Hacettepe Journal of
Mathematics & Statistics

Hacet. J. Math. Stat.
Volume XX (x) (XXXX), 1 – 8
DOI : 10.15672/hujms.xx

Research Article

Spectral properties of the finite system of Klein-Gordon S-wave equations with condition depends on spectral parameter

Elgiz Bairamov[1], Esra Kir Arpat*[2]

[1]Department of Mathematics, Ankara University, 06100, Ankara, Türkiye
[2]Department of Mathematics, Gazi University, 06500, Ankara, Türkiye

Abstract

The spectral characteristics of the operator L are studied where L is defined within the Hilbert space $L_2(\mathbb{R}_+, \mathbb{C}^V)$ given by a finite system of Klein-Gordon type differential equations and boundary condition depends on spectral parameter. The research of the Klein-Gordon type operator continues to be an important topic for researchers due to the range of applicability of them in numerous branches of mathematics and quantum physics. Contrary to the previous works, we take the potential as complex valued and generalize the problem to the matrix Klein-Gordon operator case. The spectrum is derived by determining the Jost function and resolvent operator of the prescribed operator. Further, we provide the conditions that must be met for the certain quantitative properties of the spectrum.

Mathematics Subject Classification (2020). 39A70, 47A10, 47A75

Keywords. eigenvalues, spectral singularities, Klein-Gordon S-wave equation

1. Introduction

Discoveries in quantum physics have a remarkable role on the understanding the subatomic particles. This theory was among the most powerful physics theories in history when special relativity was integrated to it. In relativistic particle physics, the Klein–Gordon (KG) equations are the most generally utilized wave equations for modeling particle movements. Therefore, the equation has gotten enormous interest in the various investigation fields of physics and mathematics like solutions and wavelet theory, nonlinear wave equations, as well as studies of numerical methods developed for the solutions of KG equations [13, 14, 20, 21].

Take into consideration the differential operator L_o, defined in the complete inner product space (Hilbert space) $L_2(\mathbb{R}_+)$ for $x \in \mathbb{R}_+ := [0, \infty)$,

$$l_o(z) = -z^{''} + q(x)z, \tag{1.1}$$

*Corresponding Author.
Email addresses: bairamov@science.ankara.edu.tr (E.Bairamov), esrakir@gazi.edu.tr (E.K.Arpat)
Received: 19.11.2023; Accepted: 16.11.2024

with the initial condition $z^{'}(0) - hz(0) = 0$. Also, assume that the potential function q takes complex values and the constant h is also complex constant. Clearly, L_o is a non-selfadjoint operator due to the complex valued potential. Naimark [19] was the first to realize the apperance of an extraordinary set of spectrum that is the spectral singularities embedded in the continuous spectrum. Later, Schwartz [22] characterized the spectral singularities as points where the resolvent of a non-selfadjoint operator has a pole however it is not the operator's eigenvalue.

In addition to these developments, Naimark also determined significant qualitative features of the operator L_o's spectrum. In particular, if the complex valued potential yields

$$\int_0^\infty e^{\varepsilon x} |q(x)| \, dx < \infty, \, \varepsilon > 0,$$

then the operator's discrete spectrum may contain only finite number of elements. In a similar manner, this condition also guarantees that there exists finitely many spectral singularities. Lyance expanded upon the influence of spectral singularities on spectral expansion by means of the spectral expansion's principal functions of L_o in [16],[17].

These developments pushed researches to investigate under what conditions imposed on the potential the operators may have finitely many of eigenvalues and spectral singularities. Also, the structure of the obtained conditions have became an interesting question, too. For instance, to what extend we can strict the conditions so that quantitative properties still remains finite. To solve these problems for a novel type of non-hermitian operators, boundary uniqueness theorems of analytic functions served as a great tool. For instance, quadratic pencil of Schrödinger type equations, Dirac and Klein-Gordon type operators for both in differential and difference operator versions including complex valued potential have been examined in [1],[2],[6–11],[15],[18]. Clearly, the spectral singularities have an impact on the spectral expansion of Sturm-Liouville type differential equations. This issue has been solved by the method of regularizing a divergent integral in the studies [4],[23].

