mu,\qquad j=0,1,\ldots,v-1. \tag{2.6}$$

by using Eq. (2.3).

Finally, when the matrix in the summation in the left-hand side of Eq. (2.6) is called as $\mathbf{U}_j$ that is an $1\times(N+1)$ vector matrix, Eq. (2.6) transforms into

$$\mathbf{U}_{j}\mathbf{A}=\mu_{j},\qquad j=0,1,\ldots,v-1.$$

3. Solution Method

In this part of the paper, we maintain the approximate solution method which can be specified as a collocation method, because we use the collocation points at the end to solve the matrix equation. In other words, we determine the unknown coefficients a_i's in Eq. (1.3) to obtain the solution of Equations (1.1)-(1.2) using a collocation method.

Theorem 3.1. *Suppose that the fractional FVIDE defined by Eq. (1.1) is given. Utilizing the collocation points $x_s>0$ and $x_s\in[a,b]$, this IDE can be abbreviated as the following matrix equation:*

$$\left\{\sum_{i=0}^{m}\boldsymbol{P}_{i}\boldsymbol{X}_{\alpha_i}\boldsymbol{L}\boldsymbol{S}_{\alpha_i}+\sum_{i=0}^{l}\boldsymbol{Q}_{i}\boldsymbol{L}\boldsymbol{M}^{i}-\lambda_{1}\boldsymbol{F}-\lambda_{2}\boldsymbol{V}\right\}\boldsymbol{A}=\boldsymbol{G}.$$

Here, the matrices $\boldsymbol{M}$ and $\boldsymbol{S}_{\alpha_i}$ are in forms as in Eq. (2.2) and (2.4), respectively. In addition, $\boldsymbol{G}=[g(x_s)]$ is an $(N+1)\times 1$ dimensional matrix; $\boldsymbol{X}_{\alpha_i}=diag[x_s^{1-\alpha_i}]$, $\boldsymbol{P}_i=diag[p_i(x_s)]$, $\boldsymbol{Q}_i=diag[q_i(x_s)]$, $\boldsymbol{L}=[\boldsymbol{L}(x_s)]$, $\boldsymbol{F}=[\boldsymbol{f}(x_s)]$, and $\boldsymbol{V}=[\boldsymbol{v}(x_s)]$ are $(N+1)\times(N+1)$ dimensional square matrices. Moreover, $\boldsymbol{L}(x)$ corresponds for the Laguerre matrix, as described in Eq. (2.1), $\boldsymbol{f}(x_s)$ and $\boldsymbol{v}(x_s)$ represent the given integrals; $\boldsymbol{f}(x_s)=\int_a^b F(x_s,t)\boldsymbol{L}(t)dt$ and $\boldsymbol{v}(x_s)=\int_a^{x_s} V(x_s,t)\boldsymbol{L}(t)dt$

Proof. Firstly, substituting matrix relations (2.1), (2.3) and (2.5) into the Eq. (1.1), the following matrix equation has been obtained

$$\sum_{i=0}^{m}p_{i}(x)x^{1-\alpha_i}\mathbf{L}(x)\mathbf{S}_{\alpha_i}\mathbf{A}+\sum_{i=0}^{l}q_{i}(x)\mathbf{L}(x)\mathbf{M}^{i}\mathbf{A}=g(x)+\lambda_{1}\int_{a}^{b}F(x,t)\mathbf{L}(t)\mathbf{A}dt+\lambda_{2}\int_{0}^{x}V(x,t)\mathbf{L}(t)\mathbf{A}dt. \tag{3.1}$$

By substituting the non-negative collocation points $x_s(s=0,1,\ldots,N)$ into Eq. (3.1), the following system of linear matrix equations has been gained

$$\sum_{i=0}^{m}p_{i}(x_s){x_s}^{1-\alpha_i}\mathbf{L}(x_s)\mathbf{S}_{\alpha_i}\mathbf{A}+\sum_{i=0}^{l}q_{i}(x_s)\mathbf{L}(x_s)\mathbf{M}^{i}\mathbf{A}=g(x_s)+\lambda_{1}\mathbf{f}(x_s)\mathbf{A}+\lambda_{2}\mathbf{v}(x_s)\mathbf{A}, \tag{3.2}$$

where $\mathbf{f}(x_s)=\int_a^b F(x_s,t)\mathbf{L}(t)\,dt$ and $\mathbf{v}(x_s)=\int_a^{x_s}V(x_s,t)\mathbf{L}(t)\,dt$.

