Hacettepe Journal of
Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 54 (2) (2025), 542 – 561
DOI : 10.15672/hujms.1391474

Research Article

Local non-abelian class field theory

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In memory of my mother Silva Bedikyan

Abstract

The "local class field theory", which can be defined as the description of the extensions of a given local field K with finite residue field of $q = p^f$ elements in terms of the algebraic and analytic objects depending only on the base K is one of the central problems of modern number theory. The theory developed for the abelian extensions, around the fundamental works of Artin and Hasse in the first quarter of the 20th century.

It is natural to ask if one could construct this theory including the non-abelian extensions of the base field. There are two approches to this problem. One approach is based on the ideas of Langlands, and the other on Koch. Koch's method was later generalized by Fesenko and Koch-de Shalit for specific type of non-abelian extensions of the base field. Laubie extended Koch-de Shalit's work and constructed a local non-abelian class field theory for K. On the other hand, İkeda and Serbest extended Fesenko's works to construct a non-abelian local class field theory for K, containing a p^{th} root of unity.

In this study, we extended İkeda-Serbest's construction of the local reciprocity map for K containing a p^{th} root of unity to *any* local field. Also we have shown that the extended map satisfies the certain functoriality and ramification theoretic properties.

Mathematics Subject Classification (2020). 11S37

Keywords. local fields, local class field theory, local non-abelian reciprocity map

1. Introduction

Let K be a local field; that is a complete discrete valuation field with finite residue class field $O_K/\mathfrak{p}_K =: \kappa_K$ of $q_K = q = p^f$ elements with p a prime number: Here, O_K denotes the ring of integers in K with the unique maximal ideal $\mathfrak{p}_K$. As usual, the unit group of K is denoted by U_K and the i^{th} higher unit group of K by U_K^i, where $0 \leq i \in \mathbb{Z}$. One of the main problems of algebraic number theory is to describe the *arithmetical structure* of each Galois extensions L/K lying in the fixed separable closure K^{sep} of K, in terms of the certain invariants depending on the base field K. By the "arithmetical structure" of the extension L/K, we mean the ramification theoretic properties of the Galois group $\mathrm{Gal}(L/K)$.

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Received: 17.01.2024; Accepted: 28.05.2024

When the extensions of K are abelian, the answer of this problem, which is known as the *local abelian class field theory* establishes a unique "natural" algebraic and topological isomorphism called the *Artin reciprocity law*

$$\mathrm{Art}_K : G_K^{\mathrm{ab}} \xrightarrow{\sim} \widehat{K^\times}$$

of K, introduced by Artin and Takagi. Here, G_K^{ab} denotes the maximal abelian Hausdorff quotient group G_K/G_K' of the absolute Galois group $G_K = \mathrm{Gal}(K^{\mathrm{sep}}/K)$ of K, where G_K' denotes the closure of the 1^{st}-commutator subgroup $[G_K, G_K]$ of G_K. On the other hand $\widehat{K^\times}$ denotes the profinite completion of the multiplicative group $K^\times$. By the naturality of Art_K, we mean, it satisfies "existence", "functoriality" and "ramification theoretic" certain properties. See [6], [10], [11], [22] and [23] for details.

There are two "apparently different" approaches for the solution to the problem including non-abelian extensions. One approach, based on the idea to construct a "natural" correspondence between the set of the n dimensional representations of the absolute Galois group G_K of K, and the set of the automorphic representations of $GL(n, K)$ (Langlands' philosophy). Another approach proposes to use of the property of G_K to be a profinite group (Koch's philosophy).

The studies based on Koch's approach are as follows: Koch and de Shalit constructed the *metabelian local class field theory* to describe the arithmetic structure of 2-step abelian extensions of K ([18,19]), and Gurevich extended this theory for n-step abelian extensions (see [9]). There is a construction of non-abelian local class field theory which belongs to Laubie ([21]), by generalizing the work of Koch and de Shalit. On the other hand, Fesenko described the arithmetical structure of each totaly ramified arithmeticaly profinite (APF) Galois extension satisfying $K \subseteq L \subseteq K_{\varphi_K}$, where φ_K is a fixed extension of the Frobenius automorphism of K^{nr} to K^{sep} (Lubin-Tate splitting over K), and K_{φ_K} denotes the fixed field of φ_K. Also he showed that, that the theories of Koch-de Shalit and Gurevich can be obtained as partial cases of his theory. Note that, Fesenko's construction needs the assumption

$$\mu_p(K^{\mathrm{sep}}) \subset K \tag{1.1}$$

where, $\mu_p(K^{\mathrm{sep}})$ denotes the group of all p^{th} roots of unity ([3–5]).

Later, İkeda and Serbest generalized Fesenko's theory to APF Galois extensions lying in $K \subseteq L \subseteq K_{\varphi_K^d}$, where d equals the degree of the residue field extension κ_L/κ_K. Moreover, they introduced certain APF Galois extensions $K \subseteq \Gamma_d^{(n)}$ for each positive integers n, d, and proved that

$$G_K = \varprojlim_{(n,d)} \mathrm{Gal}(\Gamma_d^{(n)}/K)$$

Hence they constructed the *local non-abelian reciprocity map*

$$\mathbf{\Phi}_K^{(\varphi_K)} : G_K \xrightarrow{\sim} \nabla_K^{(\varphi_K)}$$

for K ([13–16]). Here, $\nabla_K^{(\varphi_K)}$ denotes the certain group, which is defined in terms of the Fontaine-Winterberger field of norms (for detailed information about field of norms, see [7,8]). Also they showed that $\mathbf{\Phi}_K^{(\varphi_K)}$ is *natural*, that is, it satisfies *existence*, *functoriality*, and *ramification theoretic* properties. Furthermore, in [15], they remarked a method to construct the local non-abelian reciprocity map $\mathbf{\Phi}_K^{(\varphi_K)}$ for a general local field K, not need to satisfy the condition (1.1). Moreover, Kazancıoğlu in his thesis has shown that Laubie reciprocity map and İkeda-Serbest reciprocity map are equivalent ([17]).

The aim of present paper is to remove the condition (1.1), and construct the non-abelian class field theory for any local field K in the light of [15].

The organization of the paper is as follows: In section 2 we construct the non-abelian reciprocity map $\mathbf{\Phi}_K^{(\varphi_{K_0})}$ of *any* local field K. We make this construction by glueing the abelian reciprocity map $\mathrm{Art}_{K_0/K}$ of K_0/K and the non-abelian reciprocity map $\mathbf{\Phi}_{K_0}^{(\varphi_{K_0})}$ of K_0. Here $K_0 = K(\zeta_p)$, where ζ_p denotes a primitive p^{th} root of unity. Note that, *the theory of extensions of profinite groups* plays fundamental role in our construction. In section 3, we prove the certain funtoriality and ramification theoretic properties of $\mathbf{\Phi}_K^{(\varphi_{K_0})}$.

2. Local non-abelian reciprocity map

From now on, K will denote *any* local field; that is, it does not need to satisfy the condition (1.1). İkeda and Serbest remarked a method to construct the local non-abelian reciprocity map for K (see section 8 of [15]). In this section, we will construct the non-abelian reciprocity map for K by following their strategy. Briefly, this can be done as follows: Consider the local field $K_0 = K(\zeta_p)$ where ζ_p denotes a primitive p^{th} root of unity. Since K_0/K is abelian, by *isomorphism theorem of local abelian class field theory*, there exists a unique topological isomorphism

$$\mathrm{Art}_{K_0/K} : \mathrm{Gal}(K_0/K) \xrightarrow{\sim} K^\times/\mathrm{N}_{K_0/K}K_0^\times \tag{2.1}$$

called *the Artin map for* K_0/K. On the other hand, since K_0 contains p^{th} roots of unity, by the *main theorem of local non-abelian class field theory of İkeda and Serbest*, there exists a unique topological group isomorphism

$$\mathbf{\Phi}_{K_0}^{(\varphi_{K_0})} : G_{K_0} \xrightarrow{\sim} \nabla_{K_0}^{(\varphi_{K_0})}$$

for K_0. Here, as usual, φ_{K_0} denotes the Lubin-Tate splitting over K_0. In order to construct the local non-abelian reciprocity map for K, the main idea is to glue properly $\mathrm{Art}_{K_0/K}$ with $\mathbf{\Phi}_{K_0}^{(\varphi_{K_0})}$. This can be done by following construction steps: As a first step, we will reconstruct the profinite group G_K in terms of $\mathrm{Gal}(K_0/K)$ and G_{K_0}. As a second step, we shall construct a topological group structure in terms of the topological groups $\nabla_{K_0}^{(\varphi_{K_0})}$ and $K^\times/\mathrm{N}_{K_0/K}K_0^\times$. We shall denote the resulting topological group by $\nabla_K^{(\varphi_{K_0})}$. As a final step, we will define a topological group isomorphism

$$\mathbf{\Phi}_K^{(\varphi_{K_0})} : G_K \xrightarrow{\sim} \nabla_K^{(\varphi_{K_0})} .$$

2.1. Construction steps of the local non-abelian reciprocity map $\mathbf{\Phi}_K^{(\varphi_{K_0})}$

Throughout this part of the section, we will adapt the methods used in [2] for our setting.

2.1.1. Reconstruction of G_K in terms of $\mathrm{Gal}(K_0/K)$ and G_{K_0}. Since $G_{K_0} \trianglelefteq G_K$ and $G_K/G_{K_0} \cong \mathrm{Gal}(K_0/K)$, one can view G_K as a group extension of G_{K_0} by $\mathrm{Gal}(K_0/K)$. Namely, there is an exact sequence of the form

$$1 \longrightarrow G_{K_0} \xrightarrow{inc.} G_K \xrightarrow{\mathrm{res}_{K_0}} \mathrm{Gal}(K_0/K) \longrightarrow 1 , \tag{2.2}$$

where res_{K_0} is the restriction map defined by $\mathrm{res}_{K_0} : \sigma \mapsto \sigma|_{K_0}$.

By Proposition 1.3.2 of [24], we fix a continuous section

$$s : \mathrm{Gal}(K_0/K) \to G_K$$

which is *normalized*, i.e. it satisfies $s(1_{\mathrm{Gal}(K_0/K)}) = 1_{G_K}$.