Consider a well-known form of Klein-Gordon s-wave equation for $x \in \mathbb{R}_+$ [5],

$$z^{''} + [\mu - Q(x)]^2 z = 0 \,. \tag{1.2}$$

Note that Q designates the static potential. This equation is used to model the behaviour of a particle having a zero mass in quantum physics.

Let us also mention that the inverse spectral theory of Sturm-Liouville equations (also called one dimensional Schrödinger equation) has been investigated in matrix form by [3]. Hence, some new class of equations with matrix form become more important in the years after.

Inspired by the above mentioned studies, we set up our research problem as in the following: Let $L_2(\mathbb{R}_+, \mathbb{C}^V)$ stands for complete inner product space including all complex vector functions

$$z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_V \end{pmatrix},$$

where the norm of the Hilbert space is defined by

$$\|z\|^2 := \int_0^\infty \sum_{n=1}^{\infty} |z_v|^2 \, dx.$$

Consider the finite system of Klein-Gordon s-wave differential expressions

$$l_v(z_v) := z_v^{''} + [\mu - q_v(x)]^2 z_v, x \in \mathbb{R}_+, \, v = 1, 2, ..., V,$$

where q_v are complex valued functions.

Symbolize with L the operator defined in $L_2(\mathbb{R}_+, \mathbb{C}^V)$ by

$$l_v(z_v) = \begin{pmatrix} l_1(z_1) \\ l_2(z_2) \\ \vdots \\ l_N(z_V) \end{pmatrix},$$

and boundary condition with spectral parameter

$$z^{'}(0) - (\alpha_0 + \alpha_1\mu + \alpha_2\mu^2)z(0) = 0, \tag{1.3}$$

such $\alpha_i \in \mathbb{C}, i = 0, 1, 2,\ \alpha_2 \neq 0$, is a complex parameter. Since, the expressions $q_v, v = 1, 2, ..., V$ are assumed to have complex values, it is quite obvious that L is a non-selfadjoint operator.

We obtain certain quantitative properties for the operator L under the conditions

$$\lim_{x\to\infty} q_v(x) = 0,\ \underset{x\in\mathbb{R}_+}{sup}\left\{\exp(\epsilon\sqrt{x})\left|q_v^{'}(x)\right|\right\} < \infty,\ \epsilon > 0,\ v = 1, 2, ..., V.$$

2. Jost solutions of $l(z) = 0$

We will take into account the equation

$$z^{''} + [\mu - Q(x)]^2 z = 0,\ x \in \mathbb{R}_+, \tag{2.1}$$

and with spectral parameter of the boundary condition

$$z^{'}(0) - (\alpha_0 + \alpha_1\mu + \alpha_2\mu^2)z(0) = 0, \tag{2.2}$$

such that

$$z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_\nu \end{pmatrix},\ Q(x) = \begin{bmatrix} q_1(x) & 0 & ... & 0 \\ 0 & q_2(x) & ... & 0 \\ ... & & & \\ ... & & & \\ ... & & & \\ 0 & 0 & ... & q_\nu(x) \end{bmatrix}.$$

Suppose that the functions q_v, $v = 1, 2, ..., V$, satisfy

$$\lim_{x\to\infty} q_v(x) = 0, \int_0^\infty x^3 |q_v^{'}(x)| \mathrm{dx} < \infty. \tag{2.3}$$

The equation (2.1) have the matrix solutions $E^+(x, \mu)$ for $\mu \in \overline{\mathbb{C}}_+ := \{\mu : \mu \in \mathbb{C}, Im\mu \geq 0\}$ and $E^-(x, \mu)$ for $\mu \in \overline{\mathbb{C}}_- := \{\mu : \mu \in \mathbb{C}, Im\mu \leq 0\}$ [3].

$E^\pm$ have the following representation

$$E^\pm(x, \mu) = \begin{bmatrix} e_1^\pm(x,\mu) & 0 & ... & 0 \\ 0 & e_2^\pm(x,\mu) & ... & 0 \\ ... & & & \\ ... & & & \\ ... & & & \\ 0 & 0 & ... & e_V^\pm(x,\mu) \end{bmatrix}. \tag{2.4}$$

$e_v^\pm(x, \mu)$ are introduced as the Jost solutions of (2.1).

