Sakarya University Journal of Science

ISSN : 2147-835X
Publisher : Sakarya University

Vol. 28, No.2, 410-417, 2024
DOI: https://doi.org/10.16984/saufenbilder.1390617

Research Article

Schwarz Problem for Model Partial Differential Equations with One Complex Variable

Bahriye Karaca

Izmir Bakircay University, Faculty of Engineering and Architecture, Department of Fundamental Sciences, Izmir, Türkiye, bahriye.karaca@bakircay.edu.tr

ARTICLE INFO

Keywords:
Schwarz Problem
Complex model homogeneous partial differential equation
Complex model inhomogeneous partial differential equations

Article History:
Received: 14.11.2023
Accepted: 19.01.2024
Online Available: 24.04.2024

ABSTRACT

This paper investigates the Schwarz problem. Initially, the focus lies on analyzing the problem for the first, second orders. Subsequently, attention shifts towards studying the same problem for equations of higher order. In the realm of second-order equations, the Schwarz problem is specifically examined for some operators; Laplace, Bitsadze and its complex conjugate. The findings demonstrate that the Schwarz problem for an n-order equation, when equipped with solely one boundary condition, exhibits an infinite number of solutions. However, by incorporating additional boundary conditions, it becomes feasible to obtain a unique solution for problem concerning n-order equations, effectively rendering it a well-posed problem.

1. Introduction

Explicit solutions to different boundary value problems are investigated in [1-8] for different domains. The Schwarz problem has an important place in both real and complex analysis as one of the fundamental problems. This boundary value problem is widely recognized and extensively studied.

This paper aims to investigate the Schwarz problem for one complex variable with arbitrary order.

We will define $\mathbb{D}$ as the unit disc with a smooth boundary $\partial\mathbb{D}$, $\varsigma \in \mathbb{C}, \varsigma = \xi + i\eta$.

The following problem was solved with $w \equiv 0$ in [1].

$$w_{\bar{z}} = 0 \; in \; \mathbb{D}, \; \Re e w = 0 \;\; on \, \partial \, \mathbb{D}, \; \Im m w(0) = 0 \qquad (1)$$

The same problem with nonhomogeneous boundary conditions has a unique solution in the same domain and proved in Theorem 9 in [1].

The following problem is the complex conjugate of the problem (1). It has a trivial solution in $\mathbb{D}$.

$$w_z = 0 \; in \; \mathbb{D}, \Re e w = 0 \; on \, \partial\mathbb{D}, \Im m w(0) = 0 \qquad (2)$$

2. Schwarz Problem for Second Order Equations

In the realm of complex analysis, Laplace and Bitsadze operators hold fundamental importance. In this section, our focus is on addressing the Schwarz problem pertaining to these operators as well as the conjugate operator of the Bitsadze.

Lemma 1: [1] Within the context of the Laplace equation, the following Schwarz problem has infinitely many solutions

$$w_{z\bar{z}} = 0 \text{ in } \mathbb{D}, \;\; \Re e w = 0 \; on \, \partial\mathbb{D}, \Im m w(0) = 0$$

Proof. Since $w_{z\bar{z}} = 0$ in $\mathbb{D}$, w_z is analytic in $\mathbb{D}$. Integrating the quantity, we get

$$w(z) = \varphi_1(z) + \overline{\varphi_2(z)},$$

where φ_1, φ_2 are both analytic functions in $\mathbb{D}$. Since $\Re e w = 0$ on $\partial\mathbb{D}$ and $\Im m w(0) = 0$, we

Cite as: Karaca B. (2024). Schwarz Problem for Model Partial Differential Equations with One Complex Variable, *Sakarya University Journal of Science,* 28(2), 410-417. https://doi.org/10.16984/saufenbilder.1390617

obtain $\Re e\varphi_1(z) = -\Re e\varphi_2(z)$ on $\partial\mathbb{D}$ and $\mathcal{I}m\varphi_1(0) = Im\varphi_2(0)$.

From Theroem 9 in [1], we write

$$\varphi_1(z) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} -\Re e\varphi_2(\varsigma)\frac{\varsigma+z}{\varsigma-z}\frac{d\varsigma}{\varsigma} + i\,\mathcal{I}m\varphi_2(0) \quad (3)$$

It follows that

$$\varphi_1(z) = -\varphi_2(z) + \varphi_2(0) - \overline{\varphi_2(0)}.$$

Substituting $\varphi_1(z)$ into the $w(z)$, we have $w(z) = -2\,\mathcal{I}m\varphi_2(z) + 2\,\mathcal{I}m\varphi_2(0)$ for arbitrary analytic function $\varphi_2(z)$. □

The problem above has a unique solution over $\mathbb{C}$ with additional boundary conditions as shown in Theorem 1 in [1].

