INTERNATIONAL ELECTRONIC JOURNAL OF GEOMETRY
VOLUME 17 NO. 2 PAGE 437–446 (2024)
DOI: HTTPS://DOI.ORG/10.36890/IEJG.1393446

Screen Pseudo-Slant Lightlike Submersions from Indefinite Sasakian Manifolds onto Lightlike Manifolds

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(Communicated by Murat Tosun)

ABSTRACT

As a generalization of screen slant lightlike submersions, we introduce the notion of screen pseudo-slant lightlike submersions from indefinite Sasakian manifolds onto lightlike manifolds. We give examples and prove a characterization theorem for the existence of such lightlike submersions. We also obtain integrability conditions of distributions involved in the definition of this class of lightlike submersions. Further, we find necessary and sufficient conditions for foliations determined by these distributions to be totally geodesic.

Keywords: *Lightlike manifold, indefinite Sasakian manifold, screen pseudo-slant lightlike submersion.*
AMS Subject Classification (2020): *Primary: 53C15 ; 53C20 ; 53C50 ; Secondary: 53D15.*

1. Introduction

In [10], O'Neill initiated the study of Riemannian submersions and Gray [4] further continued it. Let $f : (M_1, g_1) \rightarrow (M_2, g_2)$ be a smooth map, where (M_1, g_1) and (M_2, g_2) are Riemannian manifolds. Then f is called a Riemannian submersion if f_* preserves the lengths of horizontal vectors and f has maximal rank. It is well-known that the fibers of Riemannian submersions are Riemannian submanifolds. However, the fibers of submersions between semi-Riemannian manifolds may not be semi-Riemannian submanifolds because the induced metric on fibers may also be degenerate. Therefore, O'Neill [11] introduced semi-Riemannian submersions between semi-Riemannian manifolds. Recently the geometry of some new pseudo-Riemannian submersions have been studied that can be found in [5, 6, 8, 9]. Moreover, the notion of screen lightlike submersions from lightlike manifolds onto semi-Riemannian manifolds was defined and studied by Şahin [14]. After this, Şahin and Gündüzalp [15] introduced lightlike submersions between semi-Riemannian manifolds and lightlike manifolds. They also defined O'Neill tensors for such submersions and obtained interesting results. Following this work, several geometer studied these submersions (see [7, 12, 13, 17, 18, 19, 20] and references there in). In [16], Shukla and Yadav introduced screen pseudo-slant lightlike submanifolds of indefinite Sasakian manifold. The above theories motivated us to study a new class of lightlike submersions. In the present paper, we define the notion of screen pseudo-slant lightlike submersions from indefinite Sasakian manifolds onto lightlike manifolds.

The paper is organized as follows. In Section 2, we collect basic formulae and definitions as needed for this paper. In Section 3, we study screen pseudo-slant lightlike submersions from indefinite Sasakian manifolds onto lightlike manifolds, giving two examples. In section 4, we research foliations determined by distributions on a fiber of screen pseudo-slant lightlike submersions.

Received : *20-10-2023,* ***Accepted :*** *07-09-2024*
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2. Preliminaries

In this section, we recall several definitions and formulae which will be required throughout the paper.
A smooth manifold M of dimension 2m+1 is said to have an almost contact structure (ϕ, ξ, η) if it carries a (1, 1) tensor field ϕ, a vector field ξ called characteristic vector field and a 1-form η on M, satisfying

$$\phi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1, \qquad \phi\xi = 0, \qquad \eta \circ \phi = 0, \tag{2.1}$$

where I denotes the identity tensor. Further, if there exists a semi-Riemannian metric g on M satisfying

$$g(\phi X, \phi Y) = g(X, Y) - \epsilon\eta(X)\eta(Y), \qquad \forall X, Y \in \Gamma(TM), \tag{2.2}$$

then (ϕ, ξ, η, g) is called an (ϵ)-almost contact metric structure on M [21, 2], where ϵ= 1 or -1 according as ξ is spacelike or timelike. From (2.2) it follows that

$$g(\xi, \xi) = \epsilon, \qquad \eta(X) = \epsilon g(X, \xi), \qquad g(X, \phi Y) + g(\phi X, Y) = 0. \tag{2.3}$$

An (ϵ)-almost contact metric structure (ϕ, ξ, η, g) on M is an indefinite Sasakian structure if and only if

$$(\nabla_X \phi)Y = g(X, Y)\xi - \epsilon\eta(Y)X, \tag{2.4}$$

for all $X, Y \in \Gamma(TM)$, where ∇ denotes the Riemannian connection of g [[2], Theorem 7.1.6].
If a semi-Riemannian manifold M admits an indefinite Sasakian structure (ϕ, ξ, η, g), then (M, ϕ, ξ, η, g) is called an indefinite Sasakian manifold. Setting $Y = \xi$ in (2.4), we get

$$\nabla_X \xi = -\epsilon\phi X, \qquad \forall X \in \Gamma(TM). \tag{2.5}$$

In this paper, we assume $\epsilon = 1$, i.e., the characteristic vector field ξ is spacelike.

Example 2.1. [3] Let $\mathbb{R}^{2m+1}_{2q}$ denote the manifold equipped with a semi-Riemannian metric g and its usual contact form $\eta = \frac{1}{2}\left(dz - \sum_{i=1}^{m} y_i dx_i\right)$.

The characteristics vector field ξ is given by $2\frac{\partial}{\partial z}$ and its semi-Riemannian metric g and tensor field ϕ are given by

$$g = \eta \otimes \eta + \frac{1}{4}\left(-\sum_{i=1}^{q} dx_i \otimes dx_i + dy_i \otimes dy_i + \sum_{i=q+1}^{m} dx_i \otimes dx_i + dy_i \otimes dy_i\right),$$

$$\phi\left(\sum_{i=1}^{m}\left(X_i\frac{\partial}{\partial x_i} + Y_i\frac{\partial}{\partial y_i}\right) + Z\frac{\partial}{\partial z}\right) = \sum_{i=1}^{m}\left(Y_i\frac{\partial}{\partial x_i} - X_i\frac{\partial}{\partial y_i}\right) + \sum_{i=1}^{m} Y_i y_i\frac{\partial}{\partial z},$$

where (x_i, y_i, z) $(i = 1, 2, ..., m)$ are the Cartesian coordinates on $\mathbb{R}^{2m+1}_{2q}$. This gives a contact metric structure on $\mathbb{R}^{2m+1}$. The vector fields E_i=$2\frac{\partial}{\partial y_i}$, $E_{m+i} = 2\left(\frac{\partial}{\partial x_i} + y_i\frac{\partial}{\partial z}\right)$ and ξ form a ϕ-basis for the contact metric structure. Now, it can be proved that $(\mathbb{R}^{2m+1}_{2q}, \phi, \xi, \eta, g)$ is an indefinite Sasakian manifold.