Suppose that the condition (2.3) satisfies, in this case the Jost solutions can be represented as [5]

$e_v^+(x, \mu) = e^{i\alpha(x)+i\mu x} + \int_x^\infty K_v^+(x, t)e^{i\mu t}\mathrm{dt},\ \mu \in \overline{\mathbb{C}}_+,\ v = 1, 2, ..., V,$

and
$e_v^-(x,\mu) = e^{-i\alpha(x)-i\mu x} + \int_x^\infty K_v^-(x,t)e^{i\mu t}\mathrm{d}t$, $\mu \in \overline{\mathbb{C}}_-$, $v = 1,2,...,V$ $\alpha(x) := \int_x^\infty |q_v(t)|\,\mathrm{d}t$. Furthermore, $K_v^\pm(x,t)$ are solutions of Volterra type integral equations.

Besides, $K_v^\pm(x,t)$ satisfy

$$\left|K_v^\pm(x,t)\right| \leq \int_{\frac{(x+t)}{2}}^\infty w_v(s)\mathrm{ds},\ v = 1,2,...,V, \tag{2.5}$$

$$\left|\frac{\partial}{\partial x_i}K_v^\pm(x_1,x_2)\right| \leq c\int_{\frac{(x+t)}{2}}^\infty w_v(s)\mathrm{ds} + w_v\left(\frac{x_1+x_2}{2}\right),\ v = 1,2,...,V,\ i = 1,2, \tag{2.6}$$

where $c > 0$ is a constant and $w_v(x) = |q_v(x)|^2 + \left|q_v'(x)\right|$.

Hence, the functions $e_v^\pm(x,\mu)$, $v = 1,2,...,V$ are analytic with respect to μ in $\mathbb{C}_\pm$ where $\mathbb{C}_+ := \{\mu : \mu \in \mathbb{C}, Im\mu > 0\}$, $\mathbb{C}_- := \{\mu : \mu \in \mathbb{C}, Im\mu < 0\}$, consequtively, and are also continuous up to real axis.

The solutions $e_v^\pm(x,\mu)$ also satisfy

$$e_v^\pm(x,\mu) = e^{\pm i[\alpha(x)+\mu x]} + O\left(\frac{e^{\pm x Im\mu}}{|\mu|}\right),\ \mu \in \overline{\mathbb{C}}_\pm,\ |\mu| \to \infty, \tag{2.7}$$

$$\left(e_v^\pm(x,\mu)\right)' = \pm i\left[\mu - Q(x)\right]\cdot e^{\pm i[\alpha(x)+\mu x]} + O(1),\ \mu \in \overline{\mathbb{C}}_\pm,\ |\mu| \to \infty. \tag{2.8}$$

Let $h_v^\pm(x,\mu)$ denote the solutions of (2.1) subject to the conditions

$$\lim_{x\to\infty} e^{\pm i\mu x}\cdot h_v^\pm(x,\mu) = 1,\ \lim_{x\to\infty} e^{\pm i\mu x}\cdot\left(h_v^\pm(x,\mu)\right)' = \mp i\mu,\ \mu \in \overline{\mathbb{C}}_\pm. \tag{2.9}$$

It follows from (2.9) and the definition of z_v that

$$W\left[e_v^\pm(x,\mu),\ h_v^\pm(x,\mu)\right] = \mp 2i\mu,\ \mu \in \overline{\mathbb{C}}_\pm, \tag{2.10}$$

$$W\left[e_v^+(x,\mu),\ e_v^-(x,\mu)\right] = -2i\mu,\ \mu \in \mathbb{R}. \tag{2.11}$$