By swapping the roles of z and $\bar{z}$ in the result of Theorem 1 in [1], we obtain the dual result presented in [1].

Lemma 2: [1] The Schwarz problem for the Bitsadze equation

$$w_{\bar{z}\bar{z}} = 0 \text{ in } \mathbb{D}, \Re ew = 0 \; on\, \partial\mathbb{D}, \mathcal{I}mw(0) = 0,$$

has infinitely many solutions.

Proof. $w_{\bar{z}}$ is analytic in $\mathbb{D}$. Integrating $w_{\bar{z}\bar{z}} = 0$, we have $w(z) = h_1(z) + \bar{z}h_2(z)$, where $h_1,\ h_2$ are both analytic in $\mathbb{D}$. From the boundary conditions $\Re eh_1(z) = -\Re e\bar{z}h_2(z)$ on $\partial\mathbb{D}$ and $\mathcal{I}mh_1(0) = 0$ follows. This Schwarz problem has a unique solution as

$$h_1(z) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} -\Re e\bar{\varsigma}h_2(\varsigma)\frac{\varsigma+z}{\varsigma-z}\frac{d\varsigma}{\varsigma} + i\,\mathcal{I}mh_1(0)$$

and it follows

$$h_1(z) = \frac{-1}{z}h_2(z) + \frac{1}{z}h_2(0) - z\overline{h_2(0)} + \frac{1}{2}h_2'(0) - \frac{1}{2}\overline{h_2'(0)}$$

Then, we have the following

$$w(z) = h_2(z)\left(\frac{|z|^2-1}{z}\right) + \frac{1}{z}h_2(0) - z\overline{h_2(0)} + \frac{1}{2}h_2'(0) - \frac{1}{2}\overline{h_2'(0)}$$

with arbitrary analytic function $h_2(z)$ in $\mathbb{D}$. □

In Theorem 2 in [1], it is proved that by taking additional boundary conditions, the Schwarz problem in Lemma 2 has a unique solution.

Presently, we have the opportunity to examine the Schwarz problem concerning the subsequent equation

$$w_{zz} = 0 \text{ in } \mathbb{D},\ \Re ew = 0 \; on\, \partial\mathbb{D}, \mathcal{I}mw(0) = 0. \quad (4)$$

Lemma 3 follows from this result.

Lemma 3: The Schwarz problem (4) exhibits an infinite number of solutions over the complex plane, $\mathbb{C}$.

Proof. First the following transformation is applied.

$$\overline{w}_{\bar{z}\bar{z}} = 0 \text{ in } \mathbb{D},\ \Re e\overline{w} = 0 \text{ on } \partial\mathbb{D},\ \mathcal{I}m\overline{w}(0) = 0.$$

Then from Lemma 2 applied to $\overline{w}$, we get

$$w(z) = \overline{\varphi_2(z)}\left(\frac{|z|^2-1}{\bar{z}}\right) + \frac{1}{\bar{z}}\overline{\varphi_2(0)} - \bar{z}\varphi_2(0) + \frac{1}{2}\overline{\varphi_2'(0)} - \frac{1}{2}\varphi_2'(0)$$

with arbitrary analytic function $\varphi_2(z)$ in $\mathbb{D}$. □

Using boundary conditions again, the following result is obtained.

Corollary 4: The following Schwarz problem in $\mathbb{D}$

$$w_{zz} = f(z) \text{ in } \mathbb{D},\ \Re ew = \gamma_0(z),$$
$$\Re ew_z = \gamma_1(z) \text{ on } \partial\mathbb{D}, \mathcal{I}mw(0) = c_0, \mathcal{I}mw_z(0) = c_1$$

has a unique solution for
$f \in L_1(\mathbb{D};\mathbb{C})$ *and* $\gamma_0, \gamma_1\ C(\partial\mathbb{D};\mathbb{R}), c_0, c_1 \in \mathbb{R}$
with solution

$$\mathcal{w}(z) = i\, c_0 + ic_1(z + \bar{z}) + \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma_0(\zeta)\overline{\frac{\varsigma + z}{\varsigma - z}}\frac{d\varsigma}{\varsigma} - \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma_1(\varsigma)\overline{\frac{\varsigma + z}{\varsigma - z}}\left(\varsigma - z + \overline{\varsigma - z}\right)\frac{d\varsigma}{\varsigma} + \frac{1}{2\pi}\iint_{\mathbb{D}}\left(\frac{f(\varsigma)}{\bar{\varsigma}}\overline{\frac{\varsigma + z}{\varsigma - z}} + \frac{\overline{f(\varsigma)}}{\varsigma}\frac{1 + \bar{z}\varsigma}{1 - \bar{z}\varsigma}\right)\left(\varsigma - z + \overline{\varsigma - z}\right) d\xi d\eta$$

