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ON SECOND-ORDER q-DIFFERENCE OPERATORS

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Abstract. The minimal and maximal operators defined by second-order q-difference operator are discussed in this paper. Spectrum sets of these defined operators have been determined. In addition, two extensions of the minimal operator is also mentioned.

1. Introduction

Euler [8] initiated the q-analysis in 18th cent., while Jackson [11] gave the definition of q-integral in 1910. Jackson [12] reintroduced q-derivative or $q-$difference operator as

$$D_q u(t) = \frac{u(t) - u(qt)}{(1-q)t}, \quad t \in \mathbb{K} \setminus \{0\}.$$

When the zero is an element of $\mathbb{K}$, the q-derivative, provided that it is independent of the t point, is defined for $|q| < 1$ is follows

$$D_q u(0) = \lim_{n \to +\infty} \frac{u(tq^n) - u(0)}{tq^n}, \quad t \in \mathbb{K} \setminus \{0\}.$$

q-difference operator turns into the classical derivative for $q \to 1$. Also, q-integral denoted by

$$\int_c^d u(t)\, d_q t = \int_0^d u(t)\, d_q t - \int_0^c u(t)\, d_q t, \quad 0 < c < d,$$

is given by Jackson [11] where

$$\int_0^x u(t)\, d_q t := (1-q) \sum_{n=0}^{+\infty} x q^n u(xq^n), \quad x \in \mathbb{K}$$

when two series converge. In addition, it has been proved by Bromwich [7] that the q-integral turns into a classical integral as q approaches zero in parallel with the q-derivative.

In hypergeometric functions, quantum theory, fractal geometry, the variation calculus, orthogonal polynomials and relativity theory, the q-calculus plays an unforeseen role. On addition, research in the q-calculus has been ongoing, such evidenced

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by Phillips and Aral [5, 10]. Also, many problems for second order q-difference operator are studied by many mathematicians such as [1, 2, 9, 15]. However, in our research, we have not encountered any study in terms of second-order q-difference operator, operator theory on a finite interval.

In [16], an operator T which has dense domain is said to be q-hyponormal by Ota if and only if it is ensured that $D(T) \subset D(T^*)$ and $\|T^*x\| \leqslant \sqrt{q}\|Tx\|$ for any $x \in D(T)$ with $q > 0$ and $q \neq 1$. Also, any q-hyponormal operator is closable. It can be defined an operator T as q-cohyponormal if the adjoint operator of T is q-hyponormal.

Annaby and Mansour investigated a q-analogue of Sturm-Liouville problems in $L^2_q(0,a)$, $0 < a < +\infty$ in [4]. However, they need to extend the domains of functions in $L^2_q(0,a)$ to $[0, q^{-1}a]$, because they can write the formal adjoint operator of q-difference operator as q^{-1}-difference operator. This is not necessary, since it is well known that a dense define operator has always the adjoint operator. With the same idea, a minimal operator with a definite set containing the boundary condition $u(a) = 0$ cannot be densely defined [17]. However, the definition set of the minimal operator defined by the second order expressed by the classical derivative is densely defined although it contains the same boundary condition. In some studies in the literature, the density of minimal operator domain is overlooked. For example, the minimal operator defined by the q-Sturm-Liouville expression in [3] is not dense and is not a symmetric operator since its definition set contains the condition $u(a) = 0$. However, when we look at the definition of a symmetric operator, its domain must be dense [13]. The motivation for this study is that there is some discrepancy between the results obtained and those expected according to classical theory. We address this discrepancy in this study.

In this paper we give some basic results for the q-difference operator and give the definitions of the minimal and maximal operators defined by the second order q-difference operator. Then the adjoint operators of the minimal operator is defined and the cohyponormality problem of the maximal operator is considered. In the last section the spectral problem of the minimal and maximal operators is considered. Moreover, the spectrum sets of two different closed extensions of the minimal are given.

Throughout this article, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ is considered.

