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A Generalization of Szász-Baskakov Operators by using the Appell Polynomials of Class $A^{(2)}$

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Highlights

- This paper focuses on Szász-Baskakov operators via Appell $A^{(2)}$ polynomials.
- Uniform convergence of the constructed operators is mentioned.
- The rate of convergence is obtained with the help of the Steklov function.

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Abstract

In this paper, we obtain a generalization of the Szász-Baskakov operators with the help of $A^{(2)}$ class Appell polynomials. For every compact subset of $[0,\infty)$, the uniform convergence of these operators is provided. We also mention the convergence rate of our new operators and then we find some approximation results. The rate of convergence is obtained with the help of the Steklov function.

1. INTRODUCTION

Approximation theory is one of the important research topics of mathematical analysis, which emerged and became widespread in the 19th century and has been studied by many mathematicians around the world since then. The Szász operators are formed by extending the most well-known Bernstein operators to an infinite range. Szász operators are defined by Szász [1] as follows,

$$S_\mu(f;x) = e^{-\mu x}\sum_{v=0}^{\infty}\frac{(\mu x)^v}{v!}f\left(\frac{v}{\mu}\right), \tag{1.1}$$

where $\mu\in\mathbb{N}, x\geq 0$ and $f\in C[0,\infty)$. Baskakov [2] proposed the following Baskakov operators in 1957:

$$B_\mu(f;x) = \frac{1}{(1+x)^\mu}\sum_{v=0}^{\infty}\binom{\mu+v-1}{v}\frac{x^v}{(1+x)^v}f\left(\frac{v}{\mu}\right), \tag{1.2}$$

where $\mu\in\mathbb{N}^+, x\in[0,\infty)$. Szász-Mirakyan-Baskakov operators were examined by Prasad et al. [3] in 1983 as

$$Y_\mu(f;x) = (\mu-1)\sum_{v=0}^{\infty}e^{-\mu x}\frac{(\mu x)^v}{v!}\int_0^\infty\binom{\mu+v-1}{v}\frac{t^v}{(1+t)^{\mu+v}}f(t)dt. \tag{1.3}$$

and second modulus of continuity by Steklov functions.

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Appell polynomials were used by Jakimovski and Leviatan [4] to study a generalization of Szász operators. Let $g(\omega)=\sum_{v=0}^{\infty} b_v\omega^v$ $(b_0\neq 0)$ be a function of analysis in the $disc|\omega|< Z$ $(Z>1)$ and assume that $g(1)\neq 0$. The generating functions of the Appell polynomials $p_v(x)$ are the from:

$$g(u)e^{ux}=\sum_{v=0}^{\infty} p_v(x)u^v. \tag{1.4}$$

Assuming that for any $x\in[0,\infty)$, $p_v(x)\geq 0$ Jakimovski and Leviatan introduced $P_\mu(f;x)$ via

$$P_\mu(f;x)=\frac{e^{-\mu x}}{g(1)}\sum_{v=0}^{\infty} p_v(\mu x)f\left(\frac{v}{\mu}\right), \text{ for } \mu\in\mathbb{N} \tag{1.5}$$

and obtained approximation properties of these operators. Let $B(\omega)=\sum_{v=0}^{\infty} b_v\omega^v(b_0\neq 0)$ and $H(\omega)=\sum_{v=1}^{\infty} h_v\omega^v(h_1\neq 0)$ serve as analytical processes in the $disc|\omega|<R(R>1)$, in which b_v and h_v are real. With a generating function of the type, the Sheffer polynomials $p_v(x)$

$$A(\omega)e^{xH(\omega)}=\sum_{v=0}^{\infty} p_v(x)\omega^v, \qquad |\omega|<R. \tag{1.6}$$

Using the following assumptions:

- For $x\in[0,\infty)$, $p_v(x)\geq 0$,
- $A(1)\neq 0$, and $H'(1)=1$,

Ismail [5] looked at the convergence qualities of linear positive operators given by

$$L_\mu(f;x)=\frac{e^{-\mu xH(1)}}{A(1)}\sum_{v=0}^{\infty} p_v(\mu x)f\left(\frac{v}{\mu}\right), \qquad \text{for } \mu\in\mathbb{N}. \tag{1.7}$$