Proof. The issue can be rephrased in the following manner by taking the complex conjugation:

$$\overline{\mathcal{w}}_{\bar{z}\bar{z}} = \overline{f(z)} \text{ in } \mathbb{D}, \Re e\overline{\mathcal{w}} = \gamma_0(z),$$

$$\Re e\overline{\mathcal{w}}_{\bar{z}} = \gamma_1(z) \text{ on } \partial\mathbb{D},$$
$$\mathcal{I}m\overline{\mathcal{w}(0)} = c_0, \mathcal{I}m\overline{\mathcal{w}_z(0)} = c_1$$

The desired result is obtained by applying the result in Theorem 2 in [1] to $\overline{\mathcal{w}}$. □

3. Schwarz Problem for Higher Order Differential Equations

In this section, we explored the Schwarz problem concerning n-order model differential equations in the complex domain, $\mathbb{C}$.

3.1. Auxiliary lemmas

Lemma 5: [2] Suppose that f(z) is analytic function in $\mathbb{D}$. We obtain the equation (5).

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \bar{\varsigma}^l f(\varsigma)\frac{d\varsigma}{\varsigma - z} = \frac{f(z)}{z^l} - \sum_{r=1}^{l}\frac{1}{z^{l-r+1}}\frac{f^{(r-1)}(0)}{(r-1)!},\ l \geq 0. \tag{5}$$

Lemma 6: [2] Suppose that f(z) is analytic function in $\mathbb{D}$. We get the equation (6) for k≥0.

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \varsigma^k\,\overline{f(\varsigma)}\frac{d\zeta}{\varsigma - z} = \sum_{m=0}^{k}\frac{z^{k-m}\overline{f^{(m)}(0)}}{m!} \tag{6}$$

3.2. The Schwarz problem for polyanalytic equations

The solution of $\mathcal{w}_{\bar{z}^n} = 0$ is called polyanalytic function. Theorem 2 in [1] is generalized as follows:

Lemma 7: The Schwarz problem below exhibits an infinite number of solutions in the unit disc of the complex plane. For l>2 we get,

$$\mathcal{w}_{\bar{z}^l} = 0 \text{ in } \mathbb{D},\ \Re e\mathcal{w} = 0 \text{ on } \partial\mathbb{D},\ \mathcal{I}m\mathcal{w}(0) = 0,$$

Proof. By integrating the given quantity, it follows that:

$$\mathcal{w}(z) = \sum_{n=1}^{l}\bar{z}^{n-1}\varphi_n(z),$$

for $\varphi_i,\ i = 1, \ldots, l$. They are analytic in $\mathbb{D}$. We have

$$\Re e\varphi_1(z) = -\Re e\sum_{n=2}^{l}\bar{z}^{n-1}\varphi_n(z) \text{ on } \partial\mathbb{D}$$

and

$$\mathcal{I}m\varphi_1(0) = 0.$$

We get the equation below after applying the Theorem 9 in [1].

$$\varphi_1(z) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} -\Re e\sum_{n=2}^{l}\bar{\varsigma}^{n-1}\varphi_n(\varsigma)\left(\frac{\varsigma + z}{\varsigma - z}\right)\frac{d\varsigma}{\varsigma} + i0,$$

$$\varphi_1(z) = \frac{-1}{2\pi i}\int_{\partial\mathbb{D}}\sum_{n=2}^{l}\left(\bar{\varsigma}^{n-1}\varphi_n(\varsigma) + \varsigma^{n-1}\overline{\varphi_n(\varsigma)}\right)\frac{d\varsigma}{\varsigma - z} + \frac{1}{2}\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\sum_{n=2}^{l}\left(\bar{\varsigma}^{n-1}\varphi_n(\varsigma) + \varsigma^{n-1}\overline{\varphi_n(\varsigma)}\right)\frac{d\varsigma}{\varsigma}.$$