Let (M, g) be a real m-dimensional smooth semi-Riemannian manifold. Then the radical subspace $Rad\,T_pM$ of T_pM is defined by $Rad\,T_pM = \{V \in T_pM : g(V, X) = 0, X \in T_pM\}$. Suppose dim($Rad\,T_pM$) $= r$, then the mapping $Rad\,TM : p \in M \to Rad\,T_pM$ is said to be the radical distribution of rank r on M. The manifold M is said to be an r-lightlike manifold [1] if $r > 0$.
Let $f : (M_1, g_1) \to (M_2, g_2)$ be a smooth submersion between a semi-Riemannian manifold M_1 and an r-lightlike manifold M_2 then $f^{-1}(x)$ is a submanifold (called fiber) of dimension dim M_1 - dim M_2. Further the kernel of f_* at $p \in M_1$ and its orthogonal complement are given by $Kerf_{*p} = \{X \in T_pM_1 : f_{*p}X = 0\}$ and $(Kerf_{*p})^{\perp} = \{Y \in T_pM_1 : g_1(Y, X) = 0, X \in Kerf_{*p}\}$, respectively. Since T_pM_1 is a semi-Riemannian vector space, $Kerf_*$ may not be complementary to $(Kerf_*)^{\perp}$. Hence we assume $\Delta_p = Kerf_{*p} \cap (Kerf_{*p})^{\perp} \neq \{0\}$. Then Δ and $Kerf_*$ are radical and lightlike distributions on M_1, respectively. In this case, there exists a complementary distribution to Δ in $Kerf_*$ which is non-degenerate and we denote it by $S(Kerf_*)$. Thus we have

$$Kerf_* = \Delta \perp S(kerf_*). \tag{2.6}$$

Using the above argument again for $(Kerf_*)^\perp$, we get

$$(Kerf_*)^\perp = \Delta \perp S(Kerf_*)^\perp.$$

As $S(Kerf_*)$ is non-degenerate in TM_1, we obtain

$$TM_1 = S(Kerf_*) \perp (S(Kerf_*))^\perp,$$

where $(S(Kerf_*))^\perp$ is the complementary distribution to $S(Kerf_*)$ in TM_1. Note that $S(Kerf_*)^\perp$ is a non-degenerate distribution in $(S(Kerf_*))^\perp$, we get

$$(S(Kerf_*))^\perp = S(Kerf_*)^\perp \perp (S(Kerf_*)^\perp)^\perp.$$

Since for any local basis $\{V_i\}$ of Δ, there exists a local null frame $\{N_i\}$ of sections with values in the orthogonal complement of $S(Kerf_*)^\perp$ in $(S(Kerf_*))^\perp$ such that $g_1(V_i, N_j) = \delta_{ij}$ and $g_1(N_i, N_j) = 0$. The distribution spanned by $N_1, N_2, ..., N_r$ is called lightlike transversal distribution and we denote it by $ltr(Kerf_*)$([1], page 144). Consider following vector bundle

$$tr(Kerf_*) = ltr(Kerf_*) \perp S(Kerf_*)^\perp, \tag{2.7}$$

which is complementary (but not orthogonal) vector bundle to $Kerf_*$ in $TM_{1|f^{-1}(x)}$. Therefore, we have

$$TM_{1|f^{-1}(x)} = Kerf_* \oplus tr(Kerf_*). \tag{2.8}$$

It should be noted that $ltr(Kerf_*)$ and $Kerf_*$ are not orthogonal to each other. Using (2.6), (2.7) and (2.8) we get

$$TM_{1|f^{-1}(x)} = S(Kerf_*) \perp [\Delta \oplus ltr(Kerf_*)] \perp S(Kerf_*)^\perp.$$

If we denote $\mathcal{V} = Kerf_*$, the vertical space of T_pM_1 and $\mathcal{H} = tr(Kerf_*)$, the horizontal space then we have

$$TM_1 = \mathcal{H} \oplus \mathcal{V}.$$

Moreover, we note that every $\mathcal{V}_p$ coincides with the tangent space of $f^{-1}(x)$ at p, $f(p) = x$, that is, $\mathcal{V}_p = T_pf^{-1}(x)$.

Definition 2.1. [15] Let (M_1, g_1) be a semi-Riemannian manifold and (M_2, g_2) be an r-lightlike manifold. A submersion $f : M_1 \to M_2$ is called an r-lightlike submersion if

(a) dim $\Delta = dim\{(Kerf_*) \cap (Kerf_*)^\perp\} = r$, $0 < r < min\{dim(Kerf_*), dim(Kerf_*)^\perp\}$.

(b) f_* preserves lengths of horizontal vectors, i.e., $g_1(X, Y) = g_2(f_*X, f_*Y)$ for X, Y $\in \Gamma(\mathcal{H})$.

Now, we have following three particular cases:

(i) If dim $\Delta = dim(Kerf_*) < dim(Kerf_*)^\perp$ then $\mathcal{V} = \Delta$ and $\mathcal{H} = S(Kerf_*)^\perp \perp ltr(Kerf_*)$ and f is called an isotropic submersion.

(ii) If dim $\Delta = dim(Kerf_*)^\perp < dim(Kerf_*)$ then $\mathcal{V} = S(Kerf_*) \perp \Delta$ and $\mathcal{H} = ltr(Kerf_*)$ and f is called a co-isotropic submersion.

(iii) If dim $\Delta = dim(Kerf_*)^\perp = dim(Kerf_*)$ then $\mathcal{V} = \Delta$ and $\mathcal{H} = ltr(Kerf_*)$ and f is called a totally lightlike submersion.

As we know, the geometry of Riemannian submersions is characterized by O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$. Therefore Şahin and Gündüzalp [15] defined these tensors for a lightlike submersion as

$$\mathcal{T}_XY = h\nabla_{\nu X}\nu Y + \nu\nabla_{\nu X}hY, \quad \mathcal{A}_XY = \nu\nabla_{hX}hY + h\nabla_{hX}\nu Y, \tag{2.9}$$

where $h : TM_1 \to \mathcal{H}$ and $\nu : TM_1 \to \mathcal{V}$ denote natural projections and ∇ be the Levi-Civita connection of g_1. We note that $\mathcal{T}$ and $\mathcal{A}$ are skew-symmetric in Riemannian submersions but not in lightlike submersions because the horizontal and vertical subspaces are not orthogonal to each other. $\mathcal{T}$ and $\mathcal{A}$ are vertical and horizontal, respectively and both reverses the horizontal and vertical subspaces. Moreover $\mathcal{T}$ has the symmetry property for vertical vector fields U and V, that is, $\mathcal{T}_UV = \mathcal{T}_VU$.