After that, Szász operators (1.1), which are developed using Sheffer polynomials, were proposed to be generalized by Jeelani [6]. With the help of similar methods, different generalizations of Szász operators can be found in [7-9]. We provide here a novel generalization of the operations Szász-Baskakov. Kazmin [10] define the $A^{(2)}$ class Appell polynomials $p_v(x)$ with the following generating functions:

$$A(\omega)e^{x\omega}+B(\omega)e^{-x\omega}=\sum_{v=0}^{\infty} p_v(x)\omega^v, \tag{1.8}$$

where

$$A(\omega)=\sum_{v=0}^{\infty}\frac{a_v}{v!}\omega^v \qquad and \qquad B(\omega)=\sum_{v=0}^{\infty}\frac{b_v}{v!}\omega^v \tag{1.9}$$

are formal power series defined at the $disc$ $|\omega|<R,(R>1)$ with $a_0^2-b_0^2\neq 0$. Then Varma [11] studied a generalization of Szasz operators including the Appell polynomials of class $A^{(2)}$. Sofyalıoğlu and Kanat [12] constructed Szász-Baskakov operators with Boas-Buck polynomials. By suitable substitutions $B(t)=e^t$ and $H(t)=t$ in [12]

$$T_\mu(f;x)=\frac{(\mu-1)}{A(1)e^{ux}}\sum_{v=0}^{\infty} p_v(\mu x)\int_0^\infty\binom{\mu+v-1}{v}\frac{t^v}{(1+t)^{\mu+v}}f(t)dt, \tag{1.10}$$

is obtained. In the light of these informations we define a generalization of Szász-Baskakov operators using Appell polynomials of class $A^{(2)}$ by

$$M_\mu(f;x)=\frac{(\mu-1)}{A(1)e^{\mu x}+B(1)e^{-\mu x}}\sum_{v=0}^{\infty} p_v(\mu x)\int_0^\infty\binom{\mu+v-1}{v}\frac{t^v}{(1+t)^{\mu+v}}f(t)dt, \tag{1.11}$$

where $\mu\in\mathbb{N}$, $x\geq 0$, and $f\in C[0,\infty)$. Conditions like $A(1)>0$, $B(1)\geq 0$ and $p_v(x)>0$ $(v=1,2,3,\dots)$, guarantee the positivity of the operator sequence in (1.11). If we specifically choose $A(\omega)=1$ and $B(\omega)=0$, the Szász-Baskakov-Appell polynomials (1.10) can be obtained again. Some numerical examples using certain kinds of orthogonal polynomials, like Gould-Hopper polynomials, can be created by using the sequence of operators in (1.11). Applications such as image processing, data science, and computer modeling can all benefit from the employment of these operators. The convergence qualities of the operator stated in (1.11) will be the first thing we look at in this essay. Then using Steklov functions to calculate the order of convergence with the help of the first and second modulus of continuity, we will derive various approximation results and estimations.

2. APPROXIMATION PROPERTIES OF M_μ OPERATORS

The application of the well-known Korovkin theorem and the approximation of the linear positive operator M_μ are presented in this section. Subsequently, we provide the approximation error estimate utilizing the first and second kind modulus of continuity.

Lemma 2.1. *The following equations are hold:*

$$\sum_{v=0}^{\infty} v p_v(\mu x) = A'(1)e^{\mu x} + B'(1)e^{-\mu x} + \mu x(A(1)e^{\mu x} - B(1)e^{-\mu x}),$$

$$\sum_{v=0}^{\infty} v^2 p_v(\mu x) = \left(A''(1) + A'(1)\right)e^{\mu x} + \left(B''(1) + B'(1)\right)e^{-\mu x}$$
$$+\mu x\left(\left(2A'(1) + A(1)\right)e^{\mu x} - (2B'(1) + B(1))e^{-\mu x}\right) + \mu^2 x^2 (A(1)e^{\mu x} + B(1)e^{-\mu x}).$$

Proof. After taking the derivatives of (1.8) with respect to k, we substitute μx for x then we have the desired results.