From Lemma 5 and Lemma 6, we have

$$\varphi_1(z) = -\sum_{n=2}^{l}\frac{\varphi_n(z)}{z^{n-1}} + \sum_{n=2}^{l}\sum_{m=0}^{n-2}\frac{1}{z^{n-m-1}}\frac{\varphi_n^{(m)}(0)}{m!} - \sum_{n=2}^{l}\sum_{m=0}^{n-2}z^{n-m-1}\frac{\overline{\varphi_n^{(m)}(0)}}{m!} + \frac{1}{2}\sum_{n=2}^{l}\left(\frac{\varphi_n^{(n-1)}(0)}{(n-1)!} - \frac{\overline{\varphi_n^{(n-1)}(0)}}{(n-1)!}\right).$$

Therefore, $\mathcal{w}(z)$ is equal to

$$\sum_{n=2}^{l}\varphi_n(z)\left(\frac{|z|^{2(n-1)} - 1}{z^{n-1}}\right) + \sum_{n=2}^{l}\sum_{m=0}^{n-2}\left(\frac{1}{z^{n-m-1}}\frac{\varphi_n^{(m)}(0)}{m!} - z^{n-m-1}\frac{\overline{\varphi_n^{(m)}(0)}}{m!}\right)$$

$$+\frac{1}{2}\sum_{n=2}^{l}\left(\frac{\varphi_n^{(n-1)}(0)}{(n-1)!}-\frac{\overline{\varphi_n^{(n-1)}(0)}}{(n-1)!}\right) \qquad \square$$

Unless additional boundary condtions are imposed, the Schwarz problem for higher order model equations is known to be undetermined.

Theorem 8: [1] The Schwarz problem for inhomogeneous polyanalytic equation

$$w_{\bar{z}^k} = f(z) \text{ in } \mathbb{D},\ \Re e\, \partial_{\bar{z}}^{\nu} w = \gamma_\nu \text{ on } \partial\mathbb{D},$$

$$\mathcal{I}m\, \partial_{\bar{z}}^{\nu}\, w(0) = c_\nu \ \ 0 \le \nu \le k-1,$$

has a unique solution for $\gamma_\nu \in C(\partial\mathbb{D};\mathbb{C})$, $c_\nu \in \mathbb{R}$, $0 \le \nu \le k-1$ as

$$w(z) = \mathrm{i}\sum_{\nu=0}^{k-1}\frac{c_\nu}{\nu!}(z+\bar{z})^{\nu}$$

$$+\sum_{\nu=0}^{k-1}\frac{(-1)^{\nu}}{\nu!}\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma_\nu(\varsigma)\frac{\varsigma+z}{\varsigma-z}\left(\varsigma-z+\overline{\varsigma-z}\right)^{\nu}\frac{d\varsigma}{\varsigma}$$

$$+\frac{(-1)^k}{(k-1)!}\frac{1}{2\pi}\iint_{\mathbb{D}}\left(\frac{f(\varsigma)}{\varsigma}\frac{\varsigma+z}{\varsigma-z}+\frac{\overline{f(\varsigma)}}{\bar{\varsigma}}\frac{1+z\bar{\varsigma}}{1-z\bar{\varsigma}}\right) \mathrm{x}\left(\varsigma-z+\overline{\varsigma-z}\right)^{k-1}d\xi d\eta$$

Lemma 9: The Schwarz problem

$$w_{z^k} = 0 \ \text{ in } \mathbb{D},\ \Re e w = 0 \ \text{ on } \ \partial\mathbb{D},$$
$$\mathcal{I}m w(0) = 0,\ k \in \mathbb{Z}^+,\ k > 2$$

has infinitely many solutions in the complex space.

Proof. Taking the complex conjugation of the problem in Lemma 7 and then replacing $\overline{w}$ by w leads to
$w_{z^k} = 0$ in $\mathbb{D}$, $\Re e w = 0$ on $\partial\mathbb{D}$, $\mathcal{I}m w(0) = 0$.

Therefore, we get $w(z)$ as

$$w(z) = \sum_{n=2}^{k}\overline{\varphi_n(z)}\left(\frac{|z|^{2(n-1)}-1}{\bar{z}^{n-1}}\right)$$

$$+\sum_{n=2}^{k}\sum_{m=0}^{n-2}\left(\frac{1}{\bar{z}^{n-m-1}}\frac{\overline{\varphi_n^{(m)}(0)}}{m!}-\bar{z}^{n-m-1}\frac{\varphi_n^{(m)}(0)}{m!}\right)$$

$$+\frac{1}{2}\sum_{n=2}^{k}\left(\frac{\overline{\varphi_n^{(n-1)}(0)}}{(n-1)!}-\frac{\varphi_n^{(n-1)}(0)}{(n-1)!}\right)$$

for analytic functions $\varphi_i(z), i = 2,\dots,k$ in $\mathbb{D}$. $\square$

Similarly, the unique solution for the problem

$w_{z^k} = 0$ in $\mathbb{D}$, $\Re e\, \partial_z^{\nu}\, w =$
γ_ν on $\partial\mathbb{D}$, $\mathcal{I}m\, \partial_z^{\nu}\, w(0) = c_\nu$,
$k \in \mathbb{Z}^+$, $0 \le \nu \le k-1$
can be obtained, as stated in Theorem 10.