Let f be an r-lightlike submersion from an (m+n)-dimensional semi-Riemannian manifold (M_1, g_1) onto an n-dimensional lightlike manifold (M_2, g_2), where $m, n > 1$. Then $Kerf_*$ is an m-dimensional lightlike distribution on fiber $f^{-1}(x)$. We denote the induced metric on $f^{-1}(x)$ by $\hat{g}$. Now by using (2.9) for any $U, V \in \Gamma(Kerf_*)$ and $X \in \Gamma(tr(Kerf_*))$, we get

$$\nabla_U V = \hat{\nabla}_U V + \mathcal{T}_U V, \tag{2.10}$$

$$\nabla_U X = \mathcal{T}_U X + \nabla^t_U X, \tag{2.11}$$

where $\hat{\nabla}_U V = \nu \nabla_U V$ and $\nabla^t_U X = h \nabla_U X$. Also $\{\hat{\nabla}_U V, \mathcal{T}_U X\}$ and $\{\mathcal{T}_U V, \nabla^t_U X\}$ belongs to $\Gamma(Kerf_*)$ and $\Gamma(tr(Kerf_*))$, respectively. Let $S(Kerf_*)^\perp \neq 0$. Consider projections L and S of $tr(Kerf_*)$ on $ltr(Kerf_*)$ and $S(Kerf_*)^\perp$, respectively. Then from (2.10) and (2.11), we obtain (for details see [18])

$$\nabla_U V = \hat{\nabla}_U V + \mathcal{T}^l_U V + \mathcal{T}^s_U V, \tag{2.12}$$

$$\nabla_U N = \mathcal{T}_U N + \nabla^l_U N + D^s(U, N), \tag{2.13}$$

$$\nabla_U W = \mathcal{T}_U W + D^l(U, W) + \nabla^s_U W, \tag{2.14}$$

for any $U, V \in \Gamma(Kerf_*)$, $N \in \Gamma(ltr(Kerf_*))$ and $W \in \Gamma(S(Kerf_*)^\perp)$. Here $\mathcal{T}^l_U V = L(\mathcal{T}_U V)$, $\mathcal{T}^s_U V = S(\mathcal{T}_U V)$ and ∇^l, ∇^s are linear connections on $ltr(Kerf_*)$ and $S(Kerf_*)^\perp$, respectively. Also, D^l and D^s are $\Gamma(ltr(Kerf_*))$ and $\Gamma(S(Kerf_*)^\perp)$-valued bilinear forms, respectively.

Let f be either r-lightlike or co-isotropic submersion and σ denotes the projection of $Kerf_*$ on $S(Kerf_*)$ then for any $X, Y \in \Gamma(Kerf_*)$ and $V \in \Gamma(\Delta)$, we obtain

$$\hat{\nabla}_X \sigma Y = \nabla^*_X \sigma Y + \mathcal{T}^*_X \sigma Y, \quad \hat{\nabla}_X V = \mathcal{T}^*_X V + \nabla^{*t}_X V, \tag{2.15}$$

where $\{\nabla^*_X \sigma Y, \mathcal{T}^*_X V\}$ and $\{\mathcal{T}^*_X \sigma Y, \nabla^{*t}_X V\}$ belongs to $\Gamma(S(Kerf_*))$ and $\Gamma(\Delta)$, respectively. Here ∇^* and ∇^{*t} are linear connections on $S(Kerf_*)$ and Δ, respectively.

3. Screen Pseudo-slant Lightlike Submersions

In this section, we introduce screen pseudo-slant lightlike submersions from indefinite Sasakian manifolds onto lightlike manifolds such that the characteristic vector field ξ is tangent to fiber. At first, we state the following lemma which will be useful to define the slant notion on the screen distribution.

Lemma 3.1. *[17] Let f be a 2r-lightlike submersion from an indefinite Sasakian manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a lightlike manifold (M_2, g_2). Then the screen distribution $S(Kerf_*)$ on $f^{-1}(x)$ is Riemannian, where $2r < dim(f^{-1}(x))$.*

Definition 3.1. Let f be a 2r-lightlike submersion from an indefinite Sasakian manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a lightlike manifold (M_2, g_2) such that $2r < dim(Kerf_*)$ with the characteristic vector field ξ tangent to $f^{-1}(x)$, i.e., ξ belongs to $S(Kerf_*)$. Then we say that f is a screen pseudo-slant lightlike submersion if following four conditions are satisfied:

(i) Δ is invariant with respect to ϕ, i.e., $\phi(\Delta) = \Delta$,

(ii) there exist orthogonal distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ on $f^{-1}(x)$ such that $S(Kerf_*) = \mathcal{D}_1 \perp \mathcal{D}_2 \perp \langle \xi \rangle$,

(iii) the distribution $\mathcal{D}_1$ is anti-invariant, i.e., $\phi(\mathcal{D}_1) \subset S(Kerf_*)^\perp$,

(iv) for every $p \in f^{-1}(x)$ and every non-zero vector $U \in (\mathcal{D}_2)_p$, the angle θ between ϕU and the vector subspace $(\mathcal{D}_2)_p$ is a constant($\neq \pi/2$), that is, it is independent of choice of $p \in f^{-1}(x)$ and $U \in (\mathcal{D}_2)_p$.

A screen pseudo-slant lightlike submersion is said to be proper if $\mathcal{D}_1 \neq \{0\}$, $\mathcal{D}_2 \neq \{0\}$ and $\theta \neq 0$. From Definition 3.1, we get

$$Kerf_* = \Delta \perp \mathcal{D}_1 \perp \mathcal{D}_2 \perp \langle \xi \rangle .$$

Then, we have following particular cases:

(i) If $\mathcal{D}_1 = 0$, then f is called a screen slant lightlike submersion, as studied in [17].

(ii) If $\mathcal{D}_2 = 0$, then f is called a screen real lightlike submersion.

(iii) If $\mathcal{D}_1 = 0$ and $\theta = 0$, then f is called an invariant lightlike submersion.

(iv) If $\mathcal{D}_1 \neq 0$ and $\theta = 0$, then f is called a contact SCR-lightlike submersion.

Thus, the above new class of lightlike submersions includes screen slant, screen real, invariant and contact screen Cauchy-Riemann lightlike submersions as its particular cases.

Now, we construct two examples of proper screen pseudo-slant lightlike submersions from an indefinite Sasakian manifold onto a lightlike manifold.

Example 3.1. Consider an indefinite Sasakian manifold as given in Example 2.1 for $m = 6$ and q=1, i.e., $(\mathbb{R}^{13}_2, \phi, \xi, \eta, g_1)$. Let $(\mathbb{R}^6, g_2)$ be a lightlike manifold, where $g_2 = \frac{1}{8}\{(da_2)^2 + (da_3)^2 + (da_5)^2 + 2(da_6)^2\}$ and $a_1, ..., a_6$ are usual coordinates on $\mathbb{R}^6$. Define a map $f : \mathbb{R}^{13}_2 \to \mathbb{R}^6$ by

$$f(x_1, ..., x_6, y_1, ..., y_6, z) = (x_1 - x_2, x_3 - x_4, x_5 - x_6, y_1 - y_2, y_3 + y_4, y_6).$$