3. Main results for the spectrum of L

Define

$$E_v^\pm(\mu) = \left(e_v^\pm(0,\mu)\right)' - \left(\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2\right)e_v^\pm(0,\mu) \tag{3.1}$$
$$H_v^\pm(\mu) = \left(h_v^\pm(0,\mu)\right)' - \left(\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2\right)h_v^\pm(0,\mu)$$

and

$$U_v^\pm(t,\mu) = \mp\frac{1}{2i\mu}H_v^\pm(\mu)e_v^\pm(t,\mu)$$

where $E_v^\pm(\mu)$ and $H_v^\pm(\mu)$ are diagonal matrices. Moreover, the Green's function

$$R(x,t;\mu) = \begin{cases} R^+(x,t;\mu),\ \mu \in \mathbb{C}_+, \\ R^-(x,t;\mu),\ \mu \in \mathbb{C}_-, \end{cases} \tag{3.2}$$

of the boundary value problem (2.1)-(2.2) can be calculated using the classical methods where

$$R^\pm(x,t;\mu) = R_1^\pm(x,t;\mu) + R_2^\pm(x,t;\mu). \tag{3.3}$$

Since $\det E_v^\pm(\mu) \neq 0$, we define

$$R_1^{\pm}(x,t;\mu) = -e_v^{\pm}(x,\mu)U_n^{\pm}(t,\mu)\cdot\left(E_v^{\pm}(\mu)\right)^{-1},$$
$$R_2^{\pm}(x,t;\mu) = \mp\begin{cases}\frac{e_v^{\pm}(x,\mu)\cdot U_v^{\pm}(t,\mu)}{2i\mu}, & 0\le t<x,\\ \frac{e_v^{\pm}(t,\mu)\cdot U_v^{\pm}(x,\mu)}{2i\mu}, & x\le t<\infty.\end{cases} \tag{3.4}$$

(2.10) indicates that $e_v^{\pm}$ and $h_v^{\pm}$, from (2.11) e_v^{+} and e_v^{-} are linearly independent. So the functions $\Phi_v^{\pm}(x,\mu)$ and $\Phi_v(x,\mu)$ are defined by

$$\Phi_v^{\pm}(x,\mu) = H_v^{\pm}(\mu)\cdot e_v^{\pm}(x,\mu) - E_v^{\pm}(\mu)\cdot h_v^{\pm}(x,\mu),\ \mu\in\overline{\mathbb{C}}_{\pm}\setminus\{0\}, \tag{3.5}$$

$$\Phi_v(x,\mu) = E_v^{+}(\mu)\cdot e_v^{-}(x,\mu) - E_v^{-}(\mu)\cdot e_v^{+}(x,\mu),\ \mu\in\mathbb{R}^* = \mathbb{R}\setminus\{0\}, \tag{3.6}$$

are solutions of the (2.1)-(2.2).

We designate the set of all eigenvalues and the set of all spectral singularitites of the (2.1)-(2.2) by σ_d and σ_{ss}, consequtively. Taking into account (2.7),(3.2),(3.4)-(3.6), it follows that

$$\sigma_d(L) = \left\{\mu : \mu\in\mathbb{C}_+, \det E_v^{+}(\mu) = 0\right\}\cup\left\{\mu : \mu\in\mathbb{C}_-, \det E_v^{-}(\mu) = 0\right\}, \tag{3.7}$$

$$\sigma_{ss}(L) = \left\{\mu : \mu\in\mathbb{R}^*, \det E_v^{+}(\mu) = 0\right\}\cup\left\{\mu : \mu\in\mathbb{R}^*, \det E_v^{-}(\mu) = 0\right\}, \tag{3.8}$$

where $E_v^{\pm}(\mu) := E_v^{\pm}(0,\mu)$. Since $E_v^{\pm}(\mu)$ are diagonal matrices

$$\det E_v^{\pm}(\mu) = \mathop{\Pi}_{v=1}^{V}\left[e_\nu^{'}(0) - \left(\alpha_0+\alpha_1\mu+\alpha_2\mu^2\right)e_\nu(0)\right]$$

Moreover,

$$\left\{\mu : \mu\in\mathbb{R}^*, \det E_v^{+}(\mu) = 0\right\}\cap\left\{\mu : \mu\in\mathbb{R}^*, \det E_v^{-}(\mu) = 0\right\} = \emptyset.$$

Definition 3.1. We introduce the multiplicity of a root of $\det E_v^{\pm}(v = 1,2,...,V)$ in $\mathbb{C}_{\pm}$ as the multiplicity of the corresponding eigenvalue or spectral singularity of L.