Theorem 10: The Schwarz problem for inhomogeneous equation

$$w_{z^k} = f(z) \text{ in } \mathbb{D},\ \Re e\, \partial_z^{\nu}\, w = \gamma_\nu \text{ on } \partial\mathbb{D},$$
$$\mathcal{I}m\, \partial_z^{\nu}\, w(0) = c_\nu \ \ 0 \le \nu \le k-1, \tag{7}$$

has a unique solution for $\gamma_\nu \in C(\partial\mathbb{D};\mathbb{C})$, $c_\nu \in \mathbb{R}$, $0 \le \nu \le k-1$ as

$$w(\mathrm{z}) = \mathrm{i}\sum_{\nu=0}^{k-1}\frac{c_\nu}{\nu!}(z+\bar{z})^{\nu}$$

$$+\sum_{\nu=0}^{k-1}\frac{(-1)^{\nu}}{\nu!}\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma_\nu(\zeta)\frac{\overline{\varsigma+z}}{\overline{\varsigma-z}}\left(\varsigma-z+\overline{\varsigma-z}\right)^{\nu}\frac{d\varsigma}{\varsigma}$$
$$+\frac{(-1)^k}{(k-1)!}\frac{1}{2\pi}\iint_{\mathbb{D}}\left(\frac{f(\varsigma)}{\bar{\varsigma}}\frac{\overline{\varsigma+z}}{\overline{\varsigma-z}}+\frac{\overline{f(\varsigma)}}{\varsigma}\frac{1+\bar{z}\varsigma}{1-\bar{z}\varsigma}\right)$$
$$\mathrm{x}\left(\varsigma-z+\overline{\varsigma-z}\right)^{k-1}d\xi d\eta$$

Proof. The anticipated outcome is a consequence of applying Theorem 8 to the complex conjugate of equation (7) with respect to the variable $\overline{w}$. $\square$

3.3. The Schwarz problem for mixed higher order partial differential equations

Lemma 11: The Schwarz problem
$w_{z^m\bar{z}^n} = 0$ in $\mathbb{D}$,
$\Re e w = 0$ on $\partial\mathbb{D}$, $\mathcal{I}m w(0) = 0$,
$n+m > 2$, $n \ge 1, m \ge 1$

has infinitely many solutions.

Proof. Upon integrating $w_{z^m\bar{z}^n} = 0$ with respect to both z and $\bar{z}$ we obtain

$$w(z) = \sum_{k=1}^{n} \bar{z}^{n-k} \varphi_{k-1}(z)$$

$$+ \sum_{k=n+1}^{n+m} z^{n+m-k} \overline{\varphi_{k-1}(z)}, \quad z \in \mathbb{D},$$

where $\varphi_k(z)$, $0 \leq k \leq n+m-1$ are analytic functions. Since $\Re e w = 0$ on $\partial\mathbb{D}$, and $\mathcal{I}m w(0) = 0$, we obtain

$$\Re e \varphi_{n-1}(z) = -\Re e \sum_{k=1}^{n-1} \bar{z}^{n-k} \varphi_{k-1}(z)$$

$$-\Re e \sum_{k=0}^{m-1} z^{m-k-1} \overline{\varphi_{n+k}(z)} \text{ on } \partial\mathbb{D}$$

and

$$\mathcal{I}m\varphi_{n-1}(0) = \mathcal{I}m\varphi_{n+m-1}(0).$$

$\varphi_{n-1}(z)$ is analytic in $\in \mathbb{D}$. Therefore, from Theorem 9 in [1], we obtain

$$\varphi_{n-1}(z) = \frac{-1}{2\pi i} \int_{\partial\mathbb{D}} Re\left(\sum_{k=1}^{n-1} \bar{\varsigma}^{n-k} \varphi_{k-1}(\varsigma) + \sum_{k=0}^{m-1} \varsigma^{m-k-1} \overline{\varphi_{n+k}(\varsigma)}\right) \frac{\varsigma + z}{\varsigma - z}$$