Lemma 2.2. *For every $x \in [0,\infty)$ and $e_n(t) = t^n$ for $n = 0,1,2$, we write*

$$M_\mu(e_0; x) = 1,$$

$$M_\mu(e_1; x) = \frac{\mu(A(1)e^{\mu x} - B(1)e^{-\mu x})}{(\mu-2)(A(1)e^{\mu x} + B(1)e^{-\mu x})}x + \frac{A'(1)e^{\mu x} + B'(1)e^{-\mu x}}{(\mu-2)(A(1)e^{\mu x} + B(1)e^{-\mu x})} + \frac{1}{\mu-2}, \qquad (\mu > 2),$$

$$M_\mu(e_2; x) = \frac{\left(A''(1) + 4A'(1) + 2A(1)\right)e^{\mu x} + \left(B''(1) + 4B'(1) + 2B(1)\right)e^{-\mu x}}{(\mu-2)(\mu-3)(A(1)e^{\mu x} + B(1)e^{-\mu x})} + \frac{\mu\left(\left(2A'(1) + 4A(1)\right)e^{\mu x} - \left(2B'(1) + 4B(1)\right)e^{-\mu x}\right)}{(\mu-2)(\mu-3)(A(1)e^{\mu x} + B(1)e^{-\mu x})}x + \frac{\mu^2}{(\mu-2)(\mu-3)}x^2, \qquad (\mu > 3).$$

Proof. In equation (1.8), we substitute μx instead of x and then choose $k = 1$ so we write,

$$A(1)e^{\mu x} + B(1)e^{-\mu x} = \sum_{v=0}^{\infty} p_v(\mu x).$$

The Beta-Gamma function is given as

$$\mathcal{B}(v,\mu) = \int_0^\infty \frac{t^{v-1}}{(1+t)^{\mu+v}}dt = \frac{\Gamma(v)\Gamma(\mu)}{\Gamma(v+\mu)} = \frac{(v-1)!(\mu-1)!}{(v+\mu-1)!}. \tag{2.1}$$

If we take $f(t) = 1$ in the operator (1.11), we get the following expression

$$M_\mu(e_0;x)=\frac{\mu-1}{A(1)e^{\mu x}+B(1)e^{-\mu x}}\sum_{v=0}^{\infty}p_v(\mu x)\int_0^\infty\binom{\mu+v-1}{v}\frac{t^v}{(1+t)^{\mu+v}}dt.$$

Using (2.1) to assist us solve the integral in the M_μ operator, we get

$$\begin{aligned}M_\mu(e_0;x)&=\frac{\mu-1}{A(1)e^{\mu x}+B(1)e^{-\mu x}}\sum_{v=0}^{\infty}p_v(\mu x)\binom{\mu+v-1}{v}\mathcal{B}(v+1,\mu-1)\\&=\frac{1}{A(1)e^{\mu x}+B(1)e^{-\mu x}}\sum_{v=0}^{\infty}p_v(\mu x)\\&=1.\end{aligned}$$

From Lemma 2.1, for f (t) =e_1(t) = t we write,

$$\begin{aligned}M_\mu(e_1;x)&=\frac{\mu-1}{A(1)e^{\mu x}+B(1)e^{-\mu x}}\sum_{v=0}^{\infty}p_v(\mu x)\int_0^\infty\binom{\mu+v-1}{v}\frac{t^{v+1}}{(1+t)^{\mu+v}}dt\\&=\frac{\mu-1}{A(1)e^{\mu x}+B(1)e^{-\mu x}}\sum_{v=0}^{\infty}p_v(\mu x)\binom{\mu+v-1}{v}\mathcal{B}(v+2,\mu-2)\\&=\frac{1}{(\mu-2)(A(1)e^{\mu x}+B(1)e^{-\mu x})}\left(\sum_{v=0}^{\infty}p_v(\mu x)v+\sum_{v=0}^{\infty}p_v(\mu x)\right)\\&=\frac{\mu(A(1)e^{\mu x}-B(1)e^{-\mu x})}{(\mu-2)(A(1)e^{\mu x}+B(1)e^{-\mu x})}x+\frac{A'(1)e^{\mu x}+B'(1)e^{-\mu x}}{(\mu-2)(A(1)e^{\mu x}+B(1)e^{-\mu x})}+\frac{1}{\mu-2},\qquad(\mu>2).\end{aligned}$$