Remark 2.1. *Since* $\mathrm{Gal}(K_0/K)$ *is a finite group, then one can also define a continuous section* s *as follows: Choose a complete set of representatives* $\mathcal{S} \subset G_K$ *for* G_K/G_{K_0}*, with* $1_{G_K} \in \mathcal{S}$*. The set* $\mathcal{S}$ *is finite, hence closed subset of* G_K*. For each* $\tau \in \mathrm{Gal}(K_0/K)$*, define* $s(\tau)$ *as the unique element of* $\mathcal{S}$ *satisfying* $res_{K_0}(s(\tau)) = \tau$*. Then, the map* $s : \mathrm{Gal}(K_0/K) \to G_K$ *which sends each* $\tau \in \mathrm{Gal}(K_0/K)$ *to* $s(\tau)$ *defines a normalized contionuous section.*

The continuous section s determines a pair of continuous maps (f, ψ^*)

$$f : \mathrm{Gal}(K_0/K) \times \mathrm{Gal}(K_0/K) \to G_{K_0} \tag{2.3}$$

and

$$\psi^* : \mathrm{Gal}(K_0/K) \to \mathrm{Aut}(G_{K_0}) \quad , \tag{2.4}$$

satisfying

$$\psi^*(\tau) \circ \psi^*(\tau') = \alpha(f(\tau, \tau')) \circ \psi^*(\tau\tau'); \tag{2.5}$$

$$f(\tau, \tau')f(\tau\tau', \tau'') = \left({}^{\psi^*(\tau)}f(\tau', \tau'')\right) f(\tau, \tau'\tau''), \tag{2.6}$$

$$f(1_{\mathrm{Gal}(K_0/K)}, 1_{\mathrm{Gal}(K_0/K)}) = 1_{G_{K_0}} \tag{2.7}$$

for each $\tau, \tau', \tau'' \in \mathrm{Gal}(K_0/K)$. Here $\alpha : G_{K_0} \to \mathrm{Aut}(G_{K_0})$ denotes the canonical conjugation action of G_{K_0}, and ${}^{\psi^*(\tau)}f(\tau', \tau'') = s(\tau)f(\tau', \tau'')s(\tau)^{-1}$. We call such a pair (f, ψ^*) with properties (2.5), (2.6) and (2.7) a *factor system*. We have a group structure

$$E_{f,\psi^*} := (G_{K_0} \times \mathrm{Gal}(K_0/K), *),$$

where the group operation is defined by

$$(\gamma, \tau) * (\gamma', \tau') = \left(\gamma({}^{\psi^*(\tau)}\gamma')f(\tau, \tau'), \tau\tau'\right)$$

for each $\gamma, \gamma' \in G_{K_0}$, and $\tau, \tau' \in \mathrm{Gal}(K_0/K)$. Note that, $(1_{G_{K_0}}, 1_{\mathrm{Gal}(K_0/K)})$ is the identity element, and for any element (γ, τ), one has the left inverse $\left(f(\tau^{-1}, \tau)^{-1}({}^{\psi^*(\tau^{-1})}\gamma^{-1}), \tau^{-1}\right)$, and the right inverse $(\psi^*(\tau^{-1})(\gamma f(\tau, \tau^{-1})), \tau^{-1})$. These two inverses are necessarily equal because of associativity. On the other hand, the map

$$\xi_K : E_{f,\psi^*} \to G_K$$

given by

$$\xi_K : (\gamma, \tau) \mapsto \gamma s(\tau)$$

is an isomorphism. Note that, E_{f,ψ^*} sits in the following exact sequence

$$1 \longrightarrow G_{K_0} \xrightarrow{\iota} E_{f,\psi^*} \xrightarrow{\mathrm{Pr}_2} \mathrm{Gal}(K_0/K) \longrightarrow 1 \ .$$

In particular the diagram

$$\begin{tikzcd}
 & & G_K \arrow[dr, "\mathrm{res}_{K_0}"] & & \\
1 \arrow[r] & G_{K_0} \arrow[ur, "inc."] \arrow[dr, "\iota"'] & & \mathrm{Gal}(K_0/K) \arrow[r] & 1 \\
 & & E_{f,\psi^*} \arrow[uu, "\xi_K"] \arrow[ur, "\mathrm{Pr}_2"'] & &
\end{tikzcd}$$

is commutative.

Proposition 2.2. *For each $\sigma \in G_K$, the map $\varsigma : G_K \to G_{K_0}$, defined by $\varsigma : \sigma \mapsto \sigma\left(s(\sigma|_{K_0})\right)^{-1}$ is a continuous surjection, which satisfies $\varsigma(1_{G_K}) = 1_{G_{K_0}}$ and $\varsigma(\gamma\sigma) = \gamma\varsigma(\sigma)$ for all $\gamma \in G_{K_0}$. On the other hand, the map $\rho_K : G_K \to G_{K_0} \times \mathrm{Gal}(K_0/K)$ defined by $\rho_K : \sigma \mapsto (\varsigma(\sigma), \sigma|_{K_0})$ is a homeomorphism.*

Proof. See Proposition 1.3.4 (a) and (c) of [24]. □

Note that ρ_K is the inverse of ξ_K. Thus $\xi_K : E_{f,\psi^*} \to G_K$ is a homeomorphism.

Proposition 2.3. *The group E_{f,ψ^*} is profinite.*

Proof. Clearly, the topology on E_{f,ψ^*} is locally compact and totally disconnected by Proposition 2.2. On the other hand, for each (γ,τ), $(\gamma',\tau') \in E_{f,\psi^*}$, the operation

$$(\gamma,\tau) * (\gamma',\tau')^{-1} = \Big(\gamma({}^{\psi^*(\tau\tau'^{-1})}(\gamma' f(\tau',\tau'^{-1})))f(\tau,\tau'^{-1}), \tau\tau'^{-1}\Big)$$

is continuous, since its projections to each of its components are continuous. It means E_{f,ψ^*} is a topological group. Hence it is profinite. □

2.1.2. The construction of a topological group in terms of $\nabla_{K_0}^{(\varphi_{K_0})}$ and $K^\times/\mathbf{N}_{K_0/K}K_0^\times$. For each $\Theta \in \mathrm{Aut}(G_{K_0})$, let Γ_Θ denote the automorphism of $\nabla_{K_0}^{(\varphi_{K_0})}$ defined by the composition

$$\Gamma_\Theta : \nabla_{K_0}^{(\varphi_{K_0})} \xrightarrow{\left(\mathbf{\Phi}_{K_0}^{(\varphi_{K_0})}\right)^{-1}} G_{K_0} \xrightarrow{\Theta} G_{K_0} \xrightarrow{\mathbf{\Phi}_{K_0}^{(\varphi_{K_0})}} \nabla_{K_0}^{(\varphi_{K_0})} .$$

Recall that, $\mathbf{\Phi}_{K_0}^{(\varphi_{K_0})}$ is the non-abelian reciprocity map for K_0. Thus, we have a homomorphism

$$\Gamma : \mathrm{Aut}(G_{K_0}) \to \mathrm{Aut}(\nabla_{K_0}^{(\varphi_{K_0})})$$

which is defined by $\Gamma(\Theta) = \Gamma_\Theta$. Consider *the norm-residue map*

$$\theta_{K_0/K} : K^\times / \mathrm{N}_{K_0/K}\, K_0^\times \xrightarrow{\sim} \mathrm{Gal}(K_0/K)$$

for K_0/K, which is the inverse of (2.1) . We induce the map

$$\tilde{f} : K^\times/\mathrm{N}_{K_0/K}K_0^\times \times K^\times/\mathrm{N}_{K_0/K}K_0^\times \longrightarrow \nabla_{K_0}^{(\varphi_{K_0})} \tag{2.8}$$

by the composition

$$\begin{tikzcd} K^\times/\mathrm{N}_{K_0/K}K_0^\times \times K^\times/\mathrm{N}_{K_0/K}K_0^\times \arrow[r, dotted, "\tilde{f}"] \arrow[d, "(\theta_{K_0/K},\theta_{K_0/K})"'] & \nabla_{K_0}^{(\varphi_{K_0})} \\ \mathrm{Gal}(K_0/K) \times \mathrm{Gal}(K_0/K) \arrow[r, "f"'] & G_{K_0} \arrow[u, "\mathbf{\Phi}_{K_0}^{(\varphi_{K_0})}"'] \end{tikzcd} \tag{2.9}$$

where f is given by (2.3). Also we induce,

$$\widetilde{\psi}^* : K^\times / \mathrm{N}_{K_0/K}\, K_0^\times \to \mathrm{Aut}(\nabla_{K_0}^{(\varphi_{K_0})}) \tag{2.10}$$

by the composition

$$\begin{tikzcd} K^\times / \mathrm{N}_{K_0/K}\, K_0^\times \arrow[r, dotted, "\widetilde{\psi}^*"] \arrow[d, "\theta_{K_0/K}"'] & \mathrm{Aut}(\nabla_{K_0}^{(\varphi_{K_0})}) \\ \mathrm{Gal}(K_0/K) \arrow[r, "\psi^*"'] & \mathrm{Aut}(G_{K_0}) \arrow[u, "\Gamma"'] \end{tikzcd} \tag{2.11}$$

where ψ^* is given by (2.4).

Lemma 2.4. *Let*

$$\tilde{\alpha} : \nabla_{K_0}^{(\varphi_{K_0})} \to \mathrm{Aut}(\nabla_{K_0}^{(\varphi_{K_0})})$$

denote the canonical conjugation action of $\nabla_{K_0}^{(\varphi_{K_0})}$. *Then the following diagram is comutative:*

$$\begin{tikzcd} G_{K_0} \arrow[r, "\alpha"] \arrow[d, "\mathbf{\Phi}_{K_0}^{(\varphi_{K_0})}"'] & \mathrm{Aut}(G_{K_0}) \arrow[d, "\Gamma"] \\ \nabla_{K_0}^{(\varphi_{K_0})} \arrow[r, "\tilde{\alpha}"] & \mathrm{Aut}\left(\nabla_{K_0}^{(\varphi_{K_0})}\right) \end{tikzcd}$$

where α *is the canonical conjugation action of* G_{K_0}.

Proof. For each $\gamma \in G_{K_0}$, it is easy to show that the equation

$$\left(\Gamma(\alpha(\gamma))\right)(g) = \left(\tilde{\alpha}(\mathbf{\Phi}_{K_0}^{(\varphi_{K_0})}(\gamma))\right)(g)$$

holds for all $g \in \nabla_{K_0}^{(\varphi_{K_0})}$. □

Now we are ready to prove the following.