Then we have $Kerf_* = Span\{V_1 = E_7 + E_8, V_2 = E_9 + E_{10}, V_3 = E_{11} + E_{12}, V_4 = E_1 + E_2, V_5 = E_3 - E_4, V_6 = E_5, V_7 = E_{13} = \xi\}$ and $(Kerf_*)^\perp = Span\{V_1, V_4, W_1 = E_9 - E_{10}, W_2 = E_{11} - E_{12}, W_3 = E_3 + E_4, W_4 = E_6\}$. Further we get $\Delta = Kerf_* \cap (Kerf_*)^\perp = Span\{V_1, V_4\}$. Then, we obtain $ltr(Kerf_*) = Span\{N_1 = -\frac{1}{2}(E_7 - E_8), N_2 = -\frac{1}{2}(E_1 - E_2)\}$. Thus f is a 2-lightlike submersion. Furthermore we see that $\phi(V_4) = V_1$, which implies Δ is invariant with respect to ϕ. Again we obtain three mutually orthogonal distributions $\mathcal{D}_1$, $\mathcal{D}_2$ and $\langle\xi\rangle$ such that $S(Kerf_*) = \mathcal{D}_1 \perp \mathcal{D}_2 \perp \langle\xi\rangle$, where $\mathcal{D}_1 = Span\{V_2, V_5\}$ and $\mathcal{D}_2 = Span\{V_3, V_6\}$. It is easy to see that $\mathcal{D}_1$ is anti-invariant while $\mathcal{D}_2$ is slant distribution with slant angle $\theta = \frac{\pi}{4}$. Therefore f is a proper screen pseudo-slant 2-lightlike submersion.

Example 3.2. Consider an indefinite Sasakian manifold as given in Example 2.1 for $m = 7$ and $q = 1$, i.e., $(\mathbb{R}^{15}_2, \phi, \xi, \eta, g_1)$. Let $(\mathbb{R}^8, g_2)$ be a lightlike manifold, where $g_2 = \frac{1}{8}\{(da_2)^2 + 2(da_3)^2 + 2(da_4)^2 + (da_6)^2 + 2(da_7)^2 + 2(da_8)^2\}$ and $a_1, ..., a_8$ are usual coordinates on $\mathbb{R}^8$. Define a map $f : \mathbb{R}^{15}_2 \to \mathbb{R}^8$ by

$$f(x_1, ..., x_7, y_1, ..., y_7, z) = (x_1 + x_3, x_2 + x_5, x_4 \cos\alpha + x_6 \sin\alpha, x_7, y_1 + y_3, y_2 - y_5, y_4, y_7),$$

where $\alpha \in (0, \frac{\pi}{2})$. Then we get $Kerf_* = Span\{V_1 = E_8 - E_{10}, V_2 = E_9 - E_{12}, V_3 = -E_{11}\sin\alpha + E_{13}\cos\alpha, V_4 = E_1 - E_3, V_5 = E_2 + E_5, V_6 = E_6, V_7 = \xi = E_{15}\}$ and $(Kerf_*)^\perp = Span\{V_1, V_4, W_1 = E_9 + E_{12}, W_2 = E_{11}\cos\alpha + E_{13}\sin\alpha, W_3 = E_{14}, W_4 = E_2 - E_5, W_5 = E_4, W_6 = E_7\}$. Further we get $\Delta = Kerf_* \cap (Kerf_*)^\perp = Span\{V_1, V_4\}$. Then, we obtain $ltr(Kerf_*) = Span\{N_1 = -\frac{1}{2}(E_8 + E_{10}), N_2 = -\frac{1}{2}(E_1 + E_3)\}$. Hence f is a 2-lightlike submersion. Moreover we see that $\phi(V_4) = V_1$, which gives Δ is invariant with respect to ϕ. Again we have three mutually orthogonal distributions $\mathcal{D}_1, \mathcal{D}_2$ and $\langle\xi\rangle$ such that $S(KerF_*) = \mathcal{D}_1 \perp \mathcal{D}_2 \perp \langle\xi\rangle$, where $\mathcal{D}_1 = Span\{V_2, V_5\}$ and $\mathcal{D}_2 = Span\{V_3, V_6\}$. By a simple calculation we see that $\mathcal{D}_1$ is anti-invariant distribution while $\mathcal{D}_2$ is a slant distribution with slant angle $\theta = \alpha$. Therefore f is a proper screen pseudo-slant 2-lightlike submersion.

Let f be a screen pseudo-slant lightlike submersion from an indefinite Sasakian manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a lightlike manifold (M_2, g_2). Then for $U \in \Gamma(Kerf_*)$, we can write $\phi(U) = \mathcal{P}U + FU$, where $\mathcal{P}U$ and FU are tangential and transversal components of ϕU, respectively. Suppose $\mathcal{P}_1, \mathcal{P}_2$ and $\mathcal{P}_3$ denote projections of $Kerf_*$ on Δ, $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively. Also, we denote projections of $tr(Kerf_*)$ on $ltr(Kerf_*)$, $\phi\mathcal{D}_1$ and $\mathcal{D}'$ by Q_1, Q_2 and Q_3, respectively, where $\mathcal{D}'$ is non-degenerate orthogonal complementary subbundle of $\phi\mathcal{D}_1$ in $S(Kerf_*)^\perp$. Thus for $U \in \Gamma(Kerf_*)$, we have

$$U = \mathcal{P}_1U + \mathcal{P}_2U + \mathcal{P}_3U + \eta(U)\xi. \tag{3.1}$$

On applying ϕ to (3.1), we get $\phi U = \phi\mathcal{P}_1U + \phi\mathcal{P}_2U + \phi\mathcal{P}_3U$ which gives,

$$\phi U = \phi\mathcal{P}_1U + \phi\mathcal{P}_2U + \tau\mathcal{P}_3U + \omega\mathcal{P}_3U, \tag{3.2}$$

where $\tau\mathcal{P}_3U$ and $\omega\mathcal{P}_3U$ denote tangential and transversal components of $\phi\mathcal{P}_3U$, respectively. Therefore from (3.2) and Definition 3.1, we get $\phi\mathcal{P}_1U \in \Gamma(\Delta)$, $\phi\mathcal{P}_2U \in \Gamma(\phi\mathcal{D}_1) \subset \Gamma(S(Kerf_*)^\perp)$, $\tau\mathcal{P}_3U \in \Gamma(\mathcal{D}_2)$ and $\omega\mathcal{P}_3U \in \Gamma(\mathcal{D})'$. Moreover for $W \in \Gamma(tr(Kerf_*))$, we have

$$W = Q_1W + Q_2W + Q_3W. \tag{3.3}$$

Applying ϕ to (3.3), we get $\phi W = \phi Q_1W + \phi Q_2W + \phi Q_3W$ which gives,

$$\phi W = \phi Q_1W + \phi Q_2W + \mathcal{B}Q_3W + \mathcal{C}Q_3W, \tag{3.4}$$

where $\mathcal{B}Q_3W$ and $\mathcal{C}Q_3W$ denote tangential and transversal components of ϕQ_3W, respectively. Thus we have $\phi Q_1W \in \Gamma(ltr(Kerf_*))$, $\phi Q_2W \in \Gamma(\mathcal{D}_1)$, $\mathcal{B}Q_3W \in \Gamma(\mathcal{D}_2)$ and $\mathcal{C}Q_3W \in \Gamma(\mathcal{D})'$.