Clearly, (3.1), (3.7), (3.8) indicate that to be able to search for the quantitative features of the spectrum of L, one has to take into consideration the quantitative features of the roots of $\det E_v^{\pm}$, $v = 1,2,...,V$ in the region $\overline{\mathbb{C}}_{\pm}$. Let us define

$$M_1^{\pm} = \left\{\mu : \mu\in\mathbb{C}_{\pm}, \det E_\nu^{\pm}(\mu) = 0\right\},\ M_2^{\pm} = \left\{\mu : \mu\in\mathbb{R}, \det E_\nu^{\pm}(\mu) = 0\right\}.$$

Consequently, we have

$$\sigma_d(L) = M_1^{+}\cup M_1^{-},\ \sigma_{ss}(L) = \left\{M_2^{+}\cup M_2^{-}\right\}\setminus\{0\}.$$

Lemma 3.2. *If the condition (2.3) holds,*

(i) $M_1^{\pm}$ is a bounded set. Moreover, it posseses at most countably many elements. Also, its accumulation points can only belong to a subinterval which is bounded and subset of the real axis,

(ii) $M_2^{\pm}$ is a compact set. $\mu(M_2^{\pm}) = 0$ for which $\mu(M_2^{\pm})$ represents the Lebesgue measure of $M_2^{\pm}$.

Proof. The asymptotic equality

$$e_v^{\pm}(\mu) = e^{\pm i\alpha(0)} + o(1),\ \mu\in\overline{\mathbb{C}}_{\pm},\ |\mu|\to\infty,\ v = 1,2,...,V, \tag{3.9}$$

is obtained from (2.5).

From (3.1), one can show that the sets $M_1^{\pm}$ and $M_2^{\pm}$ are bounded. As a consequence of analicity of $e_v^{\pm}$ in the region $\mathbb{C}_{\pm}$, one may see that the set $M_1^{\pm}$ has at most a countably many of elements. If we make use of the uniqueness of analytic functions, we get that the accumulation points of $M_1^{\pm}$ can only be in a bounded subinterval of the real axis. The closedness and the feature of obtaining zero Lebesgue measure of the set $M_2^{\pm}$ can be seen from the boundary uniqueness theorem of analytic functions. □

The next result can be directly written as a direct consequence of (3.7), (3.8) and Lemma (3.2).

Theorem 3.3. *Let us assume that the condition (2.3) holds, then*

(i) *The set of eigenvalues of L is bounded. It has countably many elements. Further, its accumulation points can only belong to a bounded subinterval of $\mathbb{R}_+$.*

(ii) *The set of spectral singularities of L is bounded and $\mu\left(M_2^{\pm}\right)=0$.*

From now on, let us take into account

$$\lim_{x\to\infty} q_v(x)=0,\ \underset{x\in\mathbb{R}_+}{sup}\left\{\exp(\epsilon\sqrt{x})\left[\left|q_v^{'}(x)\right|\right]\right\}<\infty,\ \epsilon>0,\ v=1,2,...,V. \tag{3.10}$$

It follows from (2.5) and (3.1) that, under the condition (3.10) the functions $E_v^{\pm}, v=1,2,...,V$ are analytic in the region $\mathbb{C}_{\pm}$. Further, whole of its derivatives are continuous in $\overline{\mathbb{C}}_{\pm}$. We obtain that following inequality

$$\left|\frac{\mathrm{d}^r}{\mathrm{d}\mu^r}E_v^{+}(\mu)\right|\le D_r^{+},\ \mu\in\overline{\mathbb{C}}_{+},\ v=1,2,...,V,\ r=0,1,...,$$

and

$$\left|\frac{\mathrm{d}^r}{\mathrm{d}\mu^r}E_v^{-}(\mu)\right|\le F_r^{-},\ \mu\in\overline{\mathbb{C}}_{-},\ v=1,2,...,V,\ r=0,1,...,$$

where

$$D_r^{+}=2^{n+1}c_1\int_0^{\infty}t^r exp\left(-\frac{\epsilon}{2}\sqrt{t}\right)\mathrm{d}t,\ r=0,1,..., \tag{3.11}$$

and

$$F_r^{-}=2^{n+1}c_2\int_0^{\infty}t^r exp\left(-\frac{\epsilon}{2}\sqrt{t}\right)\mathrm{d}t,\ r=0,1,...,$$

$c_1>0$ and $c_2>0$ are constants.