Proposition 2.5. *For each* $n, n', n'' \in K^\times / N_{K_0/K} K_0^\times$, *the properties*

$$\widetilde{\psi}^*(n) \circ \widetilde{\psi}^*(n') = \tilde{\alpha}\left(\tilde{f}(n,n')\right) \circ \widetilde{\psi}^*(nn') \tag{2.12}$$

and

$$\tilde{f}(n,n')\tilde{f}(nn',n'') = \left({}^{\widetilde{\psi}^*(n)}\tilde{f}(n',n'')\right)\tilde{f}(n,n'n'') \tag{2.13}$$

hold for $\tilde{f}$ *and* $\widetilde{\psi}^*$.

Proof. One can show that,

$$\widetilde{\psi}^*(n)\widetilde{\psi}^*(n') = \Gamma\Big(\alpha(f\left(\alpha_{K_0/K}(n), \alpha_{K_0/K}(n')\right))\Big) \circ \widetilde{\psi}^*(nn')$$

by using (2.11). On the other hand, by Lemma 2.4 we get

$$\Gamma\Big(\alpha(f\left(\theta_{K_0/K}(n), \theta_{K_0/K}(n')\right))\Big) = \tilde{\alpha}\left(\tilde{f}(n,n')\right) \ .$$

Hence (2.12) follows.

To prove (2.13) holds, note that

$$\tilde{f}(n,n')\tilde{f}(nn',n'') = \mathbf{\Phi}_{K_0}^{(\varphi_{K_0})}\Big(f(\theta_{K_0/K}(n),\theta_{K_0/K}(n'))f(\theta_{K_0/K}(n)\theta_{K_0/K}(n'),\theta_{K_0/K}(n''))\Big). \tag{2.14}$$

By the cocycle condition (2.6) of f, we have

$$f(\theta_{K_0/K}(n),\theta_{K_0/K}(n'))f(\theta_{K_0/K}(n)\theta_{K_0/K}(n'),\theta_{K_0/K}(n'')) = {}^{\psi^*(\theta_{K_0/K}(n))}f(\theta_{K_0/K}(n'),\theta_{K_0/K}(n''))f\left(\theta_{K_0/K}(n),\theta_{K_0/K}(n')\theta_{K_0/K}(n'')\right) \ .$$

It follows that,

$$\begin{aligned}
\mathbf{\Phi}_{K_0}^{(\varphi_{K_0})} & \left(f(\theta_{K_0/K}(n),\theta_{K_0/K}(n'))f(\theta_{K_0/K}(n)\theta_{K_0/K}(n'),\theta_{K_0/K}(n''))\right) = \\
& = \mathbf{\Phi}_{K_0}^{(\varphi_{K_0})}\left({}^{\psi^*(\theta_{K_0/K}(n))}f(\theta_{K_0/K}(n'),\theta_{K_0/K}(n''))\right)\circ \\
& \quad \left(\mathbf{\Phi}_{K_0}^{(\varphi_{K_0})}(f\left(\theta_{K_0/K}(n),\theta_{K_0/K}(n')\theta_{K_0/K}(n'')\right))\right) \\
& = \left(\mathbf{\Phi}_{K_0}^{(\varphi_{K_0})}\circ\psi^*(\theta_{K_0/K}(n))\circ\mathbf{\Phi}_{K_0}^{(\varphi_{K_0})^{-1}}\right)\left(\mathbf{\Phi}_{K_0}^{(\varphi_{K_0})}\left(f\left(\theta_{K_0/K}(n'),\theta_{K_0/K}(n'')\right)\right)\right) \\
& \quad \circ\mathbf{\Phi}_{K_0}^{(\varphi_{K_0})}\left(f\left(\theta_{K_0/K}(n),\theta_{K_0/K}(n')\theta_{K_0/K}(n'')\right)\right) \\
& = \left({}^{\widetilde{\psi}^*(n)}\tilde{f}(n',n'')\right)\tilde{f}(n,n'n''). \qquad (2.15)
\end{aligned}$$

Hence, (2.13) follows from (2.14) and (2.15). ☐

Thus, the pair $(\tilde{f},\widetilde{\psi}^*)$ is a factor system to the profinite groups $\nabla_{K_0}^{(\varphi_{K_0})}$ and $K^\times/\,\mathrm{N}_{K_0/K}\,K_0^\times$. So, we have the profinite group structure

$$\nabla_K^{(\varphi_{K_0})} := \left(\nabla_{K_0}^{(\varphi_{K_0})}\times K^\times/\mathrm{N}_{K_0/K}K_0^\times,\tilde{*}\right)$$

where the group operation $\tilde{*}$ is defined by

$$(g,n)\tilde{*}(g',n') = \left(g({}^{\widetilde{\psi}^*(n)}g')\tilde{f}(n,n'),nn'\right) \qquad (2.16)$$

for each (g,n), $(g',n')\in\nabla_{K_0}^{(\varphi_{K_0})}\times K^\times/\mathrm{N}_{K_0/K}K_0^\times$. Moreover, we have the group extension

$$1\longrightarrow\nabla_{K_0}^{(\varphi_{K_0})}\xrightarrow{inj.}\nabla_K^{(\varphi_{K_0})}\xrightarrow{\mathrm{Pr}_2}K^\times/\mathrm{N}_{K_0/K}K_0^\times\longrightarrow 1\,.$$

2.1.3. Definition of the local non-abelian reciprocity map $\mathbf{\Phi}_K^{(\varphi_{K_0})}$.

Theorem 2.6. *For all $(\gamma,\tau)\in E_{f,\psi^*}$ the bijection*

$$\left(\mathbf{\Phi}_{K_0}^{(\varphi_{K_0})},\mathrm{Art}_{K_0/K}\right):E_{f,\psi^*}\to\nabla_K^{(\varphi_{K_0})}$$

is a topological group isomorphism.

Proof. Obviously the map is a topological isomorphism. Now, let us show that it is also an isomorphism of groups: Let (γ,τ), $(\gamma',\tau')\in E_{f,\psi^*}$. Then,

$$\begin{aligned}
& (\mathbf{\Phi}_{K_0}^{(\varphi_{K_0})}(\gamma),\mathrm{Art}_{K_0/K}(\tau))\tilde{*}(\mathbf{\Phi}_{K_0}^{(\varphi_{K_0})}(\gamma'),\mathrm{Art}_{K_0/K}(\tau')) \\
& = \left(\mathbf{\Phi}_{K_0}^{(\varphi_{K_0})}(\gamma)\left({}^{\widetilde{\psi}^*(\mathrm{Art}_{K_0/K}(\tau))}(\mathbf{\Phi}_{K_0}^{(\varphi_{K_0})}(\gamma'))\right)\tilde{f}\left(\mathrm{Art}_{K_0/K}(\tau),\mathrm{Art}_{K_0/K}(\tau')\right),\mathrm{Art}_{K_0/K}(\tau\tau')\right).
\end{aligned}$$

On the other hand, for (γ,τ), $(\gamma',\tau')\in E_{f,\psi^*}$, one has

$$\begin{aligned}
& \left(\mathbf{\Phi}_{K_0}^{(\varphi_{K_0})},\mathrm{Art}_{K_0/K}\right)((\gamma,\tau)*(\gamma',\tau')) = \\
& \qquad = \left(\mathbf{\Phi}_{K_0}^{(\varphi_{K_0})},\mathrm{Art}_{K_0/K}\right)\left(\gamma({}^{\psi^*(\tau)}\gamma')f(\tau,\tau'),\tau\tau'\right) \\
& \qquad = \left(\mathbf{\Phi}_{K_0}^{(\varphi_{K_0})}(\gamma)\mathbf{\Phi}_{K_0}^{(\varphi_{K_0})}({}^{\psi^*(\tau)}(\gamma'))\mathbf{\Phi}_{K_0}^{(\varphi_{K_0})}(f(\tau,\tau')),\mathrm{Art}_{K_0/K}(\tau\tau')\right).
\end{aligned}$$

Hence, it's enough to show that, the equations

$$\mathbf{\Phi}_{K_0}^{(\varphi_{K_0})}({}^{\psi^*(\tau)}(\gamma')) = {}^{\widetilde{\psi}^*(\mathrm{Art}_{K_0/K}(\tau))}\left(\mathbf{\Phi}_{K_0}^{(\varphi_{K_0})}(\gamma')\right) \tag{2.17}$$

and

$$\mathbf{\Phi}_{K_0}^{(\varphi_{K_0})}(f(\tau,\tau')) = \tilde{f}(\mathrm{Art}_{K_0/K}(\tau), \mathrm{Art}_{K_0/K}(\tau')) \tag{2.18}$$

hold. To show (2.17), for each $\tau \in \mathrm{Gal}(K_0/K)$ we get,

$$\widetilde{\psi}^*(\mathrm{Art}_{K_0/K}(\tau)) = \Gamma_{\mathbf{\Phi}_{K_0}^{(\varphi_{K_0})}}(\psi^*(\tau))$$

by using (2.11). On the other hand, by definition of $\Gamma_{\mathbf{\Phi}_{K_0}^{(\varphi_{K_0})}}$ we get

$$\Gamma_{\mathbf{\Phi}_{K_0}^{(\varphi_{K_0})}}(\psi^*(\tau)) = \mathbf{\Phi}_{K_0}^{(\varphi_{K_0})}(\psi^*(\tau))\mathbf{\Phi}_{K_0}^{(\varphi_{K_0})^{-1}}.$$

Hence,

$$\begin{aligned} {}^{\widetilde{\psi}^*(\mathrm{Art}_{K_0/K}(\tau))}\mathbf{\Phi}_{K_0}^{(\varphi_{K_0})}(\gamma') &= \mathbf{\Phi}_{K_0}^{(\varphi_{K_0})}(\psi^*(\tau))\mathbf{\Phi}_{K_0}^{(\varphi_{K_0})^{-1}}(\mathbf{\Phi}_{K_0}^{(\varphi_{K_0})}(\gamma')) \\ &= \mathbf{\Phi}_{K_0}^{(\varphi_{K_0})}\left({}^{\psi^*(\tau)}\gamma'\right). \end{aligned}$$

The equation (2.18) follows by puting $\mathrm{Art}_{K_0/K}(\tau)$, $\mathrm{Art}_{K_0/K}(\tau')$ respectively instead of n and n' in the composition given by (2.9). □

Corollary 2.7. *The following composition*

$$\mathbf{\Phi}_{K}^{(\varphi_{K_0})} : G_K \xrightarrow{\rho_K} E_{f,\psi^*} \xrightarrow{\left(\mathbf{\Phi}_{K_0}^{(\varphi_{K_0})}, \mathrm{Art}_{K_0/K}\right)} \nabla_K^{(\varphi_{K_0})} \tag{2.19}$$

defines a topological group isomorphism between G_K and $\nabla_K^{(\varphi_{K_0})}$.