The system given by Eq. (3.2) can be written in the compact forms in the form

$$\left\{\sum_{i=0}^{m}\mathbf{P}_{i}\mathbf{X}_{\alpha_i}\mathbf{L}\mathbf{S}_{\alpha_i}+\sum_{i=0}^{l}\mathbf{Q}_{i}\mathbf{L}\mathbf{M}^{i}-\lambda_{1}\mathbf{F}-\lambda_{2}\mathbf{V}\right\}\mathbf{A}=\mathbf{G}, \tag{3.3}$$

where the matrices mentioned above are given as follows:

$$\mathbf{X}_{\alpha_i}=\begin{bmatrix}{x_0}^{1-\alpha_i}&0&&0\\0&{x_1}^{1-\alpha_i}&\cdots&0\\&\vdots&\ddots&\vdots\\0&0&\cdots&{x_N}^{1-\alpha_i}\end{bmatrix},\mathbf{L}=\begin{bmatrix}\mathbf{L}(x_0)\\\mathbf{L}(x_1)\\\vdots\\\mathbf{L}(x_N)\end{bmatrix},\mathbf{P}_{i}=\begin{bmatrix}p_i(x_0)&0&&0\\0&p_i(x_0)&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&p_i(x_N)\end{bmatrix},$$

$$\mathbf{Q}_i = \begin{bmatrix} q_i(x_0) & 0 & & 0 \\ 0 & q_i(x_0) & \cdots & 0 \\ & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & q_i(x_N) \end{bmatrix}, \mathbf{F} = \begin{bmatrix} \mathbf{f}(x_0) \\ \mathbf{f}(x_1) \\ \vdots \\ \mathbf{f}(x_N) \end{bmatrix}, \mathbf{V} = \begin{bmatrix} \mathbf{v}(x_0) \\ \mathbf{v}(x_1) \\ \vdots \\ \mathbf{v}(x_N) \end{bmatrix}, \mathbf{G} = \begin{bmatrix} \mathbf{g}(x_0) \\ \mathbf{g}(x_1) \\ \vdots \\ \mathbf{g}(x_N) \end{bmatrix}.$$

□

For simplicity, symbolizing the expression in the parenthesis of Eq. (3.3) by $\mathbf{W}$, the fundamental matrix equation associated with Eq. (1.1) can be abbreviated to $\mathbf{WA} = \mathbf{G}$. Apparently, this equation substitutes for a $(N+1)$ dimensional linear algebraic equations system with the unknown coefficients a_i's for $i = 0, 1, \ldots, N$ hich we can call as Laguerre coefficients.

Consequently, to find the solution of Eq. (1.1) with given conditions (1.2), the n rows of the obtained augmented matrix $[\mathbf{W}; \mathbf{G}]$ are stacked or replaced by the n rows of the augmented matrix $[\mathbf{U}_j; \mu_j]$. Therefore, because the unknown Laguerre coefficients are discovered by resolving this system, we obtain the solution of Eq. (1.1) under Conditions (1.2).

4. Numerical Examples

In this section, four examples have been tried to solve by the proposed method. All the numerical calculations were executed with the aid of Mathcad 15.

Example 4.1. *Consider the given fractional Fredholm IDE*

$$y''(x) + D^{\frac{1}{2}} y(x) + y(x) = \frac{9}{4} - \frac{1}{3}x - \frac{2}{\Gamma(\frac{5}{2})} x^{\frac{3}{2}} + x^2 + \int_0^1 (x-t) y(t) dt$$

with the conditions $y(0) = y'(0) = 0$. *This problem has the exact solution* $y(x) = x^2$.