$$+ i\,\mathcal{I}m\varphi_{n+m-1}(0).$$

From Lemma 5 and Lemma 6, we have

$$\varphi_{n-1}(z) = -\sum_{k=1}^{n-1}\left(\frac{\varphi_{k-1}(z)}{z^{n-k}} - \sum_{m=1}^{n-k} \frac{1}{z^{n-k-m+1}} \frac{\varphi_{k-1}^{(m-1)}(0)}{(m-1)!}\right)$$

$$-\sum_{k=1}^{n-1}\left(\sum_{m=0}^{n-k} z^{n-k-m} \frac{\overline{\varphi_{k-1}^{(m-1)}(0)}}{m!}\right)$$

$$-\sum_{k=0}^{m-1}\left(\sum_{l=0}^{m-k-1} z^{m-k-1-l} \frac{\overline{\varphi_{n+k}^{(l)}(0)}}{l!}\right)$$

$$-\sum_{k=0}^{m-1}\left(\frac{\varphi_{n+k}(z)}{z^{m-k-1}} - \sum_{l=1}^{m-k-1} \frac{1}{z^{m-k-l}} \frac{\varphi_{n+k}^{(l-1)}(0)}{(l-1)!}\right)$$

$$+ \sum_{k=1}^{n-1}\left(\frac{\varphi_{k-1}^{(n-k)}(0)}{(n-k)!} + \frac{\overline{\varphi_{k-1}^{(n-k)}(0)}}{(n-k)!}\right)$$

$$+ \sum_{k=0}^{m-1}\left(\frac{\overline{\varphi_{n+k}^{(m-k-1)}(0)}}{(m-k-1)!} + \frac{\varphi_{n+k}^{(m-k-1)}(0)}{(m-k-1)!}\right)$$

$$+ \frac{\varphi_{n+m-1}(0)}{2} - \frac{\overline{\varphi_{n+m-1}(0)}}{2}.$$

Substituting $\varphi_{n-1}(z)$ into $w(z)$, we get

$$w(z) = \sum_{k=1}^{n-1}\left(\frac{|z|^{2(n-k)} - 1}{z^{n-k}}\right)\varphi_{k-1}(z)$$

$$+ \sum_{k=1}^{n-1}\sum_{m=0}^{n-k-1}\left(\frac{1}{z^{n-k-m}} \frac{\varphi_{k-1}^{(m)}(0)}{m!} - z^{n-k-m} \frac{\overline{\varphi_{k-1}^{(m)}(0)}}{m!}\right)$$

$$+ \sum_{k=0}^{m-1}\sum_{l=0}^{m-k-2}\left(\frac{1}{z^{m-k-l-1}} \frac{\varphi_{n+k}^{(l)}(0)}{l!} - z^{m-k-l-1} \frac{\overline{\varphi_{n+k}^{(l)}(0)}}{l!}\right)$$

$$+ \sum_{k=0}^{m-1}\left(z^{m-k-1}\overline{\varphi_{n+k}(z)} - \frac{1}{z^{m-k-1}} \varphi_{n+k}(z)\right)$$

$$+ \frac{\varphi_{n+m-1}(0)}{2} - \frac{\overline{\varphi_{n+m-1}(0)}}{2}$$

for arbitrary analytic functions $\varphi_k(z)$,
$0 \leq k \leq n-2,\ n \leq k \leq n+m-1.$ □

Theorem 12: The Schwarz problem for the inhomogeneous higher order equation in $\mathbb{D}$

$$w_{z^m\bar{z}^n} = f(z) \text{ in } \mathbb{D}, \quad \Re e\, \partial_z^{\nu} w = \gamma_\nu,$$

$$\Re e\, \partial_{\bar{z}}^{\mu}(\partial_z^m w) = \Upsilon_\mu \text{ on } \partial\mathbb{D},$$

$$\mathcal{I}m\, \partial_z^{\nu} w(0) = c_\nu, \ \mathcal{I}m\, \partial_{\bar{z}}^{\mu}(\partial_z^m w)(0) = c_\mu,$$

$$0 \le \nu \le m-1, \ 0 \le \mu \le n-1,$$
$$m, n \in \mathbb{Z}^+, \ \ m+n > 2, \ \ n, m \ge 1$$

is uniquely solvable for $\gamma_\nu, \Upsilon_\mu \in C(\partial\mathbb{D}; \mathbb{C})$as