Proof. Obvious, since $\mathbf{\Phi}_K^{(\varphi_{K_0})}$ is defined as a composition of topological group isomorphisms. □

Definition 2.8 (non-abelian reciprocity map)**.** The topological group isomorphism

$$\mathbf{\Phi}_K^{(\varphi_{K_0})} : G_K \xrightarrow{\sim} \nabla_K^{(\varphi_{K_0})}$$

defined by the composition (2.19) in Theorem 2.7 is called *the local non-abelian reciprocity map* for K.

The diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G_{K_0} & \hookrightarrow & G_K & \xrightarrow{\mathrm{res}_{K_0}} & \mathrm{Gal}(K_0/K) & \longrightarrow & 1 \\ & & \Big\downarrow{\scriptstyle \mathbf{\Phi}_{K_0}^{(\varphi_{K_0})}} & & \Big\downarrow{\scriptstyle \mathbf{\Phi}_{K}^{(\varphi_{K_0})}} & & \Big\downarrow{\scriptstyle \mathrm{Art}_{K_0/K}} & & \\ 1 & \longrightarrow & \nabla_{K_0}^{(\varphi_{K_0})} & \xrightarrow{inj.} & \nabla_K^{(\varphi_{K_0})} & \xrightarrow{\mathrm{Pr}_2} & K^\times/\mathrm{N}_{K_0/K}K_0^\times & \longrightarrow & 1 \end{array}$$

is commutative.

Remark 2.9. The inverse $\left(\mathbf{\Phi}_K^{(\varphi_{K_0})}\right)^{-1}$ of local non-abelian reciprocity map $\mathbf{\Phi}_K^{(\varphi_{K_0})}$ is called *the local non-abelian norm-residue homomorphism* for K. It satisfies the following equality

$$\left(\mathbf{\Phi}_K^{(\varphi_{K_0})}\right)^{-1}(g,n) = \left(\mathbf{\Phi}_{K_0}^{(\varphi_{K_0})}\right)^{-1}(g) \cdot s(\theta_{K_0/K}(n))$$

for each $(g, n) \in \nabla_K^{(\varphi_{K_0})}$.

3. Properties of the local non-abelian reciprocity map

In this section, we will show that, the local non-abelian reciprocity map

$$\mathbf{\Phi}_K^{(\varphi_{K_0})} : G_K \xrightarrow{\sim} \nabla_K^{(\varphi_{K_0})}$$

satisfies the certain functoriality and ramification theoretic properties.

3.1. Functoriality

Let F/K be a finite extension of the local field K such that $F_0 = F(\zeta_p)$ is a φ_{K_0}-compatible extension of K_0. Fix the Lubin-Tate splitting φ_{F_0} over F_0. Following the same reasoning as in Section 2.1.1 we induce a group structure

$$E_{f_F, \psi_F^*} := (G_{F_0} \times \mathrm{Gal}(F_0/F), *)$$

isomorphic to G_F. We denote the corresponding isomorphism by

$$\xi_F : E_{f_F, \psi_F^*} \xrightarrow{\sim} G_F.$$

Now, the diagram

$$\begin{tikzcd}
\mathrm{Gal}(L_0/L) \arrow[r, "\mathrm{Art}_{L_0/L}"] \arrow[d, "\mathrm{res}_{K_0}"'] & F^\times/\mathrm{N}_{F_0/F}F_0^\times \arrow[d, "\mathrm{N}_{F/K_*}"] \\
\mathrm{Gal}(K_0/K) \arrow[r, "\mathrm{Art}_{K_0/K}"] & K^\times/\mathrm{N}_{K_0/K}K_0^\times
\end{tikzcd} \tag{3.1}$$

is commutative *by functoriality property of* $\mathrm{Art}_K : G_K^{\mathrm{ab}} \xrightarrow{\sim} \widehat{K^\times}$. Here the left vertical arrow res_{K_0} denotes the map given by the restriction of $\sigma \in \mathrm{Gal}(L_0/L)$ to K_0, and the right vertical arrow N_{F/K_*} denotes the map induced from the norm map $\mathrm{N}_{F/K}$. Also, by *functoriality property of the local non-abelian reciprocity map* $\mathbf{\Phi}_{K_0}^{(\varphi_{K_0})} : G_{K_0} \xrightarrow{\sim} \nabla_{K_0}^{(\varphi_{K_0})}$, the following diagram

$$\begin{tikzcd}
G_{F_0} \arrow[r, "\mathbf{\Phi}_{F_0}^{(\varphi_{F_0})}"] \arrow[d, "inc."'] & \nabla_F^{(\varphi_F)} \arrow[d, "\mathcal{N}_{F_0/K_0}^{\infty}"] \\
G_{K_0} \arrow[r, "\mathbf{\Phi}_{K_0}^{(\varphi_{K_0})}"] & \nabla_{K_0}^{(\varphi_{K_0})}
\end{tikzcd} \tag{3.2}$$

is commutative. Thus, we induce the following diagram,

$$\begin{tikzcd}
G_F \arrow[r, "\rho_F"] \arrow[d, "inc."'] \arrow[rr, dotted, bend left, "\mathbf{\Phi}_F^{(\varphi_{F_0})}"] & E_{f_F, \psi_F^*} \arrow[r, "(\mathbf{\Phi}_{F_0}^{(\varphi_{F_0})}{,}\mathrm{Art}_{F_0/F})"] \arrow[d, "(inc.{,}\mathrm{res}_{K_0})"'] & \nabla_F^{(\varphi_{F_0})} \arrow[d, "(\mathcal{N}_{F_0/K_0}^{\infty}{,}\mathrm{N}_{F/K_*})"] \\
G_K \arrow[r, "\rho_K"] \arrow[rr, dotted, bend right, "\mathbf{\Phi}_K^{(\varphi_{K_0})}"'] & E_{f, \psi^*} \arrow[r, "(\mathbf{\Phi}_{K_0}^{(\varphi_{K_0})}{,}\mathrm{Art}_{K_0/K})"] & \nabla_K^{(\varphi_{K_0})}
\end{tikzcd} \tag{3.3}$$

which is commutative, since the diagrams (3.1) and (3.2) are commutative.

We denote $\mathcal{N}_{F/K} := (\mathcal{N}^{\infty}_{F_0/K_0}, \mathrm{N}_{F/K_*})$. If $K \subseteq F \subseteq F'$ is a tower of extensions of finite degree, such that F_0/K_0 and F'_0/K_0 are compatible with φ_{K_0} (cf 0.4 of [19]), the transitivity

$$\mathcal{N}_{F'/K} = \mathcal{N}_{F/K} \circ \mathcal{N}_{F'/F}$$

follows from the commutativity of the diagram (3.3). We denote

$$\mathcal{N}_F := \mathcal{N}_{F/K}(\nabla_F^{(\varphi_{F_0})}) = \mathcal{N}^{\infty}_{F_0} \times \mathrm{N}_{F/K}\, F^{\times} / \, \mathrm{N}_{K_0/K}\, K_0^{\times},$$

which is a closed subgroup of $\nabla_K^{(\varphi_{K_0})}$. Here, $\mathcal{N}^{\infty}_{F_0}$ is the closed subgroup of $\nabla_{F_0}^{(\varphi_{F_0})}$ defined by the functoriality property of the non-abelian map $\mathbf{\Phi}_{K_0}^{(\varphi_{K_0})}$ (cf. (7.6) of [15]).

When L/K is an infinite extension, such that L_0/K_0 is a union of finite φ_{K_0}-compatible subextensions E_0/K_0, we have the closed subgroup $\mathcal{N}^{\infty}_{L_0} = \bigcap_{E_0} \mathcal{N}^{\infty}_{E_0}$ of $\nabla_{K_0}^{(\varphi_{K_0})}$, where E_0 runs all over finite φ_{K_0}-compatible subextensions of L_0/K_0 (cf. (7.7) of [15]). Also we have the closed subgroup $\mathrm{N}_{L/K}\, L^{\times} / \, \mathrm{N}_{K_0/K}\, K_0^{\times}$ of the group $K^{\times} / \, \mathrm{N}_{K_0/K}\, K_0^{\times}$, with

$$\mathrm{N}_{L/K}\, L^{\times} = \bigcap_{E} \mathrm{N}_{E/K}\, E^{\times},$$

where E runs over all finite subextensions of L/K, such that E_0/K_0 is a φ_{K_0}-compatible extension. We denote the closed subgroup $\mathcal{N}^{\infty}_{L_0} \times \mathrm{N}_{L/K}\, L^{\times} / \, \mathrm{N}_{K_0/K}\, K_0^{\times}$ of $\nabla_K^{(\varphi_{K_0})}$ by

$$\mathcal{N}_L := \mathcal{N}^{\infty}_{L_0} \times \mathrm{N}_{L/K}\, L^{\times} / \, \mathrm{N}_{K_0/K}\, K_0^{\times}.$$

Observe that, $\mathcal{N}_L$ satisfies

$$\mathcal{N}_L = \bigcap_{E} \mathcal{N}_E,$$

where E runs over all finite Galois subextensions of L/K, such that $E_0 = E(\zeta_p)$ is a φ_{K_0}-compatible extension.