One can find $M_\mu(e^2;x)$ in a similiar way.

Lemma 2.3. *If we calculate the central moments, using advantage of the linearity of the operator, we can write it as,*

$$M_\mu(t-x;x)=\left\{\frac{\mu(A(1)e^{\mu x}-B(1)e^{-\mu x})}{(\mu-2)(A(1)e^{\mu x}+B(1)e^{-\mu x})}-1\right\}x+\frac{A\prime(1)e^{\mu x}+B\prime(1)e^{-\mu x}}{(\mu-2)(A(1)e^{\mu x}+B(1)e^{-\mu x})}+\frac{1}{\mu-2},\qquad(\mu>2),$$

$$\begin{aligned}M_\mu((t-x)^2;x)=&\left\{\frac{\mu^2}{(\mu-2)(\mu-3)(A(1)e^{\mu x}+B(1)e^{-\mu x})}\right.\\&\left.-\frac{2(A'(1)e^{\mu x}-B'(1)e^{-\mu x})+2(A(1)e^{\mu x}-B(1)e^{-\mu x})u}{(\mu-2)(A(1)e^{\mu x}+B(1)e^{-\mu x})}+1\right\}x^2\\&+\left\{\frac{((2A'(1)+4A(1))e^{\mu x}-(2B'(1)+4B(1))e^{-\mu x})\mu}{(\mu-2)(\mu-3)(A(1)e^{\mu x}+B(1)e^{-\mu x})}-\frac{2}{(\mu-2)}\right\}x\\&+\frac{(A''(1)+4A'(1)+2A(1))e^{\mu x}+(B''(1)+4B'(1)+2B(1))e^{-\mu x}}{(\mu-2)(\mu-3)(A(1)e^{\mu x}+B(1)e^{-\mu x})},\qquad(\mu>3).\end{aligned}$$

Proof. From the linearity of the operators and Lemma 2.2, we get

$M_\mu(t-x;x)=M_\mu(e_1;x)-xM_\mu(e_0;x),\qquad(\mu>2),$

$M_\mu((t-x)^2;x)=M_\mu(e_2;x)-2xM_\mu(e_1;x)+x^2M_\mu(e_0;x),\qquad(\mu>3).$

As a result, the lemma yields the intended outcome.

Theorem 2.1.

Given a continuous function f on the interval $[0,\infty)$, a class member

$$E=\left\{f:\frac{f(x)}{1+x^2} \text{ is convergent as } x\to\infty\right\}.$$

Then $M_\mu(f;x)$ operators converge uniformly on each compact subset of $[0,\infty)$ as

$$\lim_{\mu\to\infty} M_\mu(f;x)=f(x).$$

Proof. From Lemma 2.2, we get

$$\lim_{\mu\to\infty} M_\mu(t^i;x)=x^i, \qquad i=0,1,2.$$

When considering the compact subset of the interval $[0,\infty)$, these convergences are uniformly satisfied. Therefore, the universal Korovkin theorem [13] provides the proof.

Definition 2.1. *Let $f\in C_D[0,\infty)$ and $\delta>0$. The modulus of continuity $\omega(f,\delta)$ of the function f is defined by*

$$\omega(f,\delta)=\sup_{\substack{x,t\in[0,\infty)\\ |x-t|\le\delta}} |f(x)-f(t)|, \tag{2.2}$$

where the space of uniformly continuous functions on $[0,\infty)$ is denoted by $C_D[0,\infty)$.