Implementing the methodology explained in Section 3, the expected fundamental matrix equation of the given problem and its conditions can be presented as

$$\left\{ \mathbf{X}_{\frac{1}{2}} \mathbf{L} \mathbf{S}_{\frac{1}{2}} + \mathbf{L} + \mathbf{L}\mathbf{M}^2 - \mathbf{V} \right\} \mathbf{A} = \mathbf{G}$$

and

$$\mathbf{U}_0 \mathbf{A} = \mathbf{L}(0) \mathbf{A} = 0, \quad \mathbf{U}_1 \mathbf{A} = \mathbf{L}(0) \mathbf{M} \mathbf{A} = 0.$$

Here, the collocation points for $N = 2$ such as $x_0 = 0.25$, $x_1 = 0.75$, $x_2 = 1$ were used. Then the matrices mentioned above are

$$\mathbf{X}_{\frac{1}{2}} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{L} = \begin{bmatrix} 1 & \frac{3}{4} & \frac{17}{32} \\ 1 & \frac{1}{4} & \frac{-7}{32} \\ 1 & 0 & \frac{-1}{2} \end{bmatrix}, \quad \mathbf{S}_{\frac{1}{2}} = \begin{bmatrix} 0 & \frac{-2}{\sqrt{\Pi}} & \frac{-8}{3\sqrt{\Pi}} \\ 0 & 0 & \frac{-4}{3\sqrt{\Pi}} \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{F} = \begin{bmatrix} \frac{-1}{4} & \frac{-1}{24} & \frac{1}{12} \\ \frac{1}{4} & \frac{5}{24} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{3} & \frac{5}{24} \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} \frac{1}{3\sqrt{\Pi}} + \frac{107}{48} \\ \frac{\sqrt{3}}{\sqrt{\Pi}} + \frac{41}{16} \\ \frac{8}{3\sqrt{\Pi}} + \frac{35}{12} \end{bmatrix}, \quad \mathbf{U}_0 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \mathbf{U}_1 = \begin{bmatrix} 0 & -1 & 2 \end{bmatrix}$$

By solving this system, we obtain $a_0 = 2$, $a_1 = -4$, $a_2 = 2$. In the final step, we substitute these coefficients into the approximate Eq. (1.3) and obtain the exact solution. This problem was solved by Ordokhani et al. [25] by using the Bessel collocation method. They found an approximate solution with absolute maximum errors 3.70×10^{-3} for $N = 2$, 3.28×10^{-4} for $N = 4$ and 8.58×10^{-5} for $N = 6$. We found the exact solution for $N = 2$ with symbolic evaluation in Mathcad 15 using the proposed method. Clearly, the proposed method is more accurate than the other method.

Example 4.2. *Let us consider the fractional FVIDE having the exact solution* $y(x) = x^2 + x^3$,

$$D^{1.7} y(x) = g(x) + \int_0^x (x-t) y(t) dt + \int_0^1 (x+t) y(t) dt$$

with the given initial conditions $y(0) = y'(0) = 0$ *where*

$$g(x) = \frac{6}{\Gamma(2.3)} x^{1.3} + \frac{2}{\Gamma(1.3)} x^{0.3} - \frac{x^5}{20} - \frac{x^4}{12} - \frac{7x}{12} - \frac{9}{20}.$$

Implementing the methodology explained in Section 3, the expected fundamental matrix equation of the given problem and its conditions can be presented as

$$\{\mathbf{X}_{1.7}\mathbf{LS}_{1.7}-\mathbf{F}-\mathbf{V}\}\mathbf{A}=\mathbf{G}$$

and

$$\mathbf{U}_0\mathbf{A}=0, \qquad \mathbf{U}_1\mathbf{A}=0.$$

Here, we use the collocation points for $N=3$ such as $x_0=0.25$, $x_1=0.5$, $x_2=0.75$, $x_3=1$. We obtain the Laguerre coefficients as $a_0=8$, $a_1=-22$, $a_2=20$, $a_3=-6$ by solving this system. In the final step, we substitute these coefficients into the approximate Eq. (1.3), then we obtain the exact solution.