2. The Minimal and the Maximal Operators Definitions

In the literature, $L^2_q(0,1)$ is defined as the set of complex-valued functions defined on $[0,1]$ such that

$$\|v\|^2_{L^2_q(0,1)} = \int_0^1 |v(x)|^2 d_q x := (1-q)\sum_{k=0}^{\infty} q^k \left|v(q^k)\right|^2 < +\infty.$$

It can be easily seen that $L^2_q(0,1)$ is a linear vector space of classes $[v]$. Besides, u and v are in the same class iff $v\left(q^k\right) = u\left(q^k\right)$, $k \in \mathbb{N}_0$. $L^2_q(0,1)$ is a Hilbert space and its inner product [4] is defined as

$$(u,v)_{L_q^2(0,1)} = \int_0^1 u(t)\,\overline{v(t)}d_qt.$$

Lemma 1. *If $D_q^2u(t)$ is an element in $L_q^2(0,1)$, then the limits $\lim_{n\to+\infty} D_qu(q^n)$ and $\lim_{n\to+\infty} u(q^n)$ exist in $\mathbb{C}$.*

Proof. Suppose $D_q^2u(t) \in L_q^2(0,1)$, since the constant function $f(t) = 1$ is an element of $L_q^2(0,1)$ then

$$\begin{aligned}\left(D_q^2u(t), f(t)\right)_{L_q^2(0,1)} &= \int_0^1 D_q^2u(t)\,d_q \\ &= \sum_{k=0}^{\infty} D_qu(q^n) - D_qu(q^{n+1}) \\ &= \frac{u(1)-u(q)}{1-q} - \lim_{n\to\infty} D_qu(q^n)\end{aligned}$$

is true. This means that the limit $\lim_{n\to+\infty} D_qu(q^n)$ exists. Since the sequence $\{D_qu(q^n)\}$ is bounded, from the definition of $L_q^2(0,1)$ it is obtained that $D_qu(t)$ is in $L_q^2(0,1)$. Similarly, the existence of the limit $\lim_{n\to+\infty} u(q^n)$ is also proved. □

Corollary 2. *If $D_q^2u(t) \in L_q^2(0,1)$, then $u(t)$ and $D_qu(t)$ are elements in the Hilbert space $L_q^2(0,1)$.*

Corollary 3. *If $D_q^mu(t)$, $m \in \mathbb{N}$ is an element in $L_q^2(0,1)$, then the limits $\lim_{n\to+\infty} D_q^ku(q^n)$ exist in $\mathbb{C}$ and $D_q^ku(t) \in L_q^2(0,1)$ for $0 \le k \le m-1$.*

The operator $L_0 : D_0 \subset L_q^2(0,1) \to L_q^2(0,1)$ is defined as the form $Lu(t) = D_q^2u(t)$ such that

$$D_0 = \left\{u(t) \in L_q^2(0,1) : D_q^2u(t) \in L_q^2(0,1),\ \lim_{n\to+\infty} u(q^n) = \lim_{n\to+\infty} D_qu(q^n) = 0\right\}.$$

We call that $L_0 : D_0 \subset L_q^2(0,1) \to L_q^2(0,1)$ is the minimal operator introduced by second order q-difference derivative.

Theorem 4. *The operator $L_0 : D_0 \subset L_q^2(0,1) \to L_q^2(0,1)$ has dense domain and is closed in $L_q^2(0,1)$.*

Proof. Firstly, it is obviously seen that D_0 is dense in $L_q^2(0,1)$ because, D_0 contains the set of functions

$$\phi_n(t) := \begin{cases} \dfrac{1}{q^{\frac{n}{2}}\sqrt{1-q}}, & t = q^n \\ 0, & otherwise \end{cases}, \quad n \in \mathbb{N}_0$$

which is an orthogonal basis of $L_q^2(0,1)$.