Definition 2.2. *The function $f\in C[a,b]$ has a second modulus of continuity that is defined as*

$$\omega_2(f,\delta)=\sup_{0<t\le\delta} \|f(.+2t)-2f(.+t)+f(.)\|, \tag{2.3}$$

where $\| f \|=\max_{x\in[a,b]}|f(x)|$.

Lemma 2.4. *(Gavrea and Raşa [14]) Let $g\in C^2[0,a]$ and $(K_\mu)_{\mu\ge 0}$ be a sequence of linear positive operators with the property $K_\mu(1;x)=1$. Then*

$$|K_\mu(g;x)-g(x)|\le\| g' \|\sqrt{K_\mu((t-x)^2;x)}+\frac{1}{2}\| g'' \| K_\mu((t-x)^2;x). \tag{2.4}$$

Definition 2.3. *(Zhuk [15]) The function f_h is called Steklov function of the function f if the following holds:*

$$f_h(t)=\frac{1}{h}\int_{t-\frac{h}{2}}^{t+\frac{h}{2}} f(u)du=\frac{1}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}} f(t+v)dv, \tag{2.5}$$

where f is integrable function on the closed and bounded interval $[a,b]$. Derivative of this function at almost every point is given as:

$$f'_h(\mathrm{t})=\frac{1}{h}f\left(t+\frac{h}{2}\right)-f\left(t-\frac{h}{2}\right). \tag{2.6}$$

If the derivative f is uniformly continuous on the real axis then we have the following relations.

$$\sup_{t\in(-\infty,\infty)}|f(t)-f_h(t)|\le\omega\left(\frac{h}{2},f\right)$$

$$\sup_{t\in(-\infty,\infty)}|f'_h(t)|\le\frac{1}{2}\omega(h,f).$$

Lemma 2.5. *(Zhuk [15]) Let $f \in C[a,b]$ and $\lambda \in \left(0, \frac{(b-a)}{2}\right)$. Let f_λ be the second-order Steklov function associated with the function f. Then the inequalities listed below are met:*

- $\| f_\lambda - f \| \leq \frac{3}{4}\omega_2(f,\lambda)$
- $\| f_\lambda'' \| \leq \frac{3}{2\lambda^2}\omega_2(f,\lambda)$.

In general we will minimize the margin of error in linear positive operators by using the first and second modulus of continuity. We shall determine the convergence rate in the two theorems that follow with the use of the earlier definitions and theorems.

Theorem 2.2. *Let $f \in C_D[0,\infty) \cap E$. M_μ operators confirm this inequality:*

$$|M_\mu(f;x) - f(x)| \leq 2\omega(f, \sqrt{\delta_\mu(x)}), \tag{2.7}$$

in which

$$\begin{aligned}\delta := \delta_\mu(x) = &\left\{\frac{\mu^2}{(\mu-2)(\mu-3)(A(1)e^{\mu x} + B(1)e^{-\mu x})}\right.\\ &\left.-\frac{2(A'(1)e^{\mu x} - B'(1)e^{-\mu x}) + 2(A(1)e^{\mu x} - B(1)e^{-\mu x})u}{(\mu-2)(A(1)e^{\mu x} + B(1)e^{-\mu x})} + 1\right\}x^2\\ &+\left\{\frac{((2A'(1)+4A(1))e^{\mu x} - (2B'(1)+4B(1))e^{-\mu x})\mu}{(\mu-2)(\mu-3)(A(1)e^{\mu x} + B(1)e^{-\mu x})} - \frac{2}{(\mu-2)}\right\}x\\ &+\frac{(A''(1)+4A'(1)+2A(1))e^{\mu x} + (B''(1)+4B'(1)+2B(1))e^{-\mu x}}{(\mu-2)(\mu-3)(A(1)e^{\mu x} + B(1)e^{-\mu x})}, \qquad (\mu > 3).\end{aligned}$$

