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Fuzzy Closure Hopf Space

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Abstract. In this study, we introduce the concepts of fuzzy closure Hopf space and fuzzy closure Hopf group within the framework of fuzzy closure spaces, using homotopy theory. We investigate the relationships between the fuzzy closure Hopf group and its homotopy equivalence. Furthermore, we demonstrate the existence of a contravariant functor from the category of fuzzy closure Hopf spaces and the continuous functions, to the category of groups and homomorphisms. This is demonstrated by illustrating that the set of homotopy function classes among fuzzy closure Hopf groups constitutes a group.

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Keywords: Fuzzy Set, closure space, fuzzy closure space, Hopf space, fuzzy closure Hopf group.

1. Introduction

A Hopf space is a topological space X with an identity element e and a multiplication $m : X \times X \to X$ such that $m \circ (1_X, c)$ and $m(c, 1_X)$ are homotop to the identity map 1_X where $c : X \to X, c(x) = e$ [1] . If $m(m \times 1_X)$ homotop to $m(1_X \times m)$, the m is called homotopy associative. If there exists a map $n : X \to X$ such that $m(n, 1_X)$ and $m(1_X, n)$ homotop to c then n is called homotopy inverse. A Hopf group, a group structure in the homotopy category, is a Hopf space with homotopy associative multiplication and homotopy inverse. A well-known example of the Hopf group is the topological group. Hopf groups have facilitated the solution of complex topological problems by effectively transforming them into algebraic contexts. To comprehensively understand Hopf spaces, please refer to [18, 23, 27].

Hopf space and Hopf group structures have attracted interest and have been the subject of research in various fields of mathematics. In [8, 10, 15–17], the digital counterparts of these concepts have been defined. The study of Hopf group structures extends to fuzzy topological spaces in [5, 6]. The concept of co-Hopf space, which serves as the dual notion of Hopf space, has been examined in the context of digital images in [9, 17], fuzzy topological spaces in [7], and closure spaces in [4]. In this study, the Hopf group structure is constructed on fuzzy closure spaces, a generalization of fuzzy topological space.

Closure space is defined in [3] by a closure operator C satisfies

c_1) $C(\varnothing) = \varnothing$
c_2) $A \subset C(A)$
c_3) $C(A \cup B) = C(A) \cup C(B)$.

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A closure space is a topological space when C is Kuratowski closure operator, i.e. satisfies c_4) $C(C(A)) = C(A)$. See [2, 3, 11] for detailed knowledge about closure spaces.

Topological concepts, such as separation axioms, continuity, and compactness, are generalized to closure spaces [20]. The notation of homotopy is defined in [25] in closure spaces.

2. Preliminaries

This section presents some basic concepts of fuzzy sets and fuzzy closure space.

[22] A fuzzy set on a nonempty set X is a function A from X to $I = [0, 1]$. The set of all fuzzy sets on X is denoted by I^X [13].

Let A and B be fuzzy sets on X. Then, $\forall x \in X$

Subset: $A \prec B \Leftrightarrow A(x) < B(x)$
Equility: $A = B \Leftrightarrow A \prec B$ and $B \prec A$.
Union: $(A \vee B)(x) = sup\{A(x), B(x)\}$
Intersection: $(A \wedge B)(x) = inf\{A(x), B(x)\}$
Complement: $A^c(x) = (\overline{1} - A)(x) = 1 - A(x)$
Empty fuzzy set: $\overline{0}(x) = 0$
Universal fuzzy set: $\overline{1}(x) = 1$.

Let $A \in I^X$ and $B \in I^Y$. The cartesian product of $A \times B$ is a fuzzy set on $X \times Y$ defined as $(A \times B)(x, y) = inf\{A(x), B(y)\}$ [21].

A fuzzy point p_x^r in X is a fuzzy set defined as $p_x^r = \{(x, r)\} \cup \{(y, 0) \mid y \neq x, y \in X\}$ [24]. Then, x is called the support, and r is called the degree of membership of p_x^r. Let $A \in I^X$ and p_α^r is a fuzzy point in X. Then, $p_\alpha^r \in A \Longleftrightarrow r \leq A(x), \forall x \in X$. It is clear that any fuzzy set A can be written as the union of fuzzy points in A, i.e. $A = \bigvee_{p_x^r \in A} p_x^r$. Cartesian product of two fuzzy points $p_x^r \in X$ and $p_y^t \in Y$ is a fuzzy point in $X \times Y$ defined as $p_{(x,y)}^{min\{r,t\}}$ [19]. Let π_i be projections from ΠX_i to X_i. Then, $\pi_i(p_x^r)$ is a fuzzy point in X_i with the support $\pi_i(x)$ and with the degree of membership of $\pi_i(x)$. Support of a fuzzy set $A \in I^X$ is a crisp set in X defined as $suppA = \{x \in X \mid A(x) > 0\}$.