More generally, if L/K is *any* finite Galois extension, then L has a finite extension L' such that, L'_0/L_0 is a finite unramified extension compatible with φ_{K_0}. Following the same reasoning as in Section 2.1.1 we have the topological group structures

$$E_{f_L, \psi_L^*} := (G_{L_0} \times \mathrm{Gal}(L_0/L), *),$$

and

$$E_{f_{L'}, \psi_{L'}^*} := (G_{L'_0} \times \mathrm{Gal}(L'_0/L'), *)$$

isomorphic with G_L, and $G_{L'}$ respectively. Also we denote the corresponding isomorphisms by

$$\rho_L : G_L \xrightarrow{\sim} E_{f_L, \psi_L^*},$$

and

$$\rho_{L'} : G_{L'} \xrightarrow{\sim} E_{f' L, \psi_{L'}^*}.$$

Now, by the functoriality property of $\mathbf{\Phi}_{K_0}^{(\varphi_{K_0})}$, the following diagram

$$\begin{tikzcd}
G_{L'} \arrow[r, "\mathbf{\Phi}_{L'}^{(\varphi_{L'_0})}"] \arrow[d, "\text{inc.}"'] & \nabla_{L'}^{(\varphi_{L'_0})} \arrow[dd, "\mathcal{N}^{\infty}_{L'/K}"] \\
G_L \arrow[d, "\text{inc.}"'] & \\
G_K \arrow[r, "\mathbf{\Phi}_K^{(\varphi_{K_0})}"] & \nabla_K^{(\varphi_{K_0})}
\end{tikzcd} \tag{3.4}$$

is commutative. If we combine (3.4) with (3.1), we induce the diagram

$$\begin{tikzcd}
 & & \mathrm{Gal}(L_0'/L') \arrow[rr, "\theta_{L_0'/L'}", "\sim"'] \arrow[dl, "inj"'] \arrow[dd, "\mathrm{res}_{L_0}"] & & L'^{\times}/\,\mathrm{N}_{L_0'/L'}\,L_0'^{\times} \arrow[dl, "inj"'] \arrow[dd, "\mathrm{N}_{L'/L}"] \\
G_{L'} \arrow[r, "\rho_{L'}", "\sim"'] \arrow[dd, "inc"'] & E_{f_{L'},\psi^*_{L'}} \arrow[rr, "\left(\mathbf{\Phi}_{L_0'}^{(\varphi_{L_0'})}{,}\mathrm{Art}_{L_0'/L'}\right)"] \arrow[dd, "(inj{,}\mathrm{res}_{L_0})"'] & & \nabla_{L'}^{(\varphi_{L_0'})} \arrow[dddd, dotted, "\mathrm{N}_{L'/K}"] & \\
 & & \mathrm{Gal}(L_0/L) \arrow[rr, "\theta_{L_0/L}", "\sim"'] \arrow[dl, "inj"] \arrow[dd, "\mathrm{res}_{K_0}"] & & L^{\times}/\,\mathrm{N}_{L_0/L}\,L_0^{\times} \arrow[dd, "\mathrm{N}_{L/K}"] \\
G_L \arrow[r, "\rho_L", "\sim"'] \arrow[dd, "inc"'] & E_{f_L,\psi^*_L} \arrow[dd, "(inj{,}\mathrm{res}_{K_0})"'] & & & \\
 & & \mathrm{Gal}(K_0/K) \arrow[rr, "\sim", "\theta_{K_0/K}"'] \arrow[dl, "inj"] & & K^{\times}/\,\mathrm{N}_{K_0/K}\,K_0^{\times} \arrow[dl, "inj"] \\
G_K \arrow[r, "\sim", "\rho_K"'] & E_{f,\psi^*} \arrow[rr, "\sim", "\left(\mathbf{\Phi}_{K_0}^{(\varphi_{K_0})}{,}\mathrm{Art}_{K_0/K}\right)"'] & & \nabla_K^{(\varphi_{K_0})} &
\end{tikzcd} \tag{3.5}$$

which is commutative. Thus the closed subgroup $\mathbf{\Phi}_K^{(\varphi_{K_0})}(G_L)$ of $\nabla_K^{(\varphi_{K_0})}$ satisfies

$$\mathbf{\Phi}_K^{(\varphi_{K_0})}(G_L) = \mathcal{N}_{L_0}^{\infty} \times \mathrm{N}_{L/K}\,L^{\times}/\,\mathrm{N}_{K_0/K}\,K_0^{\times},$$

where $\mathcal{N}_{L_0}^{\infty} = \mathbf{\Phi}_{K_0}^{(\varphi_{K_0})}(G_{L_0})$ is the closed subgroup of $\nabla_{K_0}^{(\varphi_{K_0})}$, which is defined by the functoriality property of the local non-abelian reciprocity map $\mathbf{\Phi}_{K_0}^{(\varphi_{K_0})}$ (cf. (7.13) of [15]). We denote this closed subgroup by

$$\mathcal{N}_L := \mathbf{\Phi}_K^{(\varphi_{K_0})}(G_L).$$

If L/K is an infinite Galois extension, we have the closed subgroup $\mathcal{N}_{L_0}^{\infty} = \bigcap_{E_0} \mathcal{N}_{E_0}^{\infty}$ of $\nabla_{K_0}^{(\varphi_{K_0})}$, where E_0 runs all over finite subextensions of L_0/K_0 (cf. (7.14) of [15]). Again, we have

$$\mathrm{N}_{L/K}\,L^{\times} = \bigcap_{E} \mathrm{N}_{E/K}\,E^{\times},$$

where E runs over all finite subextensions of L/K. We denote the closed subgroup $\mathcal{N}_{L_0}^{\infty} \times \mathrm{N}_{L/K}\,L^{\times}/\,\mathrm{N}_{K_0/K}\,K_0^{\times}$ of $\nabla_K^{(\varphi_{K_0})}$ by

$$\mathcal{N}_L := \mathcal{N}_{L_0}^{\infty} \times \mathrm{N}_{L/K}\,L^{\times}/\,\mathrm{N}_{K_0/K}\,K_0^{\times}.$$

Observe that,

$$\mathcal{N}_L = \bigcap_{E} \mathcal{N}_E,$$

where E runs over all finite Galois subextensions of L/K.

3.2. Isomorphism Theorem

Let L be any Galois extension of the local field K. In this section, we shall calculate the kernel of the continuous surjection

$$\mathbf{\Phi}_{L/K}^{(\varphi_{K_0})} : \nabla_K^{(\varphi_{K_0})} \xrightarrow[\sim]{(\mathbf{\Phi}_K^{(\varphi_{K_0})})^{-1}} G_K \xrightarrow{\mathrm{res}_L} \mathrm{Gal}(L/K). \tag{3.6}$$

Consider the subextension $L \cap K_0$ of L/K. We have the following diagram

$$\begin{tikzcd}[column sep=small]
 & LK_0 = L_0 \arrow[dl, dash] \arrow[dr, dash] & \\
L \arrow[dr, dash] & & K_0 \arrow[dl, dash] \\
 & L \cap K_0 \arrow[d, dash] & \\
 & K &
\end{tikzcd}$$

of field extensions such that, there is an isomorphism

$$r_L^{L_0} : \mathrm{Gal}(L_0/K_0) \xrightarrow{\sim} \mathrm{Gal}(L/L \cap K_0)$$

which sends each σ of $\mathrm{Gal}(L_0/K_0)$ to its restriction σ_L to L. This is well known fact from Galois theory (for the proof, see Theorem 1.14 of [20]). Now, as $K_0 = K(\zeta_p)$, we have the following continuous surjection

$$\nabla_{K_0}^{(\varphi_{K_0}))} \xrightarrow{\mathbf{\Phi}_{L_0/K_0}^{(\varphi_{K_0})}} \mathrm{Gal}(L_0/K_0) \xrightarrow[\sim]{(r_L^{L_0})^{-1}} \mathrm{Gal}(L/L \cap K_0) \tag{3.7}$$

whose kernel is the closed subgroup $\mathcal{N}_{L_0}^{\infty}$ of $\nabla_{K_0}^{(\varphi_{K_0}))}$. Here $\mathbf{\Phi}_{L_0/K_0}^{(\varphi_{K_0})}$ denotes the norm residue isomorphism for L_0/K_0, which is induced by the *isomorphism theorem for the local non-abelian reciprocity map* $\mathbf{\Phi}_{K_0}^{(\varphi_{K_0})}$ *for* K_0.

On the other hand, since $L \cap K_0/K$ is abelian, the following diagram

$$\begin{tikzcd}
K^{\times}/\,\mathrm{N}_{K_0/K}\, K_0^{\times} \arrow[r, "\mathrm{Art}_{K_0/K}", "\sim"'] \arrow[d, "\mathrm{e}^{\mathrm{CFT}}_{K_0/L\cap K_0}"'] & \mathrm{Gal}(K_0/K) \arrow[d, "\mathrm{res}_{L\cap K_0}"] \\
K^{\times}/\,\mathrm{N}_{L\cap K_0/K}(L \cap K_0)^{\times} \arrow[r, "\mathrm{Art}_{L\cap K_0/K}", "\sim"'] & \mathrm{Gal}(L \cap K_0/K)
\end{tikzcd}$$

is commutative by the *existence theorem of the local abelian class field theory.* Here, $\mathrm{e}^{\mathrm{CFT}}_{L_0/L\cap K_0}$ is the natural inclusion defined via the existence theorem of local abelian class field theory. Thus the composition

$$K^{\times}/\,\mathrm{N}_{K_0/K}\, K_0^{\times} \xrightarrow[\sim]{\mathrm{Art}_{K_0/K}} \mathrm{Gal}(K_0/K) \xrightarrow{\mathrm{res}_{L\cap K_0}} \mathrm{Gal}(L \cap K_0/K), \tag{3.8}$$

has kernel

$$\ker\left(\mathrm{res}_{L\cap K_0} \circ \mathrm{Art}_{K_0/K}\right) = \mathrm{N}_{L/K}\, L^{\times}/\,\mathrm{N}_{K_0/K}\, K_0^{\times}$$

where

$$\mathrm{N}_{L/K}\, L^{\times} = \bigcap_{\substack{K \underset{\text{finite}}{\subseteq} E \subseteq L}} \mathrm{N}_{E/K}\, E^{\times}.$$

Other observation is that, since we have

$$\mathrm{Gal}(L/L \cap K_0) \trianglelefteq \mathrm{Gal}(L/K)$$

and

$$\mathrm{Gal}(L/K)/\mathrm{Gal}(L/L \cap K_0) \cong \mathrm{Gal}(L \cap K_0/K),$$

one can view $\mathrm{Gal}(L/K)$ as a group extension of $\mathrm{Gal}(L/L\cap K_0)$ by $\mathrm{Gal}(L\cap K_0/K)$. So, it makes sense to reconstruct $\mathrm{Gal}(L/K)$ in terms of $\mathrm{Gal}(L/L\cap K_0)$ and $\mathrm{Gal}(L\cap K_0/K)$ in order to find the kernel of (3.6).