Now, from (2.4), (3.2), (2.12), (2.14), (3.4) and (2.15) and identifying the components on Δ, $\mathcal{D}_1$, $\mathcal{D}_2$, $ltr(Kerf_*)$, $\phi(\mathcal{D}_1)$, $\mathcal{D}'$ and $\langle\xi\rangle$ respectively, we obtain

$$\nabla^{*t}_U \phi\mathcal{P}_1V + \mathcal{P}_1(\hat{\nabla}_U \tau\mathcal{P}_3V) + \mathcal{P}_1(\mathcal{T}_U \omega\mathcal{P}_3V) = -\mathcal{P}_1(\mathcal{T}_U \phi\mathcal{P}_2V) + \phi\mathcal{P}_1\hat{\nabla}_U V - \eta(V)\mathcal{P}_1(U), \tag{3.5}$$

$$\begin{aligned}\mathcal{P}_2(\mathcal{T}^*_U\phi\mathcal{P}_1V) + \mathcal{P}_2(\mathcal{T}_U\phi\mathcal{P}_2V) + \mathcal{P}_2(\mathcal{T}_U\omega\mathcal{P}_3V) &= -\mathcal{P}_2(\hat{\nabla}_U\tau\mathcal{P}_3V) + \phi Q_2\mathcal{T}^s_UV - \eta(V)\mathcal{P}_2U,\\ \mathcal{P}_3(\mathcal{T}^*_U\phi\mathcal{P}_1V) + \mathcal{P}_3(\mathcal{T}_U\phi\mathcal{P}_2V) + \mathcal{P}_3(\mathcal{T}_U\omega\mathcal{P}_3V) + \mathcal{P}_3(\hat{\nabla}_U\tau\mathcal{P}_3V) &= \tau\mathcal{P}_3\hat{\nabla}_UV + \mathcal{B}Q_3\mathcal{T}^s_UV - \eta(V)\mathcal{P}_3U,\\ \mathcal{T}^l_U\phi\mathcal{P}_1V + D^l(U,\phi\mathcal{P}_2V) + \mathcal{T}^l_U\tau\mathcal{P}_3V + D^l(U,\omega\mathcal{P}_3V) &= \phi(\mathcal{T}^l_UV),\end{aligned} \tag{3.6}$$

$$Q_2\nabla^s_U\phi\mathcal{P}_2V + Q_2\nabla^s_U\omega\mathcal{P}_3V = \phi\mathcal{P}_2\hat{\nabla}_UV - Q_2\mathcal{T}^s_U\phi\mathcal{P}_1V - Q_2\mathcal{T}^s_U\tau\mathcal{P}_3V, \tag{3.7}$$

$$Q_3\nabla^s_U\phi\mathcal{P}_2V + Q_3\nabla^s_U\omega\mathcal{P}_3V - \omega\mathcal{P}_3\hat{\nabla}_UV = CQ_3\mathcal{T}^s_UV - Q_3\mathcal{T}^s_U\tau\mathcal{P}_3V - Q_3\mathcal{T}^s_U\phi\mathcal{P}_1V, \tag{3.8}$$

$$\eta(\hat{\nabla}_U\tau\mathcal{P}_3V) + \eta(\mathcal{T}^*_U\phi\mathcal{P}_1V) + \eta(\mathcal{T}_U\phi\mathcal{P}_2V) + \eta(\mathcal{T}_U\omega\mathcal{P}_3V) = g_1(U,V) - \eta(U)\eta(V).$$

Theorem 3.1. *Let f be a 2r-lightlike submersion from an indefinite Sasakian manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a lightlike manifold (M_2, g_2). Then f is a screen pseudo-slant lightlike submersion if and only if following two conditions are satisfied:*

(i) $ltr(Kerf_*)$ is invariant and $\mathcal{D}_1$ is anti-invariant with respect to ϕ,

(ii) there exists a constant $\lambda \in (0, 1]$ such that $(\mathcal{P}_3 \circ \tau)^2U = -\lambda U$, for $U \in \Gamma(\mathcal{D}_2)$. Here $\mathcal{D}_1$ and $\mathcal{D}_2$ are orthogonal distributions on $f^{-1}(x)$ such that $S(Kerf_*) = \mathcal{D}_1 \perp \mathcal{D}_2 \perp \langle\xi\rangle$.

Proof. Let $f : (M_1, \phi, \xi, \eta, g_1) \to (M_2, g_2)$ be a screen pseudo-slant lightlike submersion. Then $\mathcal{D}_1$ is anti-invariant and Δ is invariant with respect to ϕ. Now, from (2.3) and (3.2), we get $g_1(\phi N, U) = -g_1(N, \phi U) = -g_1(N, \phi\mathcal{P}_2U + \tau\mathcal{P}_3U + \omega\mathcal{P}_3U) = 0$, for $N \in \Gamma(ltr(Kerf_*))$ and $U \in \Gamma(S(Kerf_*) - \langle\xi\rangle)$. Hence ϕN does not belong to $\Gamma(S(Kerf_*) - \langle\xi\rangle)$. For $N \in \Gamma(ltr(Kerf_*))$ and $W \in \Gamma(S(Kerf_*)^\perp)$, using (2.3) and (3.4), we obtain $g_1(\phi N, W) = -g_1(N, \phi W) = -g_1(N, \phi Q_2W + \mathcal{B}Q_3W + \mathcal{C}Q_3W) = 0$. From which it follows that ϕN does not belong to $\Gamma(S(Kerf_*)^\perp)$. Now if $\phi N \in \Gamma(\Delta)$, then from (2.1), we get $\phi(\phi N) = \phi^2N = -N + \eta(N)\xi = -N \in \Gamma(ltr(Kerf_*))$, which contradicts that Δ is invariant with respect to ϕ. Therefore $ltr(Kerf_*)$ is invariant with respect to ϕ. To prove (ii) of Theorem 3.1, we have

$$\cos\theta = \frac{g_1(\phi U, \tau\mathcal{P}_3U)}{|\phi U||\tau\mathcal{P}_3U|} = -\frac{g_1(U, (\mathcal{P}_3 \circ \tau)^2U)}{|\phi U||\tau\mathcal{P}_3U|}, \tag{3.9}$$

for $U \in \Gamma(\mathcal{D}_2)$. We also have for $U \in \Gamma(\mathcal{D}_2)$

$$\cos\theta = \frac{|\tau\mathcal{P}_3U|}{|\phi U|}. \tag{3.10}$$

Now, from (3.9) and (3.10) we get

$$g_1(U, (\mathcal{P}_3 \circ \tau)^2U) = \cos^2\theta g_1(U, \phi^2U). \tag{3.11}$$

As f is a screen pseudo-slant lightlike submersion. Thus slant angle of $\mathcal{D}_2$ is constant, so we put $\cos^2\theta = \lambda \in (0, 1]$. Then (3.11) gives $g_1(U, (\mathcal{P}_3 \circ \tau)^2U - \lambda\phi^2U) = 0$, which implies

$$(\mathcal{P}_3 \circ \tau)^2U = \lambda\phi^2U = -\lambda U. \tag{3.12}$$

Conversely suppose that (i) and (ii) are satisfied. Then by using similar steps, it can be easily proved that Δ is invariant as $ltr(Kerf_*)$ is invariant. Further, using (3.9), (3.10) and (3.12) we obtain

$$\cos\theta = -\frac{g_1(U, \lambda\phi^2U)}{|\phi U||\tau\mathcal{P}_3U|} = -\lambda\frac{g_1(U, \phi^2U)}{|\phi U||\tau\mathcal{P}_3U|} = \frac{\lambda|\phi U|}{|\tau\mathcal{P}_3U|} = \frac{\lambda}{\cos\theta},$$

which implies $\cos^2\theta = \lambda$(constant). Therefore f is a screen pseudo-slant lightlike submersion. □