Let f be a map from X to Y and $A \in I^X$, $B \in I^Y$. $f(A)$ is a fuzzy set in Y defined as

$$f(A)(y) = \begin{cases} \sup_{x \in f^{-1}(\{y\})} A(x) & , f^{-1}(\{y\}) \neq \emptyset \\ 0 & , \text{otherwise} \end{cases}$$

and $f^{-1}(B)$ is a fuzzy set in X defined as $f^{-1}(B)(x) = B(f(x))$ [26].

Definition 2.1 ([12, 21]). A fuzzy closure map on a nonempty set X is a function $C : I^X \longrightarrow I^X$ which satisfies the following axioms:

(i) $C\left(\overline{0}\right) = \overline{0}$,
(ii) $A \leq C(A)$ for all $A \in I^X$,
(iii) $C(A \vee B) = C(A) \vee C(B)$ for all $A, B \in I^X$.

The pair (X, C) is called a fuzzy closure space. (X, p_α^r, C) is called a pointed fuzzy closure space with the base point $p_\alpha^r \in I^X$.

If $C(A) = A$ for $A \in I^X$, then A is called a fuzzy closed set and called a fuzzy open set if its complement is fuzzy closed, i.e. $C(A^c) = A^c$.

3. Fuzzy Closure Space

In this section, fundamental concepts related to fuzzy closure spaces have been introduced. Furthermore, certain operations among fuzzy closure maps have been defined, deriving new fuzzy closure maps.

Example 3.1. Let $X \neq \emptyset$ and $1_X : I^X \longrightarrow I^X$ be the identity map. Then, 1_X is the finest fuzzy closure operator on X. Let $C : I^X \longrightarrow I^X$ be a map defined as $C(\overline{0}) = \overline{0}$ and $C(A) = \overline{1}$ if $A \neq \overline{0}$. Then, C is the coarsest fuzzy closure operator on X.

Example 3.2. Let X be a nonempty set and $C : I^X \longrightarrow I^X$ be defined as $C(A)(x) = sup\{A(y) \mid y \in X\}$, for all $x \in X$. Then, C is a fuzzy closure operator on X.

Example 3.3. Let $X = \{1, 2, 3, 4\}$. The map $C : I^X \longrightarrow I^X$ defined as

$$C(A) = \begin{cases} \overline{0} & \text{if } A = \overline{0} \\ \wedge\{B \mid suppB = \{1,2\}\} & \text{if } suppA = \{1\}, \{2\} \text{ or } \{1,2\} \\ \wedge\{B \mid suppB = \{2,3\}\} & \text{if } suppA = \{3\} \text{ or } \{2,3\} \\ \wedge\{B \mid suppB = \{2,4\}\} & \text{if } suppA = \{4\} \text{ or } \{2,4\} \\ \wedge\{B \mid suppB = \{1,3,4\}\} & \text{if } suppA = \{1,3\}, \{1,4\} \text{ or } \{1,3,4\} \\ \wedge\{B \mid suppB = \{2,3,4\}\} & \text{if } suppA = \{3,4\} \text{ or } \{2,3,4\} \\ \overline{1} & \text{if } suppA = \{1,2,3\}, \{1,2,4\} \text{ or } X \end{cases}$$

for all $A \in I^X$ is a fuzzy closure operator on X.

The following theorem shows that we can define new fuzzy closure operators with the help of existing fuzzy closure operators.