For the closeness of the minimal operator L_0 we suppose that $\{u_n\} \subset D_0$ such that $u_n \xrightarrow[n\to\infty]{} u$ and $L_0 u_n \xrightarrow[n\to\infty]{} g$. Then

$$\|u_n - u\|^2_{L_q^2(0,1)} = (1-q)\sum_{k=0}^{+\infty} q^k \left|u_n\left(q^k\right) - u\left(q^k\right)\right|^2 \xrightarrow[n\to\infty]{} 0.$$

From this result, we have

$$\lim_{n\to\infty} u_n\left(q^k\right) = u\left(q^k\right) \tag{1}$$

for all $k \in \mathbb{N}_0$. Because of this limit, there is an integer $n_0 \in \mathbb{N}_0$ for any $\epsilon > 0$ such that

$$\left|u_n\left(q^k\right) - u\left(q^k\right)\right| < \epsilon$$

where $n_0 \leq n$, $n \in \mathbb{N}_0$ and $k \in \mathbb{N}_0$. Therefore,

$$0 \leq \left|u(q^k)\right| \leq \left|u_n(q^k) - u(q^k)\right| + \left|u_n(q^k)\right| < \epsilon + \left|u_n(q^k)\right|$$

is hold. From this relation and $\{u_n\} \subset D_0$ it is get that

$$\lim_{k\to+\infty} u\left(q^k\right) = 0.$$

Similarly, we can choose as $\epsilon = (1-q)q^{2k}$ and the following inequality

$$\begin{aligned}\left|D_q u(q^k)\right| = \left|\frac{u(q^k) - u(q^{k+1})}{(1-q)q^k}\right| &\leq \left|\frac{u_n(q^k) - u(q^k)}{(1-q)q^k}\right| + \left|\frac{u_n(q^{k+1}) - u(q^{k+1})}{(1-q)q^k}\right| + \left|D_q u_n(q^k)\right| \\ &< q^k + q^{k+2} + \left|D_q u_n(q^k)\right|\end{aligned}$$

is true. Because of this and $\left\{D_q^2 u_n\right\} \subset D_0$ is a bounded sequence,

$$\lim_{k\to+\infty} D_q u(q^k) = 0$$

is seen, and so $u \in D_0$. On the other hand, from the limit (1) and the uniqueness of the limit, it is gained that

$$\lim_{n\to+\infty} D_q^2 u_n(q^k) = D_q^2 u(q^k) = g\left(q^k\right).$$

The proof is complete with this result.

□

Theorem 5. *The adjoint operator $L_0^* : D\left(L_0^*\right) \subset L_q^2(0,1) \to L_q^2(0,1)$ is*

$$L_0^* u(t) = \begin{cases} \frac{u(1)}{(1-q)^2} & , q < t \leqslant 1 \\ -\frac{(1+q)u(1) - u(q)}{q^2(1-q)^2} & , q^2 < t \leqslant q \\ \frac{1}{q^2} D^2_{q^{-1}} u(t) & , 0 < t \leqslant q^2 \end{cases}$$

where $D\left(L_0^*\right) = \{u(t) \in L_q^2(0,1) :\ D_q^2 u(t) \in L_q^2(0,1)\}$.

Proof. Suppose $u \in D\left(L_0\right)$ and $D_q^2 v(t) \in L_q^2(0,1)$,

$$\begin{aligned}&\left(D_q^2 u(t), v(t)\right) = \lim_{n\to+\infty}(1-q)\sum_{k=0}^{n} q^k \left(\frac{qu\left(q^k\right) - (1+q)u\left(q^{k+1}\right) + u\left(q^{k+2}\right)}{q(1-q)^2 q^{2k}}\right)\overline{v\left(q^k\right)} \\ &= (1-q)u(1)\frac{\overline{v(1)}}{(1-q)^2} + (1-q)q\left(u(q)\left(-\frac{\overline{(1+q)v(1) - v(q)}}{q^2(1-q)^2}\right)\right) \\ &+ (1-q)\sum_{k=2}^{\infty} q^k u\left(q^k\right)\overline{\left(\frac{1}{q^2}D^2_{q^{-1}} v\left(q^k\right)\right)} + \lim_{n\to+\infty} u\left(q^n\right)\frac{1}{q}\overline{D_q v\left(q^n\right)} - D_q u\left(q^n\right)\overline{v\left(q^n\right)}\end{aligned}$$