Corollary 3.1. *Let f be a lightlike submersion from an indefinite Sasakian manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a lightlike manifold (M_2, g_2). Then, f is a screen-pseudo slant lightlike submersion if and only if* *(i)* *$ltr(Ker f_*)$ is invariant and $\mathcal{D}_1$ is anti-invariant with respect to ϕ,*

(ii) *there exists a constant $\alpha \in [0,1)$ such that $\mathcal{B}\omega U = -\alpha U, \forall U \in \Gamma(\mathcal{D}_2)$. Here $\mathcal{D}_1$ and $\mathcal{D}_2$ are orthogonal distributions on $f^{-1}(x)$ such that $S(Ker f_*) = \mathcal{D}_1 \perp \mathcal{D}_2 \perp \langle\xi\rangle$.*

Proof. Let $f : (M_1, \phi, \xi, \eta, g_1) \to (M_2, g_2)$ be a screen pseudo-slant lightlike submersion. Then $\mathcal{D}_1$ is anti-invariant and $ltr(Ker f_*)$ is invariant with respect to ϕ. Further, for $U \in \Gamma(D_2)$, we have

$$\phi U = \tau\mathcal{P}_3 U + \omega\mathcal{P}_3 U, \tag{3.13}$$

where $\tau\mathcal{P}_3 U$ and $\omega\mathcal{P}_3 U$ are tangential and transversal components of ϕU, respectively. On applying ϕ to (3.13) and considering tangential components of ϕU, we get

$$-U = (\mathcal{P}_3 \circ \tau)^2 U + \mathcal{B}\omega U. \tag{3.14}$$

Since f is a screen pseudo-slant lightlike submersion, from Theorem 3.1(ii) for any $U \in \Gamma(\mathcal{D}_2)$, we have $(\mathcal{P}_3 \circ \tau)^2 U = -\cos^2\theta U$, where $\cos^2\theta = \lambda$(constant)$\in [0,1)$. Therefore from (3.14), we obtain

$$\mathcal{B}\omega U = -\alpha U, \qquad \forall U \in \Gamma(\mathcal{D}_2) \tag{3.15}$$

where $1 - \lambda = \alpha$(constant)$\in [0,1)$.

Conversely, suppose that we have conditions (i) and (ii). Then by using similar steps as in the proof of Theorem 3.1(i), we can derive Δ is invariant with respect to ϕ. Using (3.14) and (3.15), we obtain

$$-U = (\mathcal{P}_3 \circ \tau)^2 U - \alpha U,$$

which gives

$$(\mathcal{P}_3 \circ \tau)^2 U = -\lambda U,$$

where $1 - \alpha = \lambda$(constant)$\in (0,1]$. This completes the proof. □

Corollary 3.2. *Let f be a screen pseudo-slant lightlike submersion from an indefinite Sasakian manifold $(M_1, \phi, \xi, \eta.g_1)$ onto a lightlike manifold (M_2, g_2) with slant angle θ, then for any $U, V \in \Gamma(\mathcal{D}_2)$, we have*

$$\hat{g}(\tau\mathcal{P}_3 U, \tau\mathcal{P}_3 V) = \cos^2\theta\{\hat{g}(U,V) - \eta(U)\eta(V)\}, \tag{3.16}$$

and

$$g_1(\omega\mathcal{P}_3 U, \omega\mathcal{P}_3 V) = \sin^2\theta\{\hat{g}(U,V) - \eta(U)\eta(V)\}. \tag{3.17}$$

Proof. Using (3.2), (2.3) and Theorem 3.1, we obtain $\hat{g}(\tau\mathcal{P}_3 U, \tau\mathcal{P}_3 V) = -\hat{g}(U, (\mathcal{P}_3 \circ \tau)^2 V)$, where $U, V \in \Gamma(\mathcal{D}_2)$. Now, from the last equation and (3.12), we get $\hat{g}(\tau\mathcal{P}_3 U, \tau\mathcal{P}_3 V) = -\hat{g}(U, \lambda\phi^2 U) = \lambda\hat{g}(\phi U, \phi V)$ which proves (3.16). Using similar steps as above, we can obtain (3.17). □

Lemma 3.2. *Let f be a screen pseudo-slant lightlike submersion from an indefinite Sasakian manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a lightlike manifold (M_2, g_2). Then for $U, V \in \Gamma(Ker f_* - \langle\xi\rangle)$, we have*

(i) $\hat{g}(\hat{\nabla}_U V, \xi) = g_1(V, \phi U)$,

(ii) $\hat{g}([U,V], \xi) = 2g_1(V, \phi U)$.

Proof. Let $f : (M_1, \phi, \xi, \eta, g_1) \to (M_2, g_2)$ be a screen pseudo-slant lightlike submersion. As ∇ is a metric connection, using (2.5) and (2.12), we have

$$\hat{g}(\hat{\nabla}_U V, \xi) = g_1(V, \phi U), \tag{3.18}$$

where $U, V \in \Gamma(Ker f_* - \langle\xi\rangle)$. Since $\hat{\nabla}$ is a symmetric connection, from (3.18) and (2.3), we obtain $\hat{g}([U,V], \xi) = 2g_1(V, \phi U)$. □

Theorem 3.2. *Let f be a screen pseudo-slant lightlike submersion from an indefinite Sasakian manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a lightlike manifold (M_2, g_2). Then Δ is integrable if and only if*

(i) $Q_2\mathcal{T}^s_V\phi\mathcal{P}_1 U = Q_2\mathcal{T}^s_U\phi\mathcal{P}_1 V$,

(ii) $Q_3\mathcal{T}^s_V\phi\mathcal{P}_1 U = Q_3\mathcal{T}^s_U\phi\mathcal{P}_1 V$,

(iii) $\mathcal{P}_3(\mathcal{T}_V^*\phi\mathcal{P}_1U) = \mathcal{P}_3(\mathcal{T}_U^*\phi\mathcal{P}_1V)$,

where $U, V \in \Gamma(\Delta)$.