The approximate solutions to this problem using the Legendre wavelet method were given by Meng et al. [23]. Therefore, the maximum absolute errors of their method were calculated as 5.3×10^{-2} for 16 terms, 2.7×10^{-2} for 32 terms, 1.2×10^{-2} for 64 terms and 9.0×10^{-4} for 128 terms. In addition, Genocchi polynomials were used by Loh et al. [28] to obtain the numerical solution of the above problem with the maximum absolute error 7.0×10^{-2} for $N=8$. Since we obtain the exact solution for $N=3$,the proposed method is faster, more efficient, and more accurate compared than the other methods.

Example 4.3. *Consider the given fractional FVIDE with the exact solution $y(x)=x^{\frac{7}{2}}$ which is nonpolynomial:*

$$D^{2.3}y(x)=g(x)+\frac{1}{4}\int_0^x(x-t)y(t)dt+\frac{1}{2}\int_0^1 xty(t)dt$$

with following three conditions $y(0)=y'(0)=y''(0)=0$ where the non-homogenous function given as $g(x)=\frac{\Gamma(4.5)}{\Gamma(2.2)}x^{1.2}-\frac{x^{5.5}}{99}-\frac{x}{11}$.

Implementing the methodology explained in Section 3, the expected fundamental matrix equation of the given fractional equation and its conditions can be presented as

$$\left\{\mathbf{X}_{2.3}\mathbf{LS}_{2.3}-\frac{1}{2}\mathbf{F}-\frac{1}{4}\mathbf{V}\right\}\mathbf{A}=\mathbf{G}$$

and

$$\mathbf{U}_0\mathbf{A}=0, \qquad \mathbf{U}_1\mathbf{A}=0, \qquad \mathbf{U}_2\mathbf{A}=0$$

This problem was solved using the collocation points with the formula $x_s=\left[1-\cos\left(\frac{(s+1)\Pi}{N+1}\right)\right]/2$ and the numerical results are given in Table 1 for $N=8$ and $N=9$. Besides, the illustration of the results for $N=9$ is given in Figure 4.1.

	LWM	ADM	FBF	GHF	GP	Present	method
x	k=2,M=5	n=5	m=8	n=32	N=9	N=8	N=9
$\frac{1}{8}$	6.6×10^{-6}	1.0	6.9×10^{-7}	4.2×10^{-6}	1.5×10^{-4}	1.3×10^{-10}	1.6×10^{-8}
$\frac{2}{8}$	4.5×10^{-5}	4.2	3.5×10^{-7}	5.6×10^{-5}	6.3×10^{-4}	9.7×10^{-10}	6.3×10^{-9}
$\frac{3}{8}$	3.1×10^{-5}	9.2	2.4×10^{-7}	6.2×10^{-5}	1.3×10^{-3}	7.0×10^{-9}	4.0×10^{-9}
$\frac{4}{8}$	7.4×10^{-5}	4.2	2.3×10^{-7}	6.9×10^{-5}	2.0×10^{-3}	3.3×10^{-8}	9.1×10^{-11}
$\frac{5}{8}$	2.4×10^{-4}	8.1	8.3×10^{-7}	3.2×10^{-4}	2.8×10^{-3}	1.0×10^{-7}	3.9×10^{-8}
$\frac{6}{8}$	3.8×10^{-4}	2.3	2.3×10^{-7}	4.5×10^{-4}	3.7×10^{-3}	2.5×10^{-7}	1.2×10^{-7}
$\frac{7}{8}$	6.0×10^{-4}	8.1	4.6×10^{-7}	6.2×10^{-4}	4.6×10^{-3}	5.0×10^{-7}	2.4×10^{-7}

Table 1: Comparison of absolute maximum errors of Example 4.3.