Theorem 3.4. *Let C and C' be fuzzy closure operators on a set X. Let define union and composition of C and C' such that $(C \cup C')(U) = C(U) \vee C'(U)$ and $(C \circ C')(U) = C(C'(U))$. Then, $C \cup C'$ and $C \circ C'$ are fuzzy closure operators on X.*

Proof. $(C \cup C')(\overline{0}) = C(\overline{0}) \vee C'(\overline{0}) = \overline{0}$ and $(C \circ C')(\overline{0}) = \overline{0}$.

$$(C \cup C')(U) = C(U) \vee C'(U) > U \vee U = U$$

and

$$(C \circ C')(U) = C(C')(U)) > C'(U) > U$$

for all $U \in I^X$. Let $U, W \in I^X$. Then,

$$\begin{aligned}(C \cup C')(U \vee W) &= C(U \vee W) \vee C'(U \vee W) \\ &= C(U) \vee C(W) \vee C'(U) \vee C'(W) \\ &= (C \cup C')(U) \vee (C \cup C')(W),\end{aligned}$$

$$\begin{aligned}(C \circ C')(U \vee W) &= C(C'(U \vee W)) = C(C'(U) \vee C'(W)) \\ &= C(C'(U)) \vee C(C'(W)) \\ &= (C \circ C')(U) \vee (C \circ C')(W).\end{aligned}$$

□

Let (X, C) and (X, C') be fuzzy closure spaces. Define an operation $\cap$ on fuzzy closure operations such that $(C \cap C')(A) = C(A) \wedge C'(B)$. The following theorem shows that $C \cap C'$ is not a fuzzy closure operator on X.

Example 3.5. Let (X, C) be the fuzzy closure space defined in Example 3.3. Define a fuzzy closure operator C' on X as follows:

$$C'(A) = \begin{cases} \overline{0} & \text{if } A = \overline{0} \\ \wedge\{B \mid suppB = \{1,2\}\} & \text{if } suppA = \{1\}, \{2\} \text{ or } \{1,2\} \\ \wedge\{B \mid suppB = \{3,4\}\} & \text{if } suppA = \{3\} \text{ or } \{4\} \\ \wedge\{B \mid suppB = \{1,3,4\}\} & \text{if } suppA = \{1,3\} \text{ or } \{1,3,4\} \\ \wedge\{B \mid suppB = \{1,2,4\}\} & \text{if } suppA = \{1,4\}, \{2,4\} \text{ or } \{1,2,4\} \\ \wedge\{B \mid suppB = \{1,2,3\}\} & \text{if } suppA = \{1,2,3\} \\ \overline{1} & \text{if } suppA = \{2,3\}, \{3,4\}, \{2,3,4\} \text{ or } X. \end{cases}$$

Let $E = p_3^{0.2}$ and $F = p_4^{0.5}$. Then,

$$\begin{aligned}(C \cap C')(E \vee F) &= C(E \vee F) \wedge C'(E \vee F)\\ &= (\wedge\{B \mid suppB = \{2,3,4\}\}) \wedge \overline{1}\\ &= (\wedge\{B \mid suppB = \{2,3,4\}\}),\end{aligned}$$

$$\begin{aligned}(C \cap C')(E) \vee (C \cap C')(F) &= \big(C(E) \wedge C'(E)\big) \vee \big(C(F) \wedge C'(F)\big)\\ &= (\wedge\{B \mid suppB = \{3\}\}) \vee (\wedge\{B \mid suppB = \{4\}\})\\ &= (\wedge\{B \mid suppB = \{3,4\}\}).\end{aligned}$$

So, $(C \cap C')(E \vee F) \neq (C \cap C')(E) \vee (C \cap C')(F)$.

The following theorem defines a fuzzy closure operator, named a quotient fuzzy closure operator, on a set Y with the help of a fuzzy closure space (X, C) and a surjective function between X and Y.

Theorem 3.6. *Let (X, C) be a fuzzy closure space and $f : X \to Y$ be a surjective map. Let define a map $C_f : I^Y \to I^Y$ such that, for all $y \in Y$*

$$C_f(A)(y) = \begin{cases} \sup\limits_{x \in f^{-1}(\{y\})} C(f^{-1}(A))(x) & , f^{-1}(\{y\}) \neq \emptyset \\ 0 & , \textit{otherwise.} \end{cases}$$

Then, C_f is a fuzzy closure operator on Y.

