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Enriched P-Contractions on Normed Space and a Fixed Point Result

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Abstract. This paper introduces the concept of enriched P-contractions on linear normed spaces, and provides illustrative examples that highlight the differences between this new concept and its previous counterparts. It then gives a research result regarding the existence and uniqueness of the fixed point of this innovative type of contractions in Banach spaces. Finally, reminds us of the concept of enriched nonexpansive mappings and also offers a simple fixed point theorem for such mappings.

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1. Introduction

Let (X, d) be a metric space, and consider a mapping $T : X \to X$. The notion of fixed points, where there exists $z \in X$ such that $Tz = z$, introduces a key concept in the study of mappings. The set of all fixed points of T is denoted by $Fix(T)$. When $Fix(T)$ reduces to a single point set, and the Picard sequence $\{T^n x_0\}$ for any initial point $x_0 \in X$ converges to the element of $Fix(T)$, we refer to T as a Picard operator.

A particularly significant result in fixed-point theory is the Banach fixed-point theorem [2], which asserts that every contraction self-mapping on a complete metric space is a Picard operator. To understand this theorem, we recall that a self-mapping T of a metric space (X, d) is termed a contraction if there exists a Lipschitz constant $\alpha \in [0, 1)$ such that the inequality:

$$d(Tx, Ty) \leq \alpha d(x, y) \tag{1.1}$$

holds for all $x, y \in X$. This inequality signifies that the images of two points under the mapping T are brought closer together, with the contraction factor α determining the degree of this convergence.

In the wake of this theorem, numerous fixed-point theorems in metric spaces such as Kannan [11], Chatterjea [9], Ćirić-Reich-Rus [10, 13, 14] and many others, have been established, with a significant proportion featuring mappings characterized as Picard operators. In this direction, Popescu [12] introduced the notion of P-contraction, demonstrating that P-contraction self mappings in complete metric spaces align with the characteristics of a Picard operator, akin to the principles laid out in Banach fixed-point theorem. A self-mapping T of a metric space (X, d) is called P-contraction if there exists $\beta \in [0, 1)$ such that the inequality:

$$d(Tx, Ty) \leq \beta \left[d(x, y) + |d(x, Tx) - d(y, Ty)| \right] \tag{1.2}$$

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holds for all $x, y \in X$. It is clear that (1.1) implies (1.2), the converse, however, may not be true as shown in the following easy example:

Example 1.1. Let $X = [0, 1] \cup [2, 3]$ with the usual metric, and define a self mapping T as

$$Tx = \begin{cases} 1 & , \quad x \in [0,1] \\ \\ 0 & , \quad x \in [2,3] \end{cases},$$

then T is not contraction since $d(T1, T2) = 1 = d(1, 2)$. However, it is a P-contraction with $\beta = \frac{1}{3}$. Except in obvious cases let $x \in [0, 1]$ and $y \in [2, 3]$, then we have

$$d(Tx, Ty) = 1$$

and

$$\begin{aligned} d(x,y) + |d(x,Tx) - d(y,Ty)| &= y - x + |1 - x - y| \\ &= 2y - 1, \end{aligned}$$

and so

$$d(Tx, Ty) = 1 \leq \frac{2y-1}{3} = \frac{1}{3}\left[d(x,y) + |d(x,Tx) - d(y,Ty)|\right].$$

On the other hand, for a fresh outlook on the realm of fixed point theory investigations, Berinde and Păcurar [4] have innovatively introduced the notion of enriched contractions. This novel concept extends its scope to encompass contraction mapping within normed spaces. While the enriched contraction concept lacks a definition within metric spaces, its introduction remains significant. This is attributed to the fact that the applications of fixed point theory are often applicable and meaningful in normed spaces, showcasing the versatility of this innovative approach.