Proof. Let $f \in C_D[0,\infty) \cap E$. Then from the property of modulus of continuity,

$$|f(x) - f(t)| \leq \left(\frac{|t-x|}{\delta} + 1\right)\omega(f,\delta) \tag{2.8}$$

is provided. Using the above the inequality, we have

$$\begin{aligned}|M_\mu(f;x) - f(x)| \leq &\left|\frac{(\mu-1)}{A(1)e^{\mu x} + B(1)e^{-\mu x}}\sum_{v=0}^{\infty} p_v(\mu x)\int_0^\infty \binom{\mu+v-1}{v}\frac{t^v}{(1+t)^{\mu+v}}f(t)dt\right.\\ &\left.-\frac{(\mu-1)}{A(1)e^{\mu x} + B(1)e^{-\mu x}}\sum_{v=0}^{\infty} p_v(\mu x)\int_0^\infty \binom{\mu+v-1}{v}\frac{t^v}{(1+t)^{\mu+v}}f(x)dt\right|\\ \leq &\frac{(\mu-1)}{A(1)e^{\mu x} + B(1)e^{-\mu x}}\sum_{v=0}^{\infty} p_v(\mu x)\int_0^\infty \binom{\mu+v-1}{v}\frac{t^v}{(1+t)^{\mu+v}}|f(t)-f(x)|dt\\ \leq &\frac{(\mu-1)}{A(1)e^{\mu x} + B(1)e^{-\mu x}}\sum_{v=0}^{\infty} p_v(\mu x)\int_0^\infty \binom{\mu+v-1}{v}\frac{t^v}{(1+t)^{\mu+v}}\left(\frac{|t-x|}{\delta}+1\right)\omega(f,\delta)dt\\ \leq &\left\{1 + \frac{1}{\delta}\left(\frac{(\mu-1)}{A(1)e^{\mu x} + B(1)e^{-\mu x}}\sum_{v=0}^{\infty} p_v(\mu x)\int_0^\infty \binom{\mu+v-1}{v}\frac{t^v}{(1+t)^{\mu+v}}|t-x|dt\right)\right\}\omega(f,\delta).\end{aligned}$$

By using the Cauchy-Schwarz inequality for the integral, the following result follows:

$$\left|M_\mu(f;x)-f(x)\right| \le \left\{1+\left(\frac{(\mu-1)}{A(1)e^{\mu x}+B(1)e^{-\mu x}}\right)\frac{1}{\delta}\sum_{v=0}^{\infty} p_v(\mu x)\left(\int_0^\infty \binom{\mu+v-1}{v}\frac{t^v}{(1+t)^{\mu+v}}dt\right)^{\frac{1}{2}}\right.$$
$$\left.\times\left(\int_0^\infty \binom{\mu+v-1}{v}\frac{t^v}{(1+t)^{\mu+v}}|t-x|^2dt\right)^{\frac{1}{2}}\right\}\omega(f,\delta).$$

By examining the Cauchy-Schwarz inequality in summation, one can easily reach the following conclusion

$$\left|M_\mu(f;x)-f(x)\right| \le \left\{1+\frac{1}{\delta}\left(\frac{(\mu-1)}{A(1)e^{\mu x}+B(1)e^{-\mu x}}\sum_{v=0}^{\infty} p_v(\mu x)\int_0^\infty \binom{\mu+v-1}{v}\frac{t^v}{(1+t)^{\mu+v}}dt\right)^{\frac{1}{2}}\right.$$
$$\left.\times\left(\frac{(\mu-1)}{A(1)e^{\mu x}+B(1)e^{-\mu x}}\sum_{v=0}^{\infty} p_v(\mu x)\int_0^\infty \binom{\mu+v-1}{v}\frac{t^v}{(1+t)^{\mu+v}}|t-x|^2dt\right)^{\frac{1}{2}}\right\}\omega(f,\delta)$$

$$=\left\{1+\frac{1}{\delta}\left(M_\mu(e_0;x)\right)^{\frac{1}{2}}\left(M_\mu((t-x)^2;x)\right)^{\frac{1}{2}}\right\}\omega(f,\delta)$$

$$=\left\{1+\tfrac{1}{\delta}\left(M_\mu((t-x)^2;x)\right)^{\frac{1}{2}}\right\}\omega(f,\delta),$$

the desired result is achieved.