Proof. $C_f(\overline{0}) = \overline{0}$ by the definition of C_f. Let $A \in I^Y$. Since C is a fuzzy closure operator on X, $f^{-1}(A) \prec C(f^{-1}(A))$. Therefore,

$$A = ff^{-1}(A) \prec fC(f^{-1}(A)) = C_f(A).$$

Let $A, B \in I^Y$. $C_f(A \vee B)(y) = 0$ if $f^{-1}(y) = \emptyset$. Then, $C_f(A \vee B)(y) = C_f(A) \vee C_f(B)$. Let $f^{-1}(y) \neq \emptyset$. Then,

$$\begin{aligned}C_f(A \vee B)(y) &= \sup_{x \in f^{-1}(\{y\})} C(f^{-1}(A \vee B))(x)\\ &= \sup_{x \in f^{-1}(\{y\})} C(A \vee B)(f(x))\\ &= \sup_{x \in f^{-1}(\{y\})} (C(A) \vee C(B))(f(x))\\ &= sup\left\{\sup_{x \in f^{-1}(\{y\})} C(A(f(x))), \sup_{x \in f^{-1}(\{y\})} C(B(f(x)))\right\}\\ &= sup\left\{\sup_{x \in f^{-1}(\{y\})} C\left(f^{-1}(A(x))\right), \sup_{x \in f^{-1}(\{y\})} C\left(f^{-1}(B(x))\right)\right\}\\ &= sup\left\{C_f(A)(y), C_f(B)(y)\right\}\\ &= \left(C_f(A) \vee C_f(B)\right)(y).\end{aligned}$$

□

The fuzzy closure space (Y, C_α) defined as Theorem 3.6 is named as fuzzy quotient closure space.

In [21], product fuzzy closure space is defined as below:

Definition 3.7. Let $\{(X_i, C_i) \mid i \in J\}$ be a family of fuzzy closure spaces and $\prod\limits_{i \in J} X_i = X$. Let $C_\Pi : I^X \longrightarrow I^X$ be defined as:

$p_x^r \in C_\Pi(A)$ if $\pi_i(p_x^r) \in C_i(\pi_i(A_j))$ for $\exists j = 1, .., n, \forall i \in J$ where $A = A_1 \vee .. \vee A_n$. Then, (X, C_Π) is called a product fuzzy closure space.

4. Fuzzy Closure Hopf Spaces

This section first defines the homotopy concept of fuzzy closure spaces. Then, with the help of fuzzy homotopy, we construct the Hopf space structure on fuzzy closure spaces.

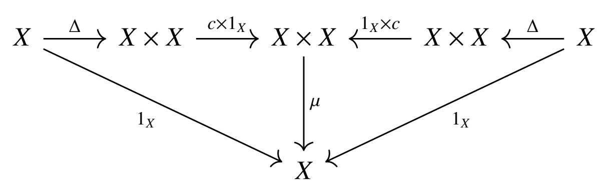
Definition 4.1. Let (X, C_X) and (Y, C_Y) be fuzzy closure spaces. Continuous functions $f, g : (X, C_X) \to (Y, C_Y)$ are called fuzzy homotopic if there exists a continuous map

$$F : (X \times I, C_\Pi) \to (Y, C_Y)$$

such that $F(x, 0) = f$ and $F(x, 1) = g$. Then, F is called fuzzy homotopy between f and g, denoted by $f \overset{F}{\sim} g$.
If $f \overset{F}{\sim} g$ then $h \circ f \overset{H}{\sim} h \circ g$ with the fuzzy homotopy $H = h \circ F$ for any continuous function $h : (Y, C_Y) \to (Z, C_Z)$. Let (X, C_X) and (Y, C_Y) be fuzzy closure spaces. If there exist functions $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \sim 1_Y$ and $g \circ f \sim 1_X$, then (X, C_X) and (Y, C_Y) are said to have the same homotopy type. Then, f and g are called homotopy equivalences. The homotopy relation " $\sim$ " is an equivalence relation. The set of homotopy class of f is denoted by $[f]$, and the set of all homotopy classes of the functions from (X, C_X) to (Y, C_Y) is denoted by $[(X, C_X), (Y, C_Y)]$.

Example 4.2. Let $(\mathbb{R}, C_\mathbb{R})$ be fuzzy closure space with $C_\mathbb{R}(A)(x) = sup\{A(y) \mid y \in \mathbb{R}\}$. Let $f, g : (\mathbb{R}, C_\mathbb{R}) \to (\mathbb{R}, C_\mathbb{R})$ be fuzzy continuous functions and $F : \mathbb{R} \times I \to \mathbb{R}$ be defined as $F(x, t) = tg(x) + (1 - t)f(x)$. Then $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$. Therefore, f and g are fuzzy homotopic.