For the sake of completeness, we recall this new concept. Let C be a nonempty subset of a linear normed space $(X, \|\cdot\|)$ and $T : C \to C$ be a mapping, then T is called enriched contraction, if there exist $b \geq 0$ and $\theta \in [0, b + 1)$ such that the inequality

$$\|b(x - y) + Tx - Ty\| \leq \theta \|x - y\|$$

holds for all $x, y \in C$. It is clear that, every contraction with constant α in a normed space is an enriched contraction with $b = 0$ and $\theta = \alpha$. An example from Berinde and Păcurar [4] demonstrates that the contrary of this circumstance is not true. For the sake of clarity, let us provide a further example here.

Example 1.2. Consider the subset $C = [0, 1]$ of usual normed space $(\mathbb{R}, |\cdot|)$. Define a mapping $T : C \to C$ as $Tx = 1 - x^2$. Then, T is not contraction (it is not even P-contraction), but it is enriched contraction with $b = \theta = 1$. Indeed, for $x = 0$ and $y = 1$, since

$$\|Tx - Ty\| = 1 = \|x - y\| + |\|x - Tx\| - \|y - Ty\||$$

T is not P-contraction. On the other hand, for $b = \theta = 1$, we have

$$\begin{aligned} \|b(x-y) + Tx - Ty\| &= \left|x - y - x^2 + y^2\right| \\ &= |1 - x - y|\,|x - y| \\ &\leq |x - y| \\ &= \theta\,\|x - y\|, \end{aligned}$$

for any $x, y \in C$.

Remark 1.3. Note that, in the previous example $Fix(T) = \left\{\frac{\sqrt{5}-1}{2}\right\}$ and the associated Picard sequence $\{x_n\}$ of T which is given by $x_{n+1} = Tx_n = 1 - x_n^2$ does not converge for $x_0 \in \{0, 1\}$. Both this example and the previous example given in [4] show that in the fixed point theorem for enriched contraction, it is not correct to proceed with the Picard iteration sequence and more appropriate iteration sequences are needed, for which Berinde and Păcurar have considered the Krasnoselskij iteration sequence.

Before presenting the first fixed point theorem for enriched contractions, we must recall that for any self mapping T on a convex subset of a normed space X we have $Fix(T) = Fix(T_\lambda)$, where T_λ is a self mapping on C defined by

$$T_\lambda x = (1-\lambda)x + \lambda T x, \tag{1.3}$$

for $\lambda \in (0,1)$.

Theorem 1.4 ([4]). *Let C be a nonempty, closed and convex subset of a Banach space $(X, \|\cdot\|)$ and $T : C \to C$ be an enriched contraction mapping. Then,*

- $Fix(T) = \{z\}$,
- *there exists $\lambda \in (0,1]$ such that the Krasnoselskij iteration sequence $\{x_n\}$, given by*

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n \tag{1.4}$$

  converges to z, for any $x_0 \in C$,
- *the following estimates hold: for $n \geq 0$ and $i \geq 1$,*

$$\|x_{n+i-1} - z\| \leq \frac{\gamma^i}{1-\gamma}\|x_n - x_{n-1}\|,$$

  where $\gamma = \frac{\theta}{b+1}$.

Following Berinde and Păcurar [4], enriched versions of Kannan, Chatterjea, Ćirić-Reich-Rus and more contraction inequalities are defined and related fixed point theorems are obtained (see, [5–8] respectively).

In this paper, we will characterize the enriched counterpart of Popescu's P-contraction inequality and provide a fixed point result on it. Then, we will recall the definition of enriched nonexpansive mapping and present a simple result.

2. Main Results

Definition 2.1. Let C be a nonempty subset of a linear normed space $(X, \|\cdot\|)$ and $T : C \to C$ be a mapping. Then, T is said to be an enriched P-contraction if there exist $b \in [0,\infty)$ and $\theta \in [0, b+1)$ such that

$$\|b(x-y) + Tx - Ty\| \leq \theta\|x-y\| + \frac{\theta}{b+1}\left|\|x-Tx\| - \|y-Ty\|\right|, \tag{2.1}$$

for all $x, y \in C$.