Proof. Let $f : (M_1, \phi, \xi, \eta, g_1) \to (M_2, g_2)$ be a screen pseudo-slant lightlike submersion. Suppose that $U, V \in \Gamma(\Delta)$. Then from (3.7), we obtain $Q_2\mathcal{T}_U^s\phi\mathcal{P}_1V = \phi\mathcal{P}_2\hat{\nabla}_UV$, which implies $Q_2\mathcal{T}_U^s\phi\mathcal{P}_1V - Q_2\mathcal{T}_V^s\phi\mathcal{P}_1U = \phi\mathcal{P}_2[U, V]$. Using (3.8), we have $Q_3\mathcal{T}_U^s\phi\mathcal{P}_1V = \mathcal{C}Q_3\mathcal{T}_U^sV + \omega\mathcal{P}_3\hat{\nabla}_UV$, which gives $Q_3\mathcal{T}_U^s\phi\mathcal{P}_1V - Q_3\mathcal{T}_V^s\phi\mathcal{P}_1U = \omega\mathcal{P}_3[U, V]$. Also from (3.6), we get $\mathcal{P}_3(\mathcal{T}_U^*\phi\mathcal{P}_1V) = \tau\mathcal{P}_3\hat{\nabla}_UV + \mathcal{B}Q_3\mathcal{T}_U^sV$. The last equation implies $\mathcal{P}_3(\mathcal{T}_U^*\phi\mathcal{P}_1V) - \mathcal{P}_3(\mathcal{T}_V^*\phi\mathcal{P}_1U) = \tau\mathcal{P}_3[U, V]$. Thus the proof follows from above equations and Lemma 3.2(ii). □

Theorem 3.3. *Let f be a screen pseudo-slant lightlike submersion from an indefinite Sasakian manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a lightlike manifold (M_2, g_2). Then $\mathcal{D}_1$ is integrable if and only if*

(i) $\mathcal{P}_1(\mathcal{T}_V\phi\mathcal{P}_2U) = \mathcal{P}_1(\mathcal{T}_U\phi\mathcal{P}_2V)$,

(ii) $\mathcal{P}_3(\mathcal{T}_V\phi\mathcal{P}_2U) = \mathcal{P}_3(\mathcal{T}_U\phi\mathcal{P}_2V)$,

(iii) $Q_3(\nabla_V^s\phi\mathcal{P}_2U) = Q_3(\nabla_U^s\phi\mathcal{P}_2V)$,

where $U, V \in \Gamma(\mathcal{D}_1)$.

Proof. Let $f : (M_1, \phi, \xi, \eta, g_1) \to (M_2, g_2)$ be a screen pseudo-slant lightlike submersion. Suppose that $U, V \in \Gamma(\mathcal{D}_1)$. Then from (3.5), we get $\mathcal{P}_1(\mathcal{T}_U\phi\mathcal{P}_2V) = \phi\mathcal{P}_1\hat{\nabla}_UV$, which gives $\mathcal{P}_1(\mathcal{T}_U\phi\mathcal{P}_2V) - \mathcal{P}_1(\mathcal{T}_V\phi\mathcal{P}_2U) = \phi\mathcal{P}_1[U, V]$. Using (3.6), we obtain $\mathcal{P}_3(\mathcal{T}_U\phi\mathcal{P}_2V) - \mathcal{B}Q_3\mathcal{T}_U^sV = \tau\mathcal{P}_3\hat{\nabla}_UV$. The last equation implies $\mathcal{P}_3(\mathcal{T}_U\phi\mathcal{P}_2V) - \mathcal{P}_3(\mathcal{T}_V\phi\mathcal{P}_2U) = \tau\mathcal{P}_3[U, V]$. Also, from (3.8), we get $Q_3(\nabla_U^s\phi\mathcal{P}_2V) - \mathcal{C}Q_3\mathcal{T}_U^sV = \omega\mathcal{P}_3\hat{\nabla}_UV$, which gives $Q_3(\nabla_U^s\phi\mathcal{P}_2V) - Q_3(\nabla_V^s\phi\mathcal{P}_2U) = \omega\mathcal{P}_3[U, V]$. Therefore we conclude the proof from Lemma 3.2(ii) and above equations. □

Theorem 3.4. *Let f be a screen pseudo-slant lightlike submersion from an indefinite Sasakian manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a lightlike manifold (M_2, g_2). Then $\mathcal{D}_2 \perp \langle\xi\rangle$ is integrable if and only if*

(i) $\mathcal{P}_1(\hat{\nabla}_U\tau\mathcal{P}_3V - \hat{\nabla}_V\tau\mathcal{P}_3U) = \mathcal{P}_1(\mathcal{T}_V\omega\mathcal{P}_3U - \mathcal{T}_U\omega\mathcal{P}_3V)$,

(ii) $Q_2(\nabla_U^s\omega\mathcal{P}_3V - \nabla_V^s\omega\mathcal{P}_3U) = Q_2(\mathcal{T}_V^s\tau\mathcal{P}_3U - \mathcal{T}_U^s\tau\mathcal{P}_3V)$,

where $U, V \in \Gamma(\mathcal{D}_2 \perp \langle\xi\rangle)$.

Proof. Let $f : (M_1, \phi, \xi, \eta, g_1) \to (M_2, g_2)$ be a screen pseudo-slant lightlike submersion. Suppose that $U, V \in \Gamma(\mathcal{D}_2 \perp \langle\xi\rangle)$. Then using (3.5), we get $\mathcal{P}_1(\hat{\nabla}_U\tau\mathcal{P}_3V) = -\mathcal{P}_1\mathcal{T}_U\omega\mathcal{P}_3V + \phi\mathcal{P}_1\hat{\nabla}_UV$, which implies $\mathcal{P}_1(\hat{\nabla}_U\tau\mathcal{P}_3V) - \mathcal{P}_1(\hat{\nabla}_V\tau\mathcal{P}_3U) = \mathcal{P}_1(\mathcal{T}_V\omega\mathcal{P}_3U) - \mathcal{P}_1(\mathcal{T}_U\omega\mathcal{P}_3V) + \phi\mathcal{P}_1[U, V]$. Also, from (3.7) we obtain $Q_2\nabla_U^s\omega\mathcal{P}_3V + Q_2\mathcal{T}_U^s\tau\mathcal{P}_3V = \phi\mathcal{P}_2\hat{\nabla}_UV$. The last equation gives $Q_2\nabla_U^s\omega\mathcal{P}_3V - Q_2\nabla_V^s\omega\mathcal{P}_3U = Q_2\mathcal{T}_V^s\tau\mathcal{P}_3U - Q_2\mathcal{T}_U^s\tau\mathcal{P}_3V + \phi\mathcal{P}_2[U, V]$. Thus the proof follows from above equations. □

Theorem 3.5. *Let f be a screen pseudo-slant lightlike submersion from an indefinite Sasakian manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a lightlike manifold (M_2, g_2). Then, the induced connection $\hat{\nabla}$ on $f^{-1}(x)$ is a metric connection if and only if*

(i) $\phi Q_2\mathcal{T}_U^sV = 0$ *and* $\mathcal{B}Q_3\mathcal{T}_U^sV = 0$,

(ii) $\mathcal{T}_U^*$ *vanishes on* $\Gamma(Kerf_*)$,

where $U \in \Gamma(Kerf_*)$ *and* $V \in \Gamma(\Delta)$.