Definition 4.3. Let (X, p^r_α, C_X) be a pointed fuzzy closure space, $\mu : X \times X \to X$ be continuous and $c : X \to X$ be a constant function such that $c(x) = \alpha$ for all $x \in X$. If the following diagram is homotopy commutative:



i.e.

$$\mu \circ (c \times 1_X) \circ \Delta \sim 1_X \sim \mu \circ (1_X \times c) \circ \Delta.$$

Then, (X, p^r_α, C_X) is called as a fuzzy closure Hopf space. We will refer to it briefly as an FCH-space. $\Delta(x) = (x, x)$ is the diagonal map. Also c is called homotopy identity of (X, p^r_α, C_X).

We use the notations μ_X and c_X for the continuous multiplication and homotopy identity of the FCH space (X, p^r_α, C_X) to confusion, in the case of more than one FCH space.

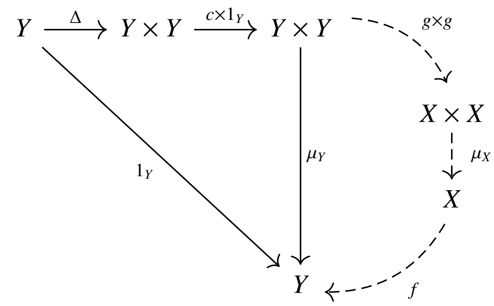
Theorem 4.4. *Let (X, p^r_α, C_X) be a FCH space and (Y, p^r_β, C_Y) has the same homotopy type with X. Then, (Y, p^r_β, C_Y) is a FCH space.*

Proof. Let $f : X \to Y$ and $g : Y \to X$ be homotopy equivalences and c_X be the homotopy identity of (X, p^r_α, C_X). Let

$$\mu_Y : Y \times Y \xrightarrow{g \times g} X \times X \xrightarrow{\mu_X} X \xrightarrow{f} Y$$

and $c_Y(y) = \beta$ for all $y \in Y$. Then,

$$\begin{aligned} \mu_Y \circ (1_Y \times c_Y) \circ \Delta &= (f \circ \mu_X \circ (g \times g)) \circ (1_Y \times c_Y) \circ \Delta \\ &= f \circ (\mu_X \circ (1_X \times c_X)) \circ \Delta \circ g \\ &\sim f \circ 1_X \circ g \\ &= f \circ g \\ &\sim 1_Y. \end{aligned}$$

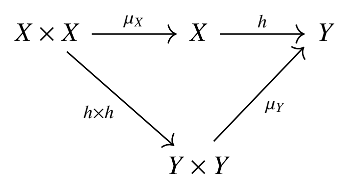


By a similar way, $\mu_Y \circ (c_Y \times 1_Y) \circ \Delta \sim 1_Y$. Therefore, (Y, p^r_β, C_Y) is a FCH space with the multiplication μ_Y and homotopy identity c_Y. □

Definition 4.5. Let (X, p^r_α, C_X) and (Y, p^t_β, C_Y) be FCH spaces. A function

$$h : (X, p^r_\alpha, C_X) \to (Y, p^t_\beta, C_Y)$$

is called fuzzy Hopf homomorphism if $h \circ \mu_X \sim \mu_Y \circ (h \times h)$, which means the following diagram is homotopy commutative:



Theorem 4.6. *Let $g : (X, p^r_\alpha, C_X) \to (Y, p^t_\beta, C_Y)$ and $h : (Y, p^t_\beta, C_Y) \to (Z, p^s_\delta, C_Z)$ be continuous fuzzy Hopf homomorphisms. Then, $h \circ g$ is a fuzzy Hopf homomorphism.*

Proof. Since g and h are fuzzy Hopf homomorphisms, there exist fuzzy homotopies $F : X \times X \times I \longrightarrow Y$ and $G : Y \times Y \times I \longrightarrow Z$ such that

$$g \circ \mu_X \overset{F}{\sim} \mu_Y \circ (g \times g) \quad \text{and} \quad h \circ \mu_Y \overset{G}{\sim} \mu_Z \circ (h \times h).$$