$$= (1-q)u(1)\overline{\frac{v(1)}{(1-q)^2}} + (1-q)q\left(u(q)\left(-\overline{\frac{(1+q)v(1)-v\left(q\right)}{q^2(1-q)^2}}\right)\right)$$

$$+ (1-q)\sum_{k=2}^{\infty} q^k u\left(q^k\right)\overline{\left(\frac{1}{q^2}D^2_{q^{-1}}v\left(q^k\right)\right)}.$$

Because the inner product definition on $L^2_q\left(0,1\right)$ and the equation

$$D^2_{q^{-1}}u\left(t\right) = q^2\frac{qu(q^{-2}t)-(1+q)u(q^{-1}t)+u(t)}{(1-q)^2t^2} = \frac{1}{q}D^2_q u\left(q^{-2}t\right), \quad 0 < t \leq q^2$$

is true,

$$L^*_0 u\left(t\right) = \begin{cases} \frac{u(1)}{(1-q)^2} & , q < t \leqslant 1 \\ -\frac{(1+q)u(1)-u(q)}{q^2(1-q)^2} & , q^2 < t \leqslant q \\ \frac{1}{q^2}D^2_{q^{-1}}u\left(t\right) & , 0 < t \leqslant q^2 \end{cases}$$

is hold and

$$D\left(L^*_0\right) = \left\{u \in L^2_q\left(0,1\right) : D^2_q u\left(t\right) \in L^2_q\left(0,1\right)\right\}$$

is obtained. □

It can be defined $D = \left\{u \in L^2_q\left(0,1\right) : D^2_q u\left(t\right) \in L^2_q\left(0,1\right)\right\}$ and $L : D \subset L^2_q\left(0,1\right) \longrightarrow L^2_q\left(0,1\right)$, $Lu(t) = D^2_q u\left(t\right)$. We call that L is the maximal operator defined by second order q-difference derivative. The maximal operator L is closed on $L^2_q\left(0,1\right)$ from Theorem 4. It is true that $L_0 \subset L$, $D\left(L^*_0\right) = D\left(L\right)$ and $D\left(L^*\right) = D\left(L_0\right)$. In addition, there are two extensions of the minimal operator L_0 different from the operator L defined as following

$$\tilde{L_1}u(t) = D^2_q u(t), \quad D\left(\tilde{L_1}\right) := \left\{u\left(t\right) \in L^2_q\left(0,1\right) : \lim_{n\to\infty} u\left(q^n\right) = 0\right\}$$

and

$$\tilde{L_2}u(t) = D^2_q u(t), \quad D\left(\tilde{L_2}\right) := \left\{u\left(t\right) \in L^2_q\left(0,1\right) : \lim_{n\to\infty} D_q u\left(q^n\right) = 0\right\}.$$

Moreover, $D\left(\tilde{L_k}^*\right) = D\left(\tilde{L_k}\right)$ is easily seen for $k = 1,2$.

Corollary 6. *The operator L is a q^4-cohyponormal on $L^2_q\left(0,1\right)$.*

Proof. It can be easily seen that $D(L^*) = D_0 \subset D = D(L)$ and for any $u \in D(L^*)$

$$\begin{aligned} \left\|Lu\left(t\right)\right\|^2_{L^2_q(0,1)} &= \int_0^1 \left|D^2_q u\left(t\right)\right|^2 d_q t = (1-q)\sum_{k=0}^{+\infty} q^k \left|D^2_q u\left(q^k\right)\right|^2 \\ &= (1-q)\sum_{k=0}^{+\infty} q^k \left|\frac{D_q u\left(q^k\right) - D_q u\left(q^{k+1}\right)}{\left(1-q\right)q^k}\right|^2 \end{aligned}$$