$$|M_\mu(f;x)-f(x)| \le \left\{1+\frac{1}{\delta}\left(M_\mu((t-x)^2;x)\right)^{\frac{1}{2}}\right\}\omega(f,\delta).$$

By choosing $\delta := \delta_\mu(x) = \sqrt{M_\mu((t-x))^2;x)}$, the intended outcome is here.

Theorem 2.3. *For $f \in C[0,a]$, the estimation that follows is valid*

$$\left|M_\mu(f;x)-f(x)\right| \le \tfrac{2}{a}\|f\|\phi^2+\tfrac{3}{4}(a+2+\phi^2)\omega_2(f,\phi), \tag{2.10}$$

where

$$\phi := \phi_\mu(x) = \sqrt[4]{M_\mu((t-x)^2;x)}. \tag{2.11}$$

Proof. Let f_ϕ be the Steklov function of second order associated with function f. Concerning the identity $M_\mu(1;x)=1$, we possess

$$\begin{aligned}|M_\mu(f;x)-f(x)| &\le \left|M_\mu(f;x)-f(x)+M_\mu(f_\phi;x)-M_\mu(f_\phi;x)+f_\phi(x)-f_\phi(x)\right|\\ &\le \left|M_\mu(f;x)-M_\mu(f_\phi;x)\right|+\left|M_\mu(f_\phi;x)-f_\phi(x)\right|+\left|f_\phi(x)-f(x)\right|\\ &\le \left\|f-f_\phi\right\|M_\mu(1;x)+\left|M_\mu(f_\phi;x)-f_\phi(x)\right|+\left\|f_\phi-f\right\|\\ &\le 2\left\|f_\phi-f\right\|+\left|M_\mu(f_\phi;x)-f_\phi(x)\right|.\end{aligned}$$

With the help of Lemma 2.3 and Lemma 2.4,

$$\left|M_\mu(f_\phi;x)-f_\phi(x)\right| \le \left\|f_\phi{}'\right\| \sqrt{M_\mu((t-x)^2;x)}+\frac{1}{2}\left\|f_\phi{}''\right\| M_\mu((t-x)^2;x).$$

The Landau inequality is defined from [16] as follows,

$$\|f'\| \le 2\|f\|^{\frac{1}{2}}\ \|f''\|^{\frac{1}{2}}.$$

If Lemma 2.5 and Landau inequality are combined,

$$\begin{aligned}\left\|f_\phi{}'\right\| &\le \frac{2}{a}\left\|f_\phi\right\|+\frac{a}{2}\left\|f_\phi{}''\right\| \\ &\le \frac{2}{a}\|f\|+\frac{3a}{4}\frac{1}{\phi^2}\omega_2(f,\phi).\end{aligned} \tag{2.12}$$

If we specifically assume $\phi=\sqrt[4]{M_\mu((t-x)^2;x)}$, then we write

$$|M_\mu(f_\phi;x)-f_\phi(x)| \le \frac{2}{a}\parallel f \parallel \phi^2+\frac{3a}{4}\omega_2(f;\phi)+\frac{3}{4}\phi^2\omega_2(f,\phi). \tag{2.13}$$

From Lemma 2.5,

$$|M_\mu(f;x)-f(x)| \le \frac{2}{a}\|f\|\phi^2+\frac{3}{4}(a+2+\phi^2)\omega_2(f,\phi)$$

is obtained and the proof is completed.

3. CONCLUSION

In this work, Szász-Baskakov operators are obtained via Appell $A^{(2)}$ polynomials. The uniform convergence of these operators is shown for each compact subset of $[0,\infty)$. The central moments of the constructed operators were obtained. Finally, the rate of convergence is obtained by using the modulus of continuity with the help of the Steklov function.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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