Remark 2.2. If we consider the definitions of contraction (C), P-contraction (PC), enriched contraction (EC) and enriched P-contraction (EPC), we can draw the diagram containing the following implications in normed linear spaces:

$$\begin{array}{ccccc} & C & \overset{\triangledown}{\longrightarrow} & PC & \\ \triangleright & \downarrow & & \downarrow & \triangleleft \\ & EC & \underset{\vartriangle}{\longrightarrow} & EPC & \end{array}.$$

Here, the Example 1.1 can be taken into consideration to show that the converse of the implication given by $\triangledown$ is not true, and the Example 1.2 can be taken into consideration to show that the converse of both the implications given by $\triangleright$ and $\triangleleft$ are not true. It remains an open problem whether the converse of the implication given by $\vartriangle$ is true.

Now, we are ready to present our main result.

Theorem 2.3. *Let C be a nonempty, closed and convex subset of a Banach space $(X, \|\cdot\|)$ and $T : C \to C$ be an enriched P-contraction mapping. Then,*

i) $Fix(T) = \{z\}$,

ii) there exists $\lambda \in (0,1]$ such that the Krasnoselskij iteration sequence $\{x_n\}$, given by (1.4) converges to z, for any $x_0 \in X$,

iii) the following estimates holds: for $n \geq 0$ and $i \geq 1$

$$\|x_{n+i-1} - z\| \leq \frac{\delta^i}{1-\delta}\|x_n - x_{n-1}\|, \tag{2.2}$$

where $\delta = \frac{2\theta}{b+\theta+1}$.

Proof. We will provide the proof by considering the following two cases of b in (2.1).

A-) Let $b > 0$ and let us take $\lambda = \frac{1}{b+1}$. Obviously, we have $0 < \lambda < 1$ and the enriched P-contraction inequality (2.1) becomes

$$\left\| \left(\frac{1}{\lambda} - 1 \right)(x - y) + Tx - Ty \right\| \leq \theta \|x - y\| + \lambda\theta \left| \|x - Tx\| - \|y - Ty\| \right|$$

for all $x, y \in C$, which can be written in an equivalent form as

$$\|T_\lambda x - T_\lambda y\| \leq \beta \left[\|x - y\| + \left| \|x - T_\lambda x\| - \|y - T_\lambda y\| \right| \right] \tag{2.3}$$

for all $x, y \in C$, where $\beta = \lambda\theta$ and T_λ is defined in (1.3). Since $\theta \in [0, b+1)$, it follows that $\beta \in [0, 1)$ and therefore by (2.3) T_λ is a P-contraction. Now, let $x_0 \in C$ be an arbitrary point. In view of (1.3), the Krasnoselskij iterative process $\{x_n\} \subseteq C$ defined by (1.4) is exactly the Picard iteration associated with T_λ, that is,

$$x_{n+1} = T_\lambda x_n$$

for all $n \in \mathbb{N}$. Now assume that $\|x_{n+1} - x_n\| > 0$ for all $n \in \mathbb{N}$ (otherwise x_n will be a fixed point of T_λ and hence of T). Take $x = x_n$ and $y = x_{n-1}$ in (2.3) to get

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|T_\lambda x_n - T_\lambda x_{n-1}\| \\ &\leq \beta \left[\|x_n - x_{n-1}\| + \left| \|x_n - T_\lambda x_n\| - \|x_{n-1} - T_\lambda x_{n-1}\| \right| \right] \\ &\leq \beta \left[\|x_n - x_{n-1}\| + \left| \|x_n - x_{n+1}\| - \|x_{n-1} - x_n\| \right| \right]. \end{aligned} \tag{2.4}$$

If there exists $k \in \mathbb{N}$ such that $\|x_k - x_{k+1}\| \geq \|x_{k-1} - x_k\|$, then from (2.4), we have

$$\|x_{k+1} - x_k\| \leq \beta \|x_{k+1} - x_k\|,$$

which is a contradiction. Therefore, $\|x_n - x_{n+1}\| < \|x_{n-1} - x_n\|$ for all $n \in \mathbb{N}$ and so from (2.4) we have

$$\|x_{n+1} - x_n\| \leq \frac{2\beta}{1+\beta} \|x_n - x_{n-1}\|.$$

Since $\beta \in [0, 1)$ by denoting $\delta = \frac{2\beta}{1+\beta}$, we have $\delta \in [0, 1)$ and therefore, the sequence $\{x_n\}$ satisfies

$$\|x_{n+1} - x_n\| \leq \delta \|x_n - x_{n-1}\| \tag{2.5}$$

for all $n \in \mathbb{N}$. By (2.5), one obtains routinely the following two estimates