and

$$\begin{aligned} \left\|L^* u\left(t\right)\right\|^2_{L^2_q(0,1)} &= (1-q)^{-1}\left|u\left(1\right)\right|^2 + q^{-3}\left(1-q\right)^{-3}\left|u(q)-(1+q)u\left(1\right)\right|^2 \\ &+ (1-q)\sum_{k=2}^{+\infty} q^k \left|\frac{1}{q^2}D^2_{q^{-1}}u\left(q^k\right)\right|^2 \\ &\geqslant (1-q)\sum_{k=2}^{+\infty} q^k \left|\frac{1}{q^2}D^2_{q^{-1}}u\left(q^k\right)\right|^2 \end{aligned}$$

$$
\begin{aligned}
&= (1-q)\sum_{k=2}^{+\infty} q^k \left|\frac{1}{q^3} D_q^2 u\left(q^{k-2}\right)\right|^2 \\
&= \frac{1}{q^4}(1-q)\sum_{k=0}^{+\infty} q^k \left|D_q^2 u\left(q^k\right)\right|^2 \\
&= \frac{1}{q^4}\left\|Lu\left(t\right)\right\|^2_{L_q^2(0,1)}.
\end{aligned}
$$

This means that

$$\left\|Lu\left(t\right)\right\| \leqslant q^2\left\|L^* u\left(t\right)\right\|, \quad u \in D(L^*)$$

and so the proof is complete. □

Remark 7. *In [17], the maximal operator introduce by first order q-difference derivative is q-cohyponormal operator in* $L_q^2(0,1)$. *Therefore, it is usual to predict that the maximal operator defined by second order q-difference derivative will be the* q^2*-cohyponormal operator. But, the maximal operator defined by the second order q-difference derivative is* q^4*-cohyponormal. This is because a consequence of the equation* $D_q D_{q^{-1}} = \frac{1}{q} D_{q^{-1}} D_q$.

3. Spectrum Sets of the Minimal and Maximal Operators

Now the spectrum problem, which is the important problem of operators, is discussed for the minimal and maximal operators that we defined in the previous section.

Theorem 8. *The continuous and residual spectrum sets of* L_0 *defined by second order q−difference derivative are*

$$\sigma_r\left(L_0\right) = \sigma_c\left(L_0\right) = \emptyset.$$

Proof. Let $\lambda^2 \in \mathbb{C} \setminus \sigma_p\left(L_0\right)$ and solve the following problem with the boundary value

$$\begin{cases} \left(L_0 - \lambda^2 E\right) u\left(t\right) = f\left(t\right) \\ \lim\limits_{n\to+\infty} u\left(q^n\right) = \lim\limits_{n\to+\infty} D_q u\left(q^n\right) = 0 \end{cases}$$

It can be written

$$\begin{cases} \left(D_q - \lambda E\right)\left(D_q + \lambda E\right) u\left(t\right) = f\left(t\right) \\ \lim\limits_{n\to+\infty} u\left(q^n\right) = \lim\limits_{n\to+\infty} D_q u\left(q^n\right) = 0 \end{cases}$$

Because $\lambda \neq \pm\lambda\left(1-q\right)q^m$, $m \in \mathbb{N}_0$ and Theorem 3.2 proof in [17] the function $g(t)$ exists such that $(D_q + \lambda E)g(t) = f(t)$,

$$
\begin{aligned}
g\left(q^{k+1}\right) &= \left(\prod_{n=0}^{k}\left(1+\lambda\left(1-q\right)q^n\right)\right) g\left(1\right) \\
&\quad -\left(1-q\right)\left(\prod_{n=1}^{k}\left(1+\lambda\left(1-q\right)q^n\right)\right) f\left(1\right) \\
&\quad -\left(1-q\right)\left(\prod_{n=2}^{k}\left(1+\lambda\left(1-q\right)q^n\right)\right) q f\left(q\right)
\end{aligned}
$$

$$-\ldots-(1-q)\left[\left(\prod_{n=k-1}^{k}(1+\lambda(1-q)q^{n})\right)q^{k-1}f\left(q^{k-1}\right)+q^{k}f\left(q^{k}\right)\right]$$