3.2.1. Reconstruction of $\mathrm{Gal}(L/K)$ in terms of $\mathrm{Gal}(L/L\cap K_0)$ and $\mathrm{Gal}(L\cap K_0/K)$. Since $\mathrm{Gal}(L/L\cap K_0) \trianglelefteq \mathrm{Gal}(L/K)$ and $\mathrm{Gal}(L/K)/\mathrm{Gal}(L/L\cap K_0) \cong \mathrm{Gal}(L\cap K_0/K)$, one can view $\mathrm{Gal}(L/K)$ as a group extension of $\mathrm{Gal}(L/L\cap K_0)$ by $\mathrm{Gal}(L\cap K_0/K)$. Namely, there is an exact sequence of the form

$$1 \longrightarrow \mathrm{Gal}(L/L\cap K_0) \hookrightarrow \mathrm{Gal}(L/K) \xrightarrow{\mathrm{res}^L_{L\cap K_0}} \mathrm{Gal}(L\cap K_0/K) \longrightarrow 1$$

where $\mathrm{res}^L_{L\cap K_0}$ is the restriction map, which sends each $\sigma \in \mathrm{Gal}(L/K)$ to the restriction σ_L, to $L\cap K_0$.

Since those groups in the above exact sequence are profinite, from the same reasoning with Section 2.1.1, we have the profinite group structure

$$H_{f_{L/K},\psi^*_{L/K}} := (\mathrm{Gal}(L/L\cap K_0) \times \mathrm{Gal}(L\cap K_0/K), *)\,,$$

isomorphic with $\mathrm{Gal}(L/K)$. We denote the corresponding isomorphism

$$\xi_{L/K} : H_{f_{L/K},\psi^*_{L/K}} \to \mathrm{Gal}(L/K)$$

which is defined by

$$(\sigma,\tau) \mapsto \sigma s_{L/K}(\tau)$$

for each $(\sigma,\tau) \in H_{f_{L/K},\psi^*_{L/K}}$. Recall that, $H_{f_{L/K},\psi^*_{L/K}}$ sits in the following exact sequence

$$1 \longrightarrow \mathrm{Gal}(L/L\cap K_0) \xrightarrow{inc.} H_{f_{L/K},\psi^*_{L/K}} \xrightarrow{\mathrm{Pr}_2} \mathrm{Gal}(L\cap K_0/K) \longrightarrow 1\,.$$

In particular the following diagram

$$\begin{tikzcd}
 & & \mathrm{Gal}(L/K) \arrow[dr, "\mathrm{res}^L_{L\cap K_0}"] & & \\
1 \arrow[r] & \mathrm{Gal}(L/L\cap K_0) \arrow[ur, "inc."] \arrow[dr, "inj."'] & & \mathrm{Gal}(L\cap K_0/K) \arrow[r] & 1 \\
 & & H_{f_{L/K},\psi^*_{L/K}} \arrow[uu, "\xi_{L/K}"'] \arrow[ur, "\mathrm{Pr}_2"'] & &
\end{tikzcd}$$

is commutative.

3.2.2. Proof of the isomorphism theorem. Before we state the isomorphism theorem of K, we have the the following lemma:

Lemma 3.1. *The square*

$$\begin{tikzcd}
G_K \arrow[r, "\mathrm{res}_L"] & \mathrm{Gal}(L/K) \\
E_{f,\psi^*} \arrow[u, "\xi"] \arrow[r, "(\mathrm{res_L},\mathrm{res_{L\cap K_0}})"] & H_{f_{L/K},\psi^*_{L/K}} \arrow[u, "\xi_{L/K}"']
\end{tikzcd}$$

is commutaive.

Proof. Let $(\sigma,\tau \in E_{f,\psi^*}$, where $\sigma \in G_{K_0}$ and $\tau \in \mathrm{Gal}(K_0/K))$. Then, by restricting $\xi(\sigma,\tau) = \sigma s(\tau)$, to L, we get

$$\mathrm{res}_L(\sigma s(\tau)) = \sigma_L s_L(\tau),$$

where σ_L and $s_L(\tau)$ denote the restriction of σ and $s(\tau)$ to L respectively. On the other hand,

$$\begin{aligned}\mathrm{res}_{L\cap K_0}(s_L(\tau)) &= \mathrm{res}_{L\cap K_0}(s(\tau)) \\ &= \mathrm{res}_{L\cap K_0}(\mathrm{res}_{\mathrm{K}_0}(s(\tau))) = \mathrm{res}_{L\cap K_0}(\tau) =: \tau_{L\cap K_0},\end{aligned}$$

which means

$$\xi_{L/K}^{-1}(\sigma_L s_L(\tau)) = \sigma_L \tau_{L\cap K_0} = (\mathrm{res}_L, \mathrm{res}_{L\cap K_0})(\sigma, \tau),$$

and this completes the proof. □

Now the *isomorphism theorem* for the local non-abelian reciprocity map $\mathbf{\Phi}_K^{(\varphi_{K_0})}$, can be stated as follows:

Theorem 3.2 (isomorphism theorem)**.** *The continuous homomorphism, given by* (3.6) *has kernel*

$$ker\left(\mathbf{\Phi}_{L/K}^{(\varphi_{K_0})}\right) = \mathcal{N}_{L_0}^{\infty} \times N_{L/K}\, L^{\times} /\, N_{K_0/K}\, K_0^{\times}.$$

Proof. Since the kernel of (3.7) is $\mathcal{N}_L^{\infty}$, and the kernel of (3.8) is $\mathrm{N}_{L/K}\, L^{\times} /\, \mathrm{N}_{K_0/K}\, K_0^{\times}$, the proof follows from Lemma 3.1. □

3.3. Existence theorem

The existence theorem for a general local field K is stated as follows:

Theorem 3.3 (existence)**.** *The rule*

$$L/K \mapsto \mathcal{N}_L$$

gives one to one correspondence between the closed subgroups of $\nabla_K^{(\varphi_{K_0})}$ *and the Galois extensions of* K*. The group* $\mathcal{N}_{L_0}^{\infty} \times \mathrm{N}_{L/K} L^{\times} / \mathrm{N}_{K_0/K} K_0^{\times}$ *is of finite index in* $\nabla_K^{(\varphi_{K_0})}$ *if and only if* L/K *is finite, and if this is the case, we have*

$$[L:K] = \left(\nabla_K^{\varphi_{K_0}} : \mathcal{N}_{L_0}^{\infty} \times \mathrm{N}_{L/K} L^{\times} / \mathrm{N}_{K_0/K} K_0^{\times}\right).$$

Proof. From the commutativity of the diagram (3.5), we have $\mathcal{N}_L = \mathbf{\Phi}_K^{(\varphi_{K_0})}(G_L)$, and we see that the correspondence $L/K \mapsto \mathcal{N}_L$ is an injection. The remaining part of the proof follows from Theorem 3.2. □

3.4. Galois conjugation

Let $\sigma : K \to K^{\mathrm{sep}}$ be any embedding of the local field K, and fix an extension

$$\tilde{\sigma} : K^{\mathrm{sep}} \xrightarrow{\sim} K^{\mathrm{sep}}$$

of σ to K^{sep}. Denote $\sigma(K)$ by

$$K^{\sigma} := \sigma(K)\ .$$

As $K_0 = K(\zeta_p)$, we have $\tilde{\sigma}(K_0) = K^{\sigma}(\zeta_p)$. We denote

$$K_0^{\sigma} := K^{\sigma}(\zeta_p).$$

In the sense of [12], we see that $\tilde{\sigma}\varphi_{K_0}\tilde{\sigma}^{-1}$ is a *Lubin-Tate splitting* of K_0^{σ}. Thus, we have the local non-abelian reciprocity map

$$\mathbf{\Phi}_{K_0^{\sigma}}^{(\tilde{\sigma}\varphi_{K_0}\tilde{\sigma}^{-1})} : G_{K_0^{\sigma}} \xrightarrow{\sim} \nabla_{K_0^{\sigma}}^{(\tilde{\sigma}\varphi_{K_0}\tilde{\sigma}^{-1})}$$

for the local field K_0^{σ} in the sense of İkeda and Serbest. Now, the correspondence

$$\phi_{\tilde{\sigma}} : \gamma \mapsto \tilde{\sigma}\gamma\tilde{\sigma}^{-1} \tag{3.9}$$

for each $\gamma \in G_K$, defines a topological group isomorphism

$$\phi_{\tilde{\sigma}} : G_K \xrightarrow{\sim} G_{K^{\sigma}}$$

and for each $x \in K^\times$, the correspondence

$$\widehat{\sigma} : x \pmod{\mathrm{N}_{K_0/K} K_0^\times} \mapsto \widetilde{\sigma}(x) \pmod{\mathrm{N}_{K_0^\sigma/K^\sigma}(K_0^\sigma)^\times}$$

defines an isomorphism

$$\widehat{\sigma} : K^\times/\,\mathrm{N}_{K_0/K} K_0^\times \xrightarrow{\sim} (K^\sigma)^\times/\,\mathrm{N}_{K_0^\sigma/K^\sigma}(K_0^\sigma)^\times\ .$$

On the other hand, there exists a topological group isomorphism

$$\widetilde{\sigma}^+ : \nabla_{K_0}^{(\varphi_{K_0})} \xrightarrow{\sim} \nabla_{K_0^\sigma}^{(\widetilde{\sigma}\varphi_{K_0}\widetilde{\sigma}^{-1})}$$

defined by the composition

$$\begin{tikzcd}
G_{K_0} \arrow[r, "\phi_{\widetilde{\sigma}}|_{G_{K_0}}"] & G_{K_0^\sigma} \arrow[d, "\mathbf{\Phi}_{K_0^\sigma}^{(\widetilde{\sigma}\varphi_{K_0}\widetilde{\sigma}^{-1})}"] \\
\nabla_{K_0}^{(\varphi_{K_0})} \arrow[u, "\mathbf{\Phi}_{K_0}^{(\varphi_{K_0})^{-1}}"] \arrow[r, dotted, "\widetilde{\sigma}^+"] & \nabla_{K_0^\sigma}^{(\widetilde{\sigma}\varphi_{K_0}\widetilde{\sigma}^{-1})}
\end{tikzcd}$$

where $\phi_{\widetilde{\sigma}}$ is the isomomorphism given by the equation (3.9).