Proof. Let $f : (M_1, \phi, \xi, \eta, g_1) \to (M_2, g_2)$ be a screen pseudo-slant lightlike submersion. From [3], the induced connection $\hat{\nabla}$ on $f^{-1}(x)$ is a metric connection if and only if Δ is a parallel distribution with respect to $\hat{\nabla}$. Using (2.4), (2.12), (2.15) and (3.4), for $U \in \Gamma(Kerf_*)$ and $V \in \Gamma(\Delta)$, we get $\nabla_U\phi V = \phi\mathcal{T}_U^*V + \phi\nabla_U^{*t}V + \phi Q_2\mathcal{T}_U^sV + \mathcal{B}Q_3\mathcal{T}_U^sV + \mathcal{C}Q_3\mathcal{T}_U^sV$. On equating tangential components of the last equation, we obtain $\hat{\nabla}_U\phi V = \phi\mathcal{T}_U^*V + \phi\nabla_U^{*t}V + \phi Q_2\mathcal{T}_U^sV + \mathcal{B}Q_3\mathcal{T}_U^sV$. This completes the proof. □

4. Geometry of Foliations

In the present section, we obtain necessary and sufficient conditions for foliations determined by distributions on a fiber of a screen pseudo-slant lightlike submersions to be totally geodesic.

Theorem 4.1. *Let f be a screen pseudo-slant lightlike submersion from an indefinite Sasakian manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a lightlike manifold (M_2, g_2). Then, Δ defines a totally geodesic foliation on $f^{-1}(x)$ if and only if $g_1(D^l(U, \phi\mathcal{P}_2W) + D^l(U, \omega\mathcal{P}_3W) + \mathcal{T}^l_U\tau\mathcal{P}_3W, \phi V) = 0$, for $U, V \in \Gamma(\Delta)$ and $W \in \Gamma(S(kerf_*))$.*

Proof. Let $f : (M_1, \phi, \xi, \eta, g_1) \to (M_2, g_2)$ be a screen pseudo-slant lightlike submersion. Then, the distribution Δ on $f^{-1}(x)$ defines a totally geodesic foliation if and only if $\hat{\nabla}_U V \in \Gamma(\Delta)$, for $U, V \in \Gamma(\Delta)$. Since ∇ is a metric connection, using (2.12) for $U, V \in \Gamma(\Delta)$ and $W \in \Gamma(S(Kerf_*))$, we obtain $g_1(\hat{\nabla}_U V, W) = -g_1(V, \nabla_U W)$. Then from (2.2) and (2.4), we get $g_1(\hat{\nabla}_U V, W) = -g_1(\nabla_U \phi W, \phi V)$. Now using (3.2), (2.12) and (2.14), the last equation gives $g_1(\hat{\nabla}_U V, W) = -g_1(D^l(U, \phi\mathcal{P}_2W) + D^l(U, \omega\mathcal{P}_3W) + \mathcal{T}^l_U\tau\mathcal{P}_3W, \phi V)$ which completes the proof. □

Theorem 4.2. *Let f be a screen pseudo-slant lightlike submersion from an indefinite Sasakian manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a lightlike manifold (M_2, g_2). Then $\mathcal{D}_1$ defines a totally geodesic foliation on $f^{-1}(x)$ if and only if*

(i) $g_1(\mathcal{T}^s_U\tau W, \phi V) = -g_1(\nabla^s_U\omega W, \phi V)$,

(ii) $D^s(U, \phi N)$ *has no component in* $\phi(\mathcal{D}_1)$,

where $U, V \in \Gamma(\mathcal{D}_1)$, $W \in \Gamma(\mathcal{D}_2)$ and $N \in \Gamma(ltr(Kerf_*))$.

Proof. Let $f : (M_1, \phi, \xi, \eta, g_1) \to (M_2, g_2)$ be a screen pseudo-slant lightlike submersion. Then $\mathcal{D}_1$ defines a totally geodesic foliation if and only if $\hat{\nabla}_U V \in \Gamma(\mathcal{D}_1)$, for $U, V \in \Gamma(\mathcal{D}_1)$. Since ∇ is a metric connection, using (2.12) for $U, V \in \Gamma(\mathcal{D}_1)$ and $W \in \Gamma(\mathcal{D}_2)$, we get $g_1(\hat{\nabla}_U V, W) = -g_1(V, \nabla_U W)$. Now using (2.2), (2.4) and (3.2), we obtain $g_1(\hat{\nabla}_U V, W) = -g_1(\nabla_U \tau W + \omega W, \phi V)$. Further using (2.12) and (2.14), we get $g_1(\hat{\nabla}_U V, W) = -g_1(\mathcal{T}^s_U\tau W + \nabla^s_U\omega W, \phi V)$. Similarly, for $U, V \in \Gamma(\mathcal{D}_1)$ and $N \in \Gamma(ltr(Kerf_*))$, we can prove $g_1(\hat{\nabla}_U V, N) = -g_1(D^s(U, \phi N), \phi V)$. Thus, the proof follows from last two equations and Lemma 3.2(i). □

Theorem 4.3. *Let f be a screen pseudo-slant lightlike submersion from an indefinite Sasakian manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a lightlike manifold (M_2, g_2). Then $\mathcal{D}_2 \perp \langle\xi\rangle$ defines a totally geodesic foliation on $f^{-1}(x)$ if and only if*

(i) $g_1(\tau V, \mathcal{T}_U\phi W) = -g_1(\omega V, \nabla^s_U\phi W)$,

(ii) $g_1(\tau V, \mathcal{T}_U\phi N) = -g_1(\omega V, D^s(U, \phi N))$,

where $U, V \in \Gamma(\mathcal{D}_2 \perp \langle\xi\rangle)$, $W \in \Gamma(\mathcal{D}_1)$ and $N \in \Gamma(ltr(Kerf_*))$.

Proof. Let $f : (M_1, \phi, \xi, \eta, g_1) \to (M_2, g_2)$ be a screen pseudo-slant lightlike submersion. Then, the distribution $\mathcal{D}_2 \perp \langle\xi\rangle$ defines a totally geodesic foliation if and only if $\hat{\nabla}_U V \in \Gamma(\mathcal{D}_2 \perp \langle\xi\rangle)$, for $U, V \in \Gamma(\mathcal{D}_2 \perp \langle\xi\rangle)$. Since ∇ is a metric connection, using (2.12), (2.2) and (2.4), for $U, V \in \Gamma(\mathcal{D}_2)$ and $W \in \Gamma(\mathcal{D}_1)$, we obtain $g_1(\hat{\nabla}_U V, W) = -g_1(\nabla_U \phi W, \phi V)$. Now using (2.14) and (3.2), the last equation gives $g_1(\hat{\nabla}_U V, W) = -g_1(\mathcal{T}_U\phi W, \tau V) - g_1(\nabla^s_U\phi W, \omega V)$. Similarly, we have $g_1(\hat{\nabla}_U V, N) = -g_1(\mathcal{T}_U\phi N, \tau V) - g_1(D^s(U, \phi N), \omega V)$, for any $U, V \in \Gamma(\mathcal{D}_2)$ and $N \in \Gamma(ltr(Kerf_*))$. Thus, we conclude the proof. □

Acknowledgements
The authors would like to express their sincere thanks to the anonymous reviewers for their helpful comments and suggestions.

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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