$k\in\mathbb{N}_0$ and $\lim\limits_{n\to+\infty}g(q^n)=0$. From the same reasons there exists a function $u(t)$ the following:

$$\begin{aligned}u\left(q^{k+1}\right) &= \left(\prod_{n=0}^{k}(1-\lambda(1-q)q^{n})\right)u(1)\\ &-(1-q)\left(\prod_{n=1}^{k}(1-\lambda(1-q)q^{n})\right)g(1)\\ &-(1-q)\left(\prod_{n=2}^{k}(1-\lambda(1-q)q^{n})\right)qg(q)\\ &-\ldots-(1-q)\left[\left(\prod_{n=k-1}^{k}(1-\lambda(1-q)q^{n})\right)q^{k-1}f\left(q^{k-1}\right)+q^{k}g\left(q^{k}\right)\right]\end{aligned}$$

$k\in\mathbb{N}_0$ and $\lim\limits_{n\to+\infty}u(q^n)=\lim\limits_{n\to+\infty}D_qu(q^n)=0$. Thus, the proof is finished. □

Theorem 9. *The minimum and maximal operators point spectrum sets are of the following forms*

$$\sigma_p(L_0)=\left\{\frac{1}{(1-q)^2q^{2k}}:k\in\mathbb{N}_0\right\},\quad \sigma_p(L)=\mathbb{C}.$$

Proof. Suppose that $\lambda^2\in\sigma_p(L_0)$. In this case, a nonzero element $u(t)$ exists in D_0 and

$$L_0u(t)=\lambda^2u(t).$$

Therefore, from [4,6]

$$(D_q-\lambda)(D_q+\lambda)u(t)=0,\quad u(t)\in D_0.$$

Because of this,

$$\frac{u\left(q^{k}\right)-u\left(q^{k+1}\right)}{(1-q)q^{k}}=\lambda u\left(q^{k}\right)$$

or

$$\frac{u\left(q^{k}\right)-u\left(q^{k+1}\right)}{(1-q)q^{k}}=-\lambda u\left(q^{k}\right)$$

$k\in\mathbb{N}_0$. From the last equations,

$$u\left(q^{k+1}\right)=c_1\left(\prod_{n=0}^{k}(1-(1-q)\lambda q^{n})\right)+c_2\left(\prod_{n=0}^{k}(1+(1-q)\lambda q^{n})\right),\ k\in\mathbb{N}_0$$

is hold where $c_1\neq0$ or $c_2\neq0$. Because of $u\in D_0$, $u\left(q^k\right)\underset{k\to+\infty}{\to}0$ and $D_qu\left(q^k\right)\underset{k\to+\infty}{\to}0$, it is true that

$$c_1\left(\prod_{n=0}^{\infty}(1-(1-q)\lambda q^{n})\right)+c_2\left(\prod_{n=0}^{\infty}(1+(1-q)\lambda q^{n})\right)=0,$$

$$\lambda c_1 \left(\prod_{n=0}^{\infty} \left(1 - (1-q)\lambda q^n\right) \right) - \lambda c_2 \left(\prod_{n=0}^{\infty} \left(1 + (1-q)\lambda q^n\right) \right) = 0.$$

From this, it must be $c_1 = 0$ or $c_2 = 0$. In this case,

$$\prod_{n=0}^{\infty} \left(1 - (1-q)\lambda q^n\right) = 0$$

or

$$\prod_{n=0}^{\infty} \left(1 + (1-q)\lambda q^n\right) = 0$$

iff there is $m \in \mathbb{N}_0$ and

$$1 - \lambda (1-q) q^m = 0$$

or

$$1 + \lambda (1-q) q^m = 0$$

[14]. Therefore, it is get that $\lambda^2 = \frac{1}{(1-q^2)q^{2m}}$, $m \in \mathbb{N}_0$ i.e.

$$\sigma_p (L_0) = \left\{ \frac{1}{(1-q)^2 q^{2k}} : k \in \mathbb{N}_0 \right\}$$

is gotten.