Consider the extension of G_{K^σ} by $\mathrm{Gal}(K_0^\sigma/K^\sigma)$

$$1 \longrightarrow G_{K_0^\sigma} \hookrightarrow G_{K^\sigma} \xrightarrow{\mathrm{res}_{K_0^\sigma}} \mathrm{Gal}(K_0^\sigma/K^\sigma) \longrightarrow 1$$

where the map $\mathrm{res}_{K_0^\sigma}$ is defined by $\mathrm{res}_{K_0^\sigma}(\gamma) = \gamma\mid_{K_0^\sigma}$. Observe that, there is an isomorphism

$$\phi_{\widetilde{\sigma}_{K_0}} : \mathrm{Gal}(K_0/K) \xrightarrow{\sim} \mathrm{Gal}(K_0^\sigma/K^\sigma)$$

defined by

$$\phi_{\widetilde{\sigma}_{K_0}} : \tau \mapsto \widetilde{\sigma}_{K_0}\tau\widetilde{\sigma}_{K_0}^{-1}$$

for each $\tau \in \mathrm{Gal}(K_0/K)$, where

$$\widetilde{\sigma}_{K_0} : K_0 \xrightarrow{\sim} K_0^\sigma$$

is the restriction $\widetilde{\sigma}\mid_{K_0}$ of the automorphism $\widetilde{\sigma} : K^{\mathrm{sep}} \to K^{\mathrm{sep}}$. Now, for each $\gamma \in \mathrm{Gal}(K_0^\sigma/K)$ define the map

$$s_{\tilde{\sigma}} : \mathrm{Gal}(K_0^\sigma/K) \to G_{K^\sigma}$$

by

$$s_{\widetilde{\sigma}}(\gamma) = \widetilde{\sigma}s(\phi_{\widetilde{\sigma}_{K_0}}^{-1}(\gamma))\widetilde{\sigma}^{-1}\ .$$

This is a normalized continuous section for $\mathrm{res}_{K_0^\sigma}$. Hence, from the same reasoning with step1 of Section 2, we construct a topological group operation "$*_{\tilde{\sigma}}$" on $\nabla_{(K^\sigma)_0}^{(\widetilde{\sigma}\varphi_{K_0}\widetilde{\sigma}^{-1})} \times (K^\sigma)^\times/\,\mathrm{N}_{K_0^\sigma/K^\sigma}(K_0^\sigma)^\times$, and we denote this topological group by $\nabla_{K^\sigma}^{(\tilde{\sigma}\varphi_{K_0}\tilde{\sigma}^{-1})}$. By Theorem 2.7, we have

$$\mathbf{\Phi}_{K^\sigma}^{(\widetilde{\sigma}\varphi_{K_0}\widetilde{\sigma}^{-1})} : G_{K^\sigma} \xrightarrow{\sim} \nabla_{K^\sigma}^{(\tilde{\sigma}\varphi_{K_0}\tilde{\sigma}^{-1})}$$

which is the *local non-abelian reciprocity map* for K^σ.

Theorem 3.4 (Galois conjugation)**.** *The following diagram*

$$\begin{tikzcd}
\nabla_K^{(\varphi_{K_0})} \arrow[r, "\mathbf{\Phi}_K^{(\varphi_{K_0})^{-1}}"] \arrow[d, "(\widetilde{\sigma}^+{,}\widehat{\sigma})"'] & G_K \arrow[d, "\phi_{\widetilde{\sigma}}"] \\
\nabla_{K^\sigma}^{(\tilde{\sigma}\varphi_{K_0}\tilde{\sigma}^{-1})} \arrow[r, "\mathbf{\Phi}_{K^\sigma}^{(\widetilde{\sigma}\varphi_{K_0}\widetilde{\sigma}^{-1})^{-1}}"'] & G_{K^\sigma}
\end{tikzcd}$$

is commutative.

Proof. From Remark 2.9, we see that, for each $(g', a) \in \nabla_{K^\sigma}^{(\tilde{\sigma}\varphi_{K_0}\tilde{\sigma}^{-1})}$, the inverse of the local non-abelian reciprocity map $\mathbf{\Phi}_{K^\sigma}^{(\widetilde{\sigma}\varphi_{K_0}\widetilde{\sigma})}$ satisfies

$$\mathbf{\Phi}_{K^\sigma}^{(\widetilde{\sigma}\varphi_{K_0}\widetilde{\sigma}^{-1})^{-1}}(g', a) = \left(\mathbf{\Phi}_{K_0^\sigma}^{(\widetilde{\sigma}\varphi_{K_0}\widetilde{\sigma}^{-1})^{-1}}(g')\right) s_{\widetilde{\sigma}}(\alpha_{K_0^\sigma/K^\sigma}(a))$$

where $\alpha_{K_0^\sigma/K^\sigma}$ is the local norm-residue map for the abelian extension K_0^σ/K^σ . This implies

$$\mathbf{\Phi}_{K^\sigma}^{(\widetilde{\sigma}\varphi_{K_0}\widetilde{\sigma}^{-1})^{-1}}\Big((\widetilde{\sigma}^+, \widehat{\sigma})(g, n)\Big) = \mathbf{\Phi}_{K_0^\sigma}^{(\widetilde{\sigma}\varphi_{K_0}\widetilde{\sigma}^{-1})^{-1}}(\widetilde{\sigma}^+(g)) \cdot s_{\widetilde{\sigma}}(\alpha_{K_0^\sigma/K^\sigma}(\widehat{\sigma}(n))) \tag{3.10}$$

for each $(g, n) \in \nabla_K^{(\varphi_{K_0})}$.

By the Galois conjugation law for the local non-abelian reciprocity map $\mathbf{\Phi}_{K_0^\sigma}^{(\tilde{\sigma}\varphi_{K_0}\tilde{\sigma}^{-1})}$, the equation

$$\mathbf{\Phi}_{K_0^\sigma}^{(\widetilde{\sigma}\varphi_{K_0}\widetilde{\sigma}^{-1})^{-1}}(\widetilde{\sigma}^+(g)) = \widetilde{\sigma}\mathbf{\Phi}_{K_0}^{(\varphi_{K_0})^{-1}}(g)\widetilde{\sigma}^{-1} \tag{3.11}$$

holds for each $g \in \nabla_{K_0}^{(\varphi_{K_0})}$, and from the Galois conjugation principle of the abelian local class field theory for the extension K_0/K,

$$\alpha_{K_0^\sigma/K^\sigma}(\widehat{\sigma}(n)) = \widetilde{\sigma}_{K_0}\alpha_{K_0/K}(n)\widetilde{\sigma}_{K_0}^{-1} \tag{3.12}$$

holds for each $n \in K^\times/\operatorname{N}_{K_0/K} K_0^\times$. If we put (3.11) and (3.12) in the equation (3.10) we get

$$\mathbf{\Phi}_{K^\sigma}^{(\widetilde{\sigma}\varphi_{K_0}\widetilde{\sigma}^{-1})^{-1}}\Big((\widetilde{\sigma}^+, \widehat{\sigma})(g, n)\Big) = \widetilde{\sigma}\mathbf{\Phi}_{K_0}^{(\varphi_{K_0})^{-1}}(g)\widetilde{\sigma}^{-1} \cdot s_{\widetilde{\sigma}}\left(\widetilde{\sigma}_{K_0}\alpha_{K_0/K}(n)\widetilde{\sigma}_{K_0}^{-1}\right) .$$

But, by the definition of $s_{\widetilde{\sigma}}$, we have

$$s_{\widetilde{\sigma}}(\alpha_{K_0^\sigma/K^\sigma}(\widehat{\sigma}(n))) = s_{\widetilde{\sigma}}(\widetilde{\sigma}_{K_0}\alpha_{K_0/K}(n)\widetilde{\sigma}_{K_0}^{-1}) = \widetilde{\sigma}s(\alpha_{K_0/K}(n))\widetilde{\sigma}^{-1}.$$

Thus, we conclude that

$$\begin{aligned}\mathbf{\Phi}_{K^\sigma}^{(\widetilde{\sigma}\varphi_{K_0}\widetilde{\sigma}^{-1})^{-1}}\Big((\widetilde{\sigma}^+, \widehat{\sigma})(g, n)\Big) &= \widetilde{\sigma}\mathbf{\Phi}_{K_0}^{(\varphi_{K_0})}(g) \cdot s(\alpha_{K_0/K}(n))\widetilde{\sigma}^{-1} \\ &= \widetilde{\sigma}\mathbf{\Phi}_K^{(\varphi_{K_0})}(g, n)\widetilde{\sigma}^{-1} .\end{aligned}$$

This completes the proof. □

3.5. Ramification Theory

Let $K_{0,d}^{\mathrm{nr}}$ denote the unique unramified degree d extension of K_0; $\Gamma_{0,d}^{(n)}$ denote the maximal n-abelian extension of $K_{0,d}^{\mathrm{nr}}$ in $(K_0)_{\varphi_{K_0}^d}$, where $(K_0)_{\varphi_{K_0}^d}$ denotes the fixed field of $\varphi_{K_0}^d$.

Following [16], we define the partial ordering " $\preceq$" on $(\mathbb{Z} \times \mathbb{Z})$ by $(n', d') \preceq (n, d)$ iff $n' \leq n$ and $d'|d$ for each (n, d), $(n', d') \in \mathbb{Z} \times \mathbb{Z}$. For an increasing net $\underline{w} := (w_{(n,d)})$ over $\mathbb{R}_{\geq -1}$ defined on the partially ordered set $(\mathbb{Z} \times \mathbb{Z}, \preceq)$,

$$\psi_{K_0/K}(\underline{w}) := (\psi_{K_0/K}(w_{(n,d)}))$$

is also an increasing net. Here, $\psi_{K_0/K}$ denotes the Herbrand function for the extension K_0/K. Thus, for each increasing net $\underline{w}$, the projective limit

$$G_{K_0}^{\psi_{K_0/K}(\underline{w})} = \varprojlim_{(n,d)} \operatorname{Gal}(\Gamma_{0,d}^{(n)}/K_0)^{\psi_{K_0/K}(w_{n,d})}$$

over the transition homomorphisms

$$r^{\psi_{K_0/K}(w_{(n,d)})}_{\psi_{K_0/K}(w_{(n',d')})} : \mathrm{Gal}(\Gamma^{(n)}_{0,d}/K_0)^{\psi_{K_0/K}(w_{(n,d)})} \to \mathrm{Gal}(\Gamma^{(n')}_{0,d'}/K_0)^{\psi_{K_0/K}(w_{(n,d)})} \hookrightarrow \mathrm{Gal}(\Gamma^{(n')}_{0,d'}/K_0)^{\psi_{K_0/K}(w_{(n',d')})}$$

is a subgroup of G_{K_0} called the *$\psi_{K_0/K}(\underline{w})$-higher ramification subgroup of G_{K_0} in upper numbering* (see [16] for the definition of *$\underline{w}$-higher ramification subgroup in upper numbering* $G_K^{\underline{w}}$ of the absolute Galois group of a local field K for each increasing net $\underline{w}$).