$$w(z) = i \sum_{\nu=0}^{m-1} \frac{c_\nu}{\nu!} (z+\bar{z})^\nu$$

$$+ \sum_{\nu=0}^{m-1} \frac{(-1)^\nu}{\nu!} \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \gamma_\nu(\zeta) \frac{\overline{\zeta+z}}{\overline{\zeta-z}} \left(\zeta - z + \overline{\zeta-z}\right)^\nu \frac{d\zeta}{\zeta}$$

$$+ i \sum_{\nu=0}^{n-1} \frac{(-1)^m}{(m-1)!} \frac{c_\nu}{\nu!} A(\zeta, z)$$

$$+ \frac{(-1)^m}{(m-1)!} \sum_{\nu=0}^{n-1} \frac{(-1)^\nu}{\nu!} \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \gamma_\nu(\zeta) B(\zeta, z) \frac{d\zeta}{\zeta}$$

$$+ \frac{(-1)^m}{(m-1)!} \frac{(-1)^n}{(n-1)!} \frac{1}{2\pi} \iint_{\mathbb{D}} \frac{f(\zeta)}{\zeta} \big(C(\zeta, z) + F(\zeta, z)\big) d\xi d\eta$$

$$+ \frac{(-1)^m}{(m-1)!} \frac{(-1)^n}{(n-1)!} \frac{1}{2\pi} \iint_{\mathbb{D}} \frac{\overline{f(\zeta)}}{\bar{\zeta}} \big(D(\zeta, z) + E(\zeta, z)\big) d\xi d\eta$$

where

$$A(\zeta, z) = \frac{1}{2\pi} \iint_{\mathbb{D}} \left(\zeta + \bar{\zeta}\right)^\nu \left(\frac{1}{\bar{\zeta}} \frac{\overline{\zeta+z}}{\overline{\zeta-z}} - \frac{1}{\zeta} \frac{1+\bar{z}\zeta}{1-\bar{z}\zeta}\right) x \left(\zeta - z + \overline{\zeta-z}\right)^{m-1} d\xi d\eta,$$

$$B(\zeta, z) = \frac{1}{2\pi} \iint_{\mathbb{D}} \left(\tilde{\zeta} - \zeta + \overline{\tilde{\zeta} - \zeta}\right)^\nu \left(\frac{\overline{\tilde{\zeta}+\zeta}}{\overline{\tilde{\zeta}-\zeta}} \frac{1}{\bar{\zeta}} \frac{\overline{\zeta+z}}{\overline{\zeta-z}} + \frac{\overline{\tilde{\zeta}+\zeta}}{\overline{\tilde{\zeta}-\zeta}} \frac{1}{\zeta} \frac{1+\bar{z}\zeta}{1-\bar{z}\zeta}\right) x \left(\zeta - z + \overline{\zeta-z}\right)^{m-1} d\xi d\eta,$$

$$C(\zeta, z) = \frac{1}{2\pi} \iint_{\mathbb{D}} \frac{\tilde{\zeta}+\zeta}{\tilde{\zeta}-\zeta} \frac{1}{\bar{\zeta}} \frac{\overline{\zeta+z}}{\overline{\zeta-z}} \left(\tilde{\zeta} - \zeta + \overline{\tilde{\zeta}-\zeta}\right)^{n-1} x \left(\zeta - z + \overline{\zeta-z}\right)^{m-1} d\xi d\eta,$$

$$D(\zeta, z) = \frac{1}{2\pi} \iint_{\mathbb{D}} \frac{1+\zeta\bar{\tilde{\zeta}}}{1-\zeta\bar{\tilde{\zeta}}} \left(\tilde{\zeta} - \zeta + \overline{\tilde{\zeta}-\zeta}\right)^{n-1} x \frac{1}{\bar{\zeta}} \frac{\overline{\zeta+z}}{\overline{\zeta-z}} \left(\zeta - z + \overline{\zeta-z}\right)^{m-1} d\xi d\eta,$$

$$E(\zeta, z) = \frac{1}{2\pi} \iint_{\mathbb{D}} \frac{\overline{\tilde{\zeta}+\zeta}}{\overline{\tilde{\zeta}-\zeta}} \frac{1}{\zeta} \left(\tilde{\zeta} - \zeta + \overline{\tilde{\zeta}-\zeta}\right)^{n-1} x \frac{1+\bar{z}\zeta}{1-\bar{z}\zeta} \left(\zeta - z + \overline{\zeta-z}\right)^{m-1} d\xi d\eta,$$

$$F(\zeta, z) = \frac{1}{2\pi} \iint_{\mathbb{D}} \frac{1+\bar{\zeta}\tilde{\zeta}}{1-\bar{\zeta}\tilde{\zeta}} \left(\tilde{\zeta} - \zeta + \overline{\tilde{\zeta}-\zeta}\right)^{n-1} x \frac{1}{\zeta} \frac{1+\bar{z}\zeta}{1-\bar{z}\zeta} \left(\zeta - z + \overline{\zeta-z}\right)^{m-1} d\xi d\eta.$$