Since there are no boundary conditions, the elements defined as

$$u\left(q^{k+1}\right) = c_1 \left(\prod_{n=0}^{k} \left(1 - (1-q)\lambda q^n\right) \right) + c_2 \left(\prod_{n=0}^{k} \left(1 + (1-q)\lambda q^n\right) \right), \ k \in \mathbb{N}_0$$

is eigenvector of L for any $\lambda^2 \in \mathbb{C}$. Thence, $\sigma_p (L) = \mathbb{C}$ is true.

□

Corollary 10. *The following relation*

$$\sigma (L_0) = \sigma_p (L_0) = \left\{ \frac{1}{(1-q)^2 q^{2k}} : k \in \mathbb{N}_0 \right\}$$

is hold.

Corollary 11. *The spectrum sets of the operators $\tilde{L}_i : D(\tilde{L}) \subset L_q^2(0,1) \to L_q^2(0,1)$, $i = 1, 2$ are*

$$\sigma_p \left(\tilde{L}_i\right) = \mathbb{C}.$$

Theorem 12. *The spectrum set of L_0^* is equal to only the point spectrum and*

$$\sigma_p (L_0^*) = \left\{ \frac{1}{(1-q)^2 q^{2k}} : k \in \mathbb{N}_0 \right\}.$$

Proof. Suppose that λ^2 is an eigenvalue of the adjoint operator L_0^*. In this case, there is a nonzero function $u(t)$ in $D(L_0^*)$ such that

$$L_0^* u(t) = \lambda^2 u(t).$$

From here,

$$\frac{1}{(1-q)^2} u(1) = \lambda^2 u(1),$$

$$\frac{u(q)-(1+q)u(1)}{q^2(1-q)^2}u\left(1\right)=\lambda^2 u\left(q\right),$$
$$\left(-\frac{1}{q}D_{q^{-1}}-\lambda\right)\left(-\frac{1}{q}D_{q^{-1}}+\lambda\right)u(t)=0,\quad 0<t\leq q^2.$$

If $u(1)\neq 0$, then $\lambda^2=\frac{1}{(1-q)^2}$ and

$$u(q)=\frac{1}{1-q}u(1),$$
$$u\left(q^k\right)=c_1\prod_{j=2}^{k}\left(1-q^j\right)^{-1}+c_2\prod_{j=2}^{k}\left(1+q^j\right)^{-1},\ k\geqslant 2$$

where $\frac{1+q}{1-q^2}u(q)-\frac{q}{1-q^2}u(1)=\frac{c_1}{1-q^4}+\frac{c_2}{1-q^4}$. In the same idea, if $u(1)=\cdots=u(q^{m-1})=0$, $m\geqslant 1$ and $u(q^m)\neq 0$, then $\lambda^2=\left(1-q\right)^2q^{2m}$ and

$$u\left(q^k\right)=c_1\prod_{j=m+1}^{k}\left(1-q^{j-m}\right)^{-1}+c_2\prod_{j=m+1}^{k}\left(1+q^{j-m}\right)^{-1},\ m\in\mathbb{N}.$$

Since there is not any boundary condition, $u(t)$ is an eigenvector of the adjoint operator L_0^*. As a result, the set

$$\sigma_p\left(L_0^*\right)=\left\{\frac{1}{\left(1-q\right)^2q^{2k}}:k\in\mathbb{N}_0\right\}$$

is gotten. □

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