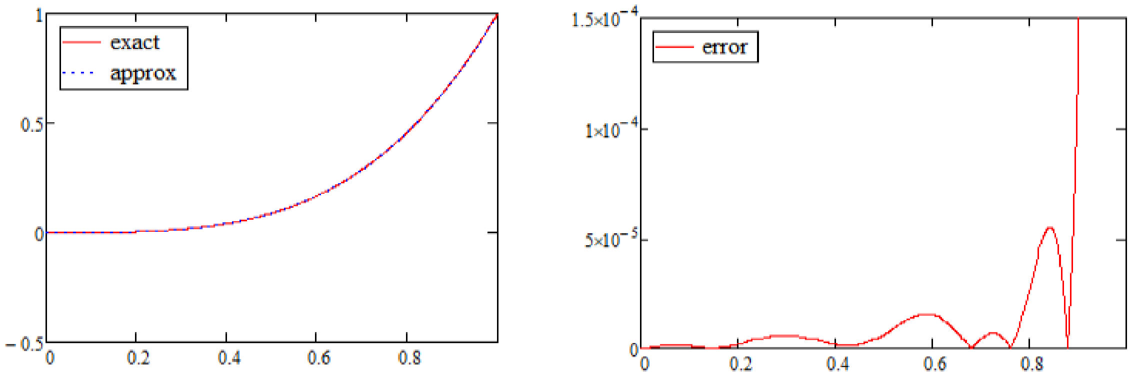

Figure 4.1: Graphical analysis of Example 4.3 for N=9

The results of the Legendre wavelet method (LWM) and the Adomian decomposition method(ADM) were provided by Meng et al. [23]. In addition, the fractional order Bernoulli functions (FBF) were used by Rahimkhani et al. [31], Genocchi polynomials (GP) wwere used by Loh et al. [28] and generalized hat functions (GHF) were used by Li [14] to obtain the approximate solution of this problem. The numerical results are presented in Table 1. It is obviously seen from the table that the proposed method is more effective and more accurate than the other methods compared.

Example 4.4. *Let us consider the following fractional IDE*

$$y''(x)+\frac{1}{x}D^{\frac{1}{2}}y(x)+\frac{1}{x^2}y(x)=g(x)+\int_0^1\cos(x-t)y(t)dt+\int_0^x\sin(x-t)y(t)dt$$

with the boundary conditions $y(0)=y(1)=0$*. The exact solution of this problem is* $y(x)=x^2-x^3$.

This problem was also solved by sinc-collocation method proposed by Alkan et al. [22]. They found an approximate solution with the maximum absolute errors 4.6×10^{-2} for $N=4$, 2.7×10^{-2} for $N=8$, 1.8×10^{-3} for $N=16$, 2.6×10^{-5} for $N=32$ and 3.9×10^{-7} for $N=64$. However, we found the exact solution using the proposed method with $N=3$. Therefore, it is evident that the proposed method is more efficient than the other methods.

5. Conclusion

In this paper, Laguerre polynomials were applied to construct a numerical approximation method to obtain the solutions of the fractional linear IDEs of the Fredholm-Volterra type. Using this approximation method a great variety of differential and integral (or both) equations has been covered since the equation in (1) has been presented in a general manner including not only the fractional IDEs of the Fredholm-Volterra type but also the fractional IDEs of the Fredholm or Volterra type and the fractional differential equations. Specifically, the given general fractional IDE of the Fredholm-Volterra type is converted into the fractional IDE of the Volterra type for $\lambda_1=0$, $\lambda_2\neq0$; the fractional IDE of the Fredholm type for $\lambda_1\neq0$, $\lambda_2=0$ and the fractional differential equation for both $\lambda_1=\lambda_2=0$. For this reason, the relation for the matrix of the Caputo fractional derivative of the Laguerre polynomials and the related exact matrix relation have been obtained for the first time in the fractional calculus literature. Utilizing suitable collocation points and the obtained matrix relations, the fractional IDE was transformed into an algebraic equations system. This method is more efficient, faster, and easier to apply than the other methods in the literature.

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