On the other hand, as $\Gamma^{(n)}_{0,d}/K_0$ is APF, so $\Gamma^{(n)}_{0,d}/K$ is an APF extension. Thus, for each increasing net $\underline{w}$

$$G_K^{\underline{w}} := \varprojlim_{(n,d)} \mathrm{Gal}(\Gamma^{(n)}_{0,d}/K)^{w_{(n,d)}}$$

over the transition homomorphisms

$$r^{(n,d)}_{(n',d')} \quad : \quad \mathrm{Gal}(\Gamma^{(n)}_{0,d}/K)^{w_{(n,d)}} \quad \to \quad \mathrm{Gal}(\Gamma^{(n')}_{0,d'}/K)^{w_{(n,d)}} \quad \hookrightarrow \quad \mathrm{Gal}(\Gamma^{(n')}_{0,d'}/K)^{w_{(n',d')}}$$

for each $(n',d') \preceq (n,d)$, is a subgroup of G_K. Again we call $G_K^{\underline{w}}$ by *$\underline{w}$-higher ramification subgroup in upper numbering* of G_K.

Proposition 3.5. *For a given increasing net $\underline{w} = (w_{(n,d)})$, the projective limit*

$$\varprojlim_{n,d} \mathrm{Gal}(K_0/K)^{w_{(n,d)}}$$

over the embeddings

$$\mathrm{Gal}(K_0/K)^{w_{(n,d)}} \hookrightarrow \mathrm{Gal}(K_0/K)^{w_{(n',d')}} \quad (w_{(n',d')} \leq w_{(n,d)})$$

satisfies

$$\varprojlim_{n,d} \mathrm{Gal}(K_0/K)^{w_{(n,d)}} = \mathrm{Gal}(K_0/K)^{w}$$

where the number $w \in \mathbb{R} \cup \{\infty\}$ is defined by $w = \sup\{w_{(n,d)}\}$. We define

$$\mathrm{Gal}(K_0/K)^{\infty} := \{1_{\mathrm{Gal}(K_0/K)}\}$$

when $w = \infty$.

Proof. Let $w < \infty$. Note that, for each real number w' satisfying

$$\lceil \psi_{K_0/K}(w) \rceil - 1 < w' \leq \lceil \psi_{K_0/K}(w) \rceil$$

we have

$$\mathrm{Gal}(K_0/K)_{\psi_{K_0/K}(w)} = \mathrm{Gal}(K_0/K)_{w'}$$

by definition of the higher ramification groups. On the other hand, since Herbrand function is increasing,

$$\psi_{K_0/K}(w) = \sup\{\psi_{K_0/K}(w_{(n,d)})\}.$$

Let us fix a couple $(n_0,d_0) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}$ satisfying

$$\lceil \psi_{K_0/K}(w) \rceil - 1 \leq \psi_{K_0/K}(w_{(n_0,d_0)}) < \psi_{K_0/K}(w).$$

For a given

$$(\sigma_{(n,d)}) \in \varprojlim_{(n,d)} \mathrm{Gal}(K_0/K)_{\psi_{K_0/K}(w_{(n,d)})}$$

the following equality

$$\sigma_{(n_0,d_0)} = \sigma \in \mathrm{Gal}(K_0/K)_{\psi_{K_0/K}(w)}$$

holds. On the other hand for each $(n,d) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}$, we have

$$\sigma = \sigma_{(n_0,d_0)} = r^{(nn_0,dd_0)}_{(n_0,d_0)}(\sigma_{(nn_0,dd_0)}) = \sigma_{(nn_0,dd_0)}$$

Hence,

$$\sigma_{(n,d)} = r_{(n,d)}^{(nn_0,dd_0)}(\sigma_{(nn_0,dd_0)}) = \sigma \ .$$

This shows

$$(\sigma_{(n,d)}) = (\sigma) \ .$$

If $w = \infty$, then there exists a $(n_1, d_1) \in \mathbb{Z}\times\mathbb{Z}$, such that $\mathrm{Gal}(K_0/K)^{w_{(n_1,d_1)}} = \{1_{\mathrm{Gal}(K_0/K)}\}$. The proof follows by taking $w = w_{(n_1,d_1)}$ and by making the preceding calculations. □

Let us consider the normalized continuous section $s : \mathrm{Gal}(K_0/K) \to G_K$ for the extension of G_{K_0} by $\mathrm{Gal}(K_0/K)$, which has given by (2.2). Now, for each $\tau \in \mathrm{Gal}(K_0/K)^w$, one can suppose that, s satisfies

$$s(\tau) \in G_K^{\underline{w}}$$

by Remark 2.1. In this case, for each τ, $\tau' \in \mathrm{Gal}(K_0/K)$ one can show that $f(\tau,\tau') \in G_{K_0}^{\underline{w}}$, and $\psi^*(\tau) \mid_{G_{K_0}} \in \mathrm{Aut}(G_{K_0})$. From these observations, it can be shown that $(G_{K_0}^{\psi_{K_0/K}(\underline{w})} \times \mathrm{Gal}(K_0/K)^w, *)$ is a subgroup of E_{f,ψ^*}. We denote the topological group $(G_{K_0}^{\psi_{K_0/K}(\underline{w})} \times \mathrm{Gal}(K_0/K)^w, *)$ by

$$E_{f,\psi^*}^{\underline{w}} := (G_{K_0}^{\psi_{K_0/K}(\underline{w})} \times \mathrm{Gal}(K_0/K)^w, *) \ .$$

Moreover, restriction of the topological group isomorphism $\xi : E_{f,\psi^*} \to G_K$ defined by (2.1.1) to $E_{f,\psi^*}^{\underline{w}}$ gives the topological group isomorphism

$$\xi_{\underline{w}} := \xi \mid_{E_{f,\psi^*}^{\underline{w}}} : E_{f,\psi^*}^{\underline{w}} \xrightarrow{\sim} G_K^{\underline{w}} \ . \tag{3.13}$$

Now, consider the topological group isomorphism

$$E_{f,\psi^*} \xrightarrow{\left(\boldsymbol{\Phi}_{K_0}^{(\varphi_{K_0})}, \mathrm{Art}_{K_0/K}\right)} \nabla_K^{(\varphi_{K_0})}$$

given in Lemma 2.6. Then for any increasing $\mathbb{R}_{\geq 0}$-net $\underline{w} = (w_{(n,d)})$, we define the subgroup $({}_1\nabla_K^{(\varphi_{K_0})})^{\underline{w}}$ of $\nabla_K^{(\varphi_{K_0})}$ by

$$({}_1\nabla_K^{(\varphi_{K_0})})^{\underline{w}} := \left(\boldsymbol{\Phi}_{K_0}^{(\varphi_{K_0})}, \mathrm{Art}_{K_0/K}\right)(E_{f,\psi^*}^{\underline{w}}).$$

Lemma 3.6. *For a given increasing net $\underline{w} = (w_{(n,d)})$, assume that $w < \infty$. Consider the subgroups*

$$({}_1\nabla_{K_0}^{(\varphi_{K_0})})^{\psi_{K_0/K}(\underline{w})} := \left\langle 1_{\widehat{\mathbb{Z}}}\right\rangle \times \varprojlim_{(n,d)} \left(U_{\widetilde{\mathbb{X}}(\Gamma_{0,d}^{(n)}/K_0)}^{\diamond}\right)^{\psi_{\Gamma_{0,d}^{(n)}/K_0{}_d^{\mathrm{nr}}}(\psi_{K_0/K}(w_{(n,d)}))} Y_{\Gamma_{0,d}^{(n)}/K_0{}_d^{\mathrm{nr}}} / Y_{\Gamma_{0,d}^{(n)}/K_0{}_d^{\mathrm{nr}}}$$

of $\nabla_K^{(\varphi_{K_0})}$, and

$$U_K^{\lceil w\rceil}\, N_{K_0/K}\, K_0^\times /\, N_{K_0/K}\, K_0^\times$$

of $K^\times /\, N_{K_0/K}\, K_0^\times$. Then,

$$({}_1\nabla_K^{(\varphi_{K_0})})^{\underline{w}} = ({}_1\nabla_{K_0}^{(\varphi_{K_0})})^{\psi_{K_0/K}(\underline{w})} \times U_K^{\lceil w\rceil}\, N_{K_0/K}\, K_0^\times /\, N_{K_0/K}\, K_0^\times \ .$$

On the other hand if $w = \infty$, then we have

$$({}_1\nabla_K^{(\varphi_{K_0})})^{\underline{w}} = ({}_1\nabla_{K_0}^{(\varphi_{K_0})})^{\psi_{K_0/K}(\underline{w})} \times \left\langle 1_{K^\times /\, N_{K_0/K}\, K_0^\times}\right\rangle \ .$$

Proof. The proof follows from Theorem 6.10 of [1], which is the sharpened version of ramification theory for $\boldsymbol{\Phi}_{K_0}^{(\varphi_{K_0})}$ of İkeda and Serbest (cf. [16]), and from ramification theory for the abelian extension K_0/K. □

Theorem 3.7 (Ramification theory for K). *Let $\underline{w} = (w_{(n,d)})$ be an increasing net. For each $\sigma \in G_K$ we have*

$$\sigma \in G_K^{\underline{w}} \Leftrightarrow \mathbf{\Phi}_K^{(\varphi_{K_0})}(\sigma) \in ({}_1\nabla_K^{(\varphi_{K_0})})^{\underline{w}}$$

Proof. As $\sigma \in G_K$, we have

$$\xi^{-1}(\sigma) = \xi_{\underline{w}}^{-1}(\sigma) \in E_{f,\psi^*}^{\underline{w}} \ ,$$

where $\xi_{\underline{w}}$ is defined in (3.13). Thus, from the definition of the group $({}_1\nabla_K^{(\varphi_{K_0})})^{\underline{w}}$, we get

$$\mathbf{\Phi}_K^{(\varphi_{K_0})}(\sigma) \in ({}_1\nabla_K^{(\varphi_{K_0})})^{\underline{w}} \ .$$

□

Acknowledgment.

The author extends special thanks to K. İlhan İkeda, whose passion and enthusiasm as a teacher inspired deep admiration and affection. Also would like to thank Erol Serbest for guiding, being patient, and supporting throughout this work. His help, advice, and encouragement have meant a lot and truly have shaped the direction of the work. The author would also like to express gratitude to the anonymous referee for their careful review and valuable feedback. Additionally, it is acknowledged that this project is based on authors doctoral dissertation, supported by TÜBİTAK project no. 107T728, and expresses gratitude for the expertise and abilities gained during doctoral studies, which played a crucial role in completing this work.

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