Proof. The problem can be stated in another way as follows:

$$\partial_z^m w = \omega \ \text{ in } \ \mathbb{D}, \ \Re e\, \partial_z^\nu w = \gamma_\nu \text{ on } \partial\mathbb{D},$$

$$\mathcal{I}m\, \partial_z^\nu w = c_\nu, \ 0 \le \nu \le m-1,$$

$$\partial_{\bar{z}}^n \omega = f(z) \ \text{ in } \mathbb{D}, \ \Re e\, \partial_{\bar{z}}^\mu(\partial_z^m w) = \Upsilon_\mu \text{ on } \partial\mathbb{D},$$

$$\mathcal{I}m\, \partial_{\bar{z}}^\mu(\partial_z^m w) = c_\mu, \ 0 \le \mu \le n-1.$$

Using theorems 10 and 8 respectively, the unique solutions are given as follows

$$w(z) = i \sum_{\nu=0}^{m-1} \frac{c_\nu}{\nu!} (z+\bar{z})^\nu + \sum_{\nu=0}^{m-1} \frac{(-1)^\nu}{\nu!} \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \gamma_\nu(\zeta) \frac{\overline{\zeta+z}}{\overline{\zeta-z}} \left(\zeta - z + \overline{\zeta-z}\right)^\nu \frac{d\zeta}{\zeta}$$
$$+ \frac{(-1)^m}{(m-1)!} \frac{1}{2\pi} \iint_{\mathbb{D}} \left(\frac{\omega(\zeta)}{\bar{\zeta}} \frac{\overline{\zeta+z}}{\overline{\zeta-z}} + \frac{\overline{\omega(\zeta)}}{\zeta} \frac{1+\bar{z}\zeta}{1-\bar{z}\zeta}\right) x \left(\zeta - z + \overline{\zeta-z}\right)^{m-1} d\xi d\eta, \quad (8)$$

$$\omega(z) = i \sum_{\nu=0}^{n-1} \frac{c_\nu}{\nu!} (z+\bar{z})^\nu$$

$$+\sum_{\nu=0}^{n-1}\frac{(-1)^{\nu}}{\nu!}\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma_{\nu}(\varsigma)\frac{\varsigma+z}{\varsigma-z}\left(\varsigma-z+\overline{\varsigma-z}\right)^{\nu}\frac{d\varsigma}{\varsigma}$$

$$+\frac{(-1)^{n}}{(n-1)!}\frac{1}{2\pi}\iint_{\mathbb{D}}\left(\frac{f(\varsigma)}{\varsigma}\frac{\varsigma+z}{\varsigma-z}+\frac{\overline{f(\varsigma)}}{\overline{\varsigma}}\frac{1+z\overline{\varsigma}}{1-z\overline{\varsigma}}\right)x\left(\varsigma-z+\overline{\varsigma-z}\right)^{n-1}d\xi d\eta$$

Inserting ω into (8), we achieve the desired outcome. □

4. Conclusion

This paper explores the analysis of higher order complex partial differential equations in $\mathbb{C}$. The focus lies on investigating the uniqueness of the Schwarz problem. The findings reveal that there exist an infinite number of solutions for the Schwarz problem in the context of higher order complex partial differential equations in $\mathbb{C}$ with only two boundary conditions. The ideas here can be extended to the multidimensional case.

Article Information Form

Funding
The author has no received any financial support for the research, authorship or publication of this study.

The Declaration of Conflict of Interest/ Common Interest
No conflict of interest or common interest has been declared by the authors.

The Declaration of Ethics Committee Approval
This study does not require ethics committee permission or any special permission.

The Declaration of Research and Publication Ethics
The author of the paper declares that she complies with the scientific, ethical and quotation rules of SAUJS in all processes of the paper and that they do not make any falsification on the data collected. In addition, they declare that Sakarya University Journal of Science and its editorial board have no responsibility for any ethical violations that may be encountered, and that this study has not been evaluated in any academic publication environment other than Sakarya University Journal of Science.

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