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq \|x_{n+m} - x_{n+m-1}\| + \|x_{n+m-1} - x_{n+m-2}\| + \cdots + \|x_{n+1} - x_n\| \\ &\leq \left(\delta^{m+n-1} + \delta^{n+m-2} + \cdots + \delta^n \right) \|x_1 - x_0\| \\ &= \delta^n \left(\delta^{m-1} + \delta^{m-2} + \cdots + 1 \right) \|x_1 - x_0\| \\ &= \delta^n \frac{1 - \delta^m}{1 - \delta} \|x_1 - x_0\| \end{aligned} \tag{2.6}$$

for $n \geq 0$, $m \geq 1$ and

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq \|x_{n+m} - x_{n+m-1}\| + \|x_{n+m-1} - x_{n+m-2}\| + \cdots + \|x_{n+1} - x_n\| \\ &\leq \delta^m \|x_n - x_{n-1}\| + \delta^{m-1} \|x_n - x_{n-1}\| + \cdots + \delta \|x_n - x_{n-1}\| \\ &= \delta \left(\delta^{m-1} + \delta^{m-2} + \cdots + 1 \right) \|x_n - x_{n-1}\| \\ &= \delta \frac{1 - \delta^m}{1 - \delta} \|x_n - x_{n-1}\| \end{aligned} \tag{2.7}$$

for $n \geq 1$, $m \geq 1$. Now, by (2.6), it follows that $\{x_n\}$ is a Cauchy sequence and hence it is convergent in C. Let us denote

$$z = \lim_{n \to \infty} x_n. \tag{2.8}$$

Take $x = x_n$ and $y = z$ in (2.3) we get

$$\begin{aligned} \|x_{n+1} - T_\lambda z\| &= \|T_\lambda x_n - T_\lambda z\| \\ &\leq \beta \left[\|x_n - z\| + \left| \|x_n - T_\lambda x_n\| - \|z - T_\lambda z\| \right| \right] \\ &= \beta \left[\|x_n - z\| + \left| \|x_n - x_{n+1}\| - \|z - T_\lambda z\| \right| \right]. \end{aligned}$$

By letting $n \to \infty$ in the last inequality, we immediately obtain

$$\|z - T_\lambda z\| \le \beta \|z - T_\lambda z\| .$$

Hence, we get $\|z - T_\lambda z\| = 0$, that is, $z \in Fix(T_\lambda)$.

Next, we prove that z is the unique fixed point of T_λ. Assume that $w \neq z$ is another fixed point of T_λ. Then, by (2.3)

$$0 < \|z - w\| \le \beta [\|z - w\| + |\|z - T_\lambda z\| - \|w - T_\lambda w\||] < \|z - w\| ,$$

a contradiction. Therefore, $Fix(T_\lambda) = \{z\} = Fix(T)$ and hence (i) holds. Conclusion (ii) follows by (2.8). To prove (iii), we let $m \to \infty$ in (2.6) and (2.7) to get

$$\|x_n - z\| \le \frac{\delta^n}{1-\delta} \|x_1 - x_0\|$$

and

$$\|x_n - z\| \le \frac{\delta}{1-\delta} \|x_n - x_{n-1}\| \tag{2.9}$$

for all $n \ge 1$, respectively, where $\delta = \frac{2\beta}{1+\beta} = \frac{2\theta}{b+\theta+1}$. Now (2.5) and (2.9), we have

$$\begin{aligned} \|x_{n+i-1} - z\| &\le \frac{\delta}{1-\delta} \|x_{n+i-1} - x_{n+i-2}\| \\ &\le \frac{\delta}{1-\delta} \delta \|x_{n+i-2} - x_{n+i-3}\| \\ &\vdots \\ &\le \frac{\delta}{1-\delta} \delta^{i-1} \|x_n - x_{n-1}\| \\ &= \frac{\delta^i}{1-\delta} \|x_n - x_{n-1}\| \end{aligned}$$

for $n \ge 0$ and $i \ge 1$. that is, the error estimate (2.2) holds.