We use the symbolizations to designate the set of all accumulation points of $M_1^{\pm}$ and $M_2^{\pm}$ by $M_3^{\pm}$ and $M_4^{\pm}$, consequtively, and the set of whole roots of $\det E_v^{\pm}$ having infinity multiplicity in $\overline{\mathbb{C}}_{\pm}$ by $M_5^{\pm}$.

Making use of the uniqueness results investigated in [12], we get

$$M_3^{\pm}\subset M_2^{\pm},\ M_4^{\pm}\subset M_2^{\pm},\ \mu(M_5^{\pm})=0.$$

It is a well-known fact that whole derivatives of $F_n^{\pm}$ are continuous on real axis. Therefore, we get

$$M_3^{\pm}\subset M_2^{\pm}. \tag{3.12}$$

Lemma 3.4. *Under the condition (3.10), the set $M_5^{\pm}=\emptyset$.*

Proof. At this stage, we will only show that $M_5^+ = \Phi$. To show $M_5^- = \Phi$, similar steps can be used. If we benefit from properties of the analytic functions in terms of the uniqueness results given in [12], we have

$$\int_0^h lnT(s)d\mu(M_{5,s}^+) > -\infty, \tag{3.13}$$

where $h > 0$ is a constant, $T(s) = \underset{r}{inf} \frac{D_r^+ s^r}{r!}$, the constant D_r^+ is defined by (3.11) and $\mu(M_{5,s}^+)$ stands for the Lebesgue measure of s-neighborhood of $M_{5,s}^+$.

It is easy to derive that

$$D_r^+ \leq Bb^r r^r r!, \tag{3.14}$$

where B and b are constants.

Using (3.14), we obtain

$$T(s) = \underset{r}{inf} \frac{D_{r,s}^+ r}{r!} \leq B\underset{r}{inf} \{b^r s^r r^r\} \leq B \exp\left\{-b^{-1}e^{-1}s^{-1}\right\},$$

or by (3.13)

$$\int_0^h \frac{1}{s}\mu\left(M_{5,s}^+\right) < \infty. \tag{3.15}$$

It is clear that, (3.15) satisfies for an orbitrary s, if and only if $\mu(M_{5,s}^+) = 0$ or $M_5^+ = \Phi$. □

Theorem 3.5. *Suppose that (3.10) holds to be true. Then, L can have only finitely many spectral singularities and eigenvalues. Further, their multiplicities can only have a finite number.*

Proof. Clearly, one needs to verify that the functions $E_v^{\pm}(\mu)$, $v = 1, 2, ..., V$ have a finitely many zeros with finite multiplicities in the region $\overline{\mathbb{C}}_{\pm}$. If we benefit from the Lemma (3.4) and (3.12), it can be written that $M_3^{\pm} = \Phi$. Hence, we see that the bounded sets $M_1^{\pm}$ and $M_2^{\pm}$ do not have no accumulation points. This implies that the functions $E_v^{\pm}(\mu)$, $v = 1, ..., V$ have only a finitely many of zeros in $\overline{\mathbb{C}}_{\pm}$. As a consequence of the fact that $M_5^{\pm} = \Phi$, they must have a finite multiplicity. □

Author contributions. All the co-authors have contributed equally in all aspects of the preparation of this submission.

Conflict of interest statement. The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Funding. This work is not funded.

Data availability. No data was used for the research described in the article.

References

[1] M. Advar and E. Bairamov, *Spectral properties of non-selfadjoint difference operators*, J. Math. Anal. Appl. **261 (2)**, 461-478, 2001.

[2] M. Advar and E. Bairamov, *Difference equations of second order with spectral singularities*, J. Math. Anal. Appl. **277 (2)**, 714-721, 2003.

[3] Z.S. Agranovich and V.A. Marchenko, *The inverse problem of scattering theory*, Gordon and Breach, New York, 1963.

[4] E.K. Arpat, *An eingenfunction expansion of the non-selfadjoint Sturm-Liouville operator with a singular potential*, J. Math. Chem. **51 (8)**, 2196-2213, 2013.

[5] E. Bairamov and A.O. Çelebi, *Spectral properties of the Klein-Gordon s-wave equation with complex potential*, Indian J. Pure Appl. Math. **28 (6)**, 813-824, 1997.

[6] E. Bairamov and A.O. Çelebi, *Spectrum and spectral expansion for the non-selfadjoint discrete Dirac operators*, Q. J. Math., Oxf. Second Ser. **50 (200)**, 371-384, 1999.

[7] E. Bairamov, Ö. Çakar and A.M. Krall, *An eigenfunction expansion for a quadratic pencil of a Schrödinger operator with spectral singularities*, J. Dier. Equ. **151 (2)**, 268-289, 1999.

[8] E. Bairamov, Ö. Çakar and A.M. Krall, *Non-selfadjoint difference operators and Jacobi matrices with spectral singularities*, Math. Nachr. **229**, 5-14, 2001.

[9] E. Bairamov, Ö. Çakar and A.M. Krall, *Spectral properties including spectral singularities of a quadratic pencil of Schrödinger operators on the whole real axis*, Quaest. Math. **26 (1)**, 15-30, 2003.

[10] E. Bairamov, Ö. Çakar and C. Yank, *Spectral singularities of the Klein-Gordon s-wave equation*, Indian J. Pure Appl. Math. **32 (6)**, 851-857, 2001.

[11] E. Bairamov and Ö. Karaman, *Spectral singularities of the Klein-Gordon s-wave equations with integral boundary conditions*, Acta Math. Hungar. **97 (1-2)**, 121-131, 2002.

[12] E.P. Dolzhenko, *Boundary value uniqueness theorems for analytic functions*, Math. Notes **25**, 437-442, 1979.

[13] E. Guariglia and S. Silvestrov, *Fractional-Wavelet Analysis of Positive definite Distributions and Wavelets on D'(C)*, Engrg. Math. II. Springer Proc. Math. Stat. **179**, 337-353, 2016.

[14] A. Hussain, S. Haq and M. Uddin, *Numerical solution of Klein–Gordon and sine-Gordon equations by meshless method of lines*, Eng. Anal. Bound. Elem. **37 (11)**, 1351-1366, 2013.

[15] A.M. Krall, E. Bairamov and Ö. Çakar, *Spectrum and spectral singularities of a quadratic pencil of a Schrödinger operator with a general boundary condition*, J. Dier. Equ. **151 (2)**, 252-267, 1999.

[16] V.E. Lyance, *A differential operator with spectral singularities I*, Trans. Amer. Math. Soc. Ser. 2 **60**, 185-225, 1967.

[17] V.E. Lyance, *A differential operator with spectral singularities II*, Trans. Amer. Math. Soc. Ser. 2 **60**, 227-283, 1967.

[18] F.G. Maksudov and B.P. Allakhverdiev, *Spectral analysis of a new class of non-selfadjoint operators with continuous and point spectrum*, Sov. Math. Dokl. **30**, 566-569, 1984.

[19] M.A. Naimark, *Investigation of the spectrum and the expansion in eigenfunctions of a non-selfadjoint operator of the second order on a semi-axis*, Trans. Amer. Math. Soc. Ser. 2 **16**, 103-193, 1960.

[20] M.A. Ragusa, *Elliptic boundary value problem in vanishing mean oscillation hypothesis*, Comment. Math. Univ. Carol. **40**, 651-663, 1999.

[21] M.A. Ragusa and A. Tachikawa, *Boundary regularity of minimizers of p (x)-energy functionals*, Ann. Inst. Henri Poincare (C) Anal. Non Linéaire **33 (2)**, 451-476, 2016.

[22] J.T. Schwartz, *Some nonselfadjoint operators*, Comm. Pure Appl. Math. **13**, 609-639, 1960.

[23] G.B. Tunca, *Spectral expansion of a non-selfadjoint differential operator on the whole axis*, J. Math. Anal. Appl. **252 (1)**, 278-297, 2000.