B-) Let $b = 0$. In this case, $\lambda = 1$ and $\beta = \theta$ and we proceed like in above with $T = T_1$ instead of T_λ, when the associated Krasnoselskij iteration reduces, in fact, to the simple Picard iteration associated with T. □

Finally we recall the concept of enriched nonexpansive mappings and provide a simple result. The notion of enriched nonexpansive mapping was first defined by Berinde [3] and some fixed point theorems for such mappings in Hilbert spaces were given. For similar results, see also the papers [1, 15].

Definition 2.4. Let C be a nonempty subset of a linear normed space $(X, \|\cdot\|)$ and $T : C \to C$ be a mapping, then T is called enriched nonexpansive, if there exists $b \ge 0$ such that the inequality

$$\|b(x - y) + Tx - Ty\| \le (b + 1) \|x - y\|$$

holds for all $x, y \in C$.

It is clear that, every nonexpansive mapping is an enriched nonexpansive. However, the converse may not be true as shown in Example 2.1 of [3]. For another simple example, let $X = C = \mathbb{R}$ be endowed usual norm and let $T : C \to C$ defined by $Tx = 1 - 2x$. Then, T is not nonexpansive mapping, but it is enriched nonexpansive with $b = 1$. Note that, enriched nonexpansive mappings are also continuous.

Theorem 2.5. *Let C be a nonempty, closed and convex subset of a Banach space $(X, \|\cdot\|)$ and $T : C \to C$ be an enriched nonexpansive mapping. If $T(C)$ is a subset of a compact set of C, then T has a fixed point.*

Proof. Let $x_0 \in C$ and define

$$T_n x = \left(1 - \frac{1}{n}\right) Tx + \frac{1}{n} x_0$$

for $n \in \{2, 3, \cdots\}$. Since C is convex and $x_0 \in C$, then $T_n : C \to C$ for all $n \in \{2, 3, \cdots\}$. Also, for all $x, y \in C$, we have

$$\begin{aligned}
\|b(x-y) + T_n x - T_n y\| &= \left\| b\left(1 - \tfrac{1}{n} + \tfrac{1}{n}\right)(x-y) + \left(1 - \tfrac{1}{n}\right)Tx - \left(1 - \tfrac{1}{n}\right)Tx \right\| \\
&\leq \left(1 - \frac{1}{n}\right)\|b(x-y) + Tx - Tx\| + \frac{b}{n}\|x-y\| \\
&\leq \left(1 - \frac{1}{n}\right)(b+1)\|x-y\| + \frac{b}{n}\|x-y\| \\
&= \left(b + 1 - \frac{b}{n} - \frac{1}{n} + \frac{b}{n}\right)\|x-y\| \\
&= \left(b + 1 - \frac{1}{n}\right)\|x-y\|
\end{aligned}$$

for all $n \in \{2, 3, \cdots\}$. That is, every T_n is an enriched contraction with $b \geq 0$ and $\theta = b + 1 - \frac{1}{n}$. Therefore, by Theorem 1.4 each T_n has a unique fixed point $z_n \in C$, that is,

$$z_n = T_n z_n = \left(1 - \frac{1}{n}\right)Tz_n + \frac{1}{n}x_0$$

for all $n \in \{2, 3, \cdots\}$. On the other hand, since $T(C)$ lies in a compact subset of C, there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $Tz_{n_k} \to z \in C$ as $k \to \infty$. Hence,

$$z_{n_k} = \left(1 - \frac{1}{n_k}\right)Tz_{n_k} + \frac{1}{n_k}x_0 \to z \text{ as } k \to \infty.$$

By the continuity of T, we have $Tz_{n_k} \to Tz$ as $k \to \infty$ and therefore $z = Tz$. □

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

Each author contributed equally to the paper.

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