Define $F' : X \times X \times I \longrightarrow Z$ such that $F' = h \circ F$. Then,

$$\begin{aligned} F'((x,y),0) &= (h \circ F)((x,y),0) \\ &= h(F((x,y),0)) \\ &= h(g \circ \mu_X)(x,y) \\ &= (h \circ g \circ \mu_X)(x,y), \end{aligned}$$

and

$$\begin{aligned} F'((x,y),1) &= (h \circ F)((x,y),1) \\ &= h(F((x,y),1))) \\ &= h(\mu_Y \circ (g \times g))(x,y) \\ &= (h \circ \mu_Y \circ (g \times g))(x,y). \end{aligned}$$

Also, define $G' : X \times X \times I \longrightarrow Z$ such that $G' = G \circ (g \times g \times 1_{[0,1]})$. Then,

$$\begin{aligned} G'((x,y),0) &= G \circ (g \times g \times 1_{[0,1]})((x,y),0) \\ &= G((g \times g)(x,y),0) \\ &= (h \circ \mu_Y \circ (g \times g))(x,y) \end{aligned}$$

and

$$\begin{aligned} G'((x,y),1) &= G \circ (g \times g \times 1_{[0,1]})((x,y),1) \\ &= G((g \times g)(x,y),1) \\ &= G((g(x),g(y)),1) \\ &= (\mu_Z \circ (h \times h))(g(x),g(y)) \\ &= (\mu_Z \circ (h \times h) \circ (g \times g))(x,y) \\ &= (\mu_Z \circ ((h \circ g) \times (h \circ g)))(x,y). \end{aligned}$$

Now, let define $H : X \times X \times I \longrightarrow Z$ such that

$$H((x,y),t) = \begin{cases} F'((x,y),2t) & , 0 \leq t \leq \frac{1}{2} \\ G'((x,y),2t-1) & , \frac{1}{2} \leq t \leq 1. \end{cases}$$

Then,

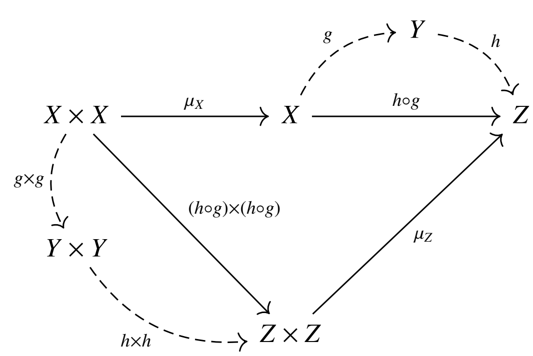
$$H((x,y),0) = F'((x,y),0) = (h \circ g \circ \mu_X)(x,y),$$

$$H((x,y),1) = G'((x,y),1) = (\mu_Z \circ ((h \circ g) \times (h \circ g)))(x,y).$$

Therefore, the homotopy

$$h \circ g \circ \mu_X \overset{H}{\sim} \mu_Z \circ ((h \circ g \times h \circ g))$$

is provided, which means the following diagram is homotopy commutative:



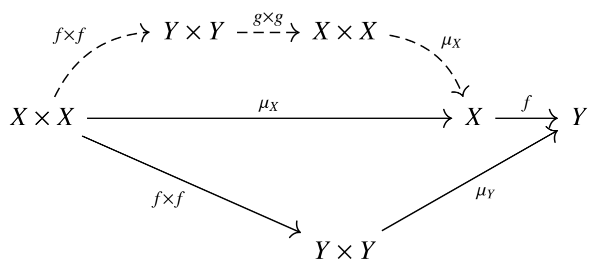
□

Theorem 4.7. *Let (X, p^r_α, C_X) be a FCH space and (Y, p^t_β, C_Y) has the same homotopy type as X. Then, homotopy equivalences are fuzzy Hopf homomorphisms.*

Proof. Let $f : X \to Y$ and $g : Y \to X$ be homotopy equivalences. Let $\mu_Y = f \circ \mu_X \circ (g \times g)$. Then, (Y, p^t_β, C_Y) is a FCH space by the Theorem 4.4. Let show f is a fuzzy Hopf homomorphism:

$$\begin{aligned} f \circ \mu_X &= f \circ \mu_X \circ 1_{X \times X} \\ &\sim f \circ \mu_X \circ ((g \circ f) \times (g \circ f)) \\ &= f \circ \mu_X \circ (g \times g) \circ (f \times f) \\ &= \mu_Y \circ (f \times f). \end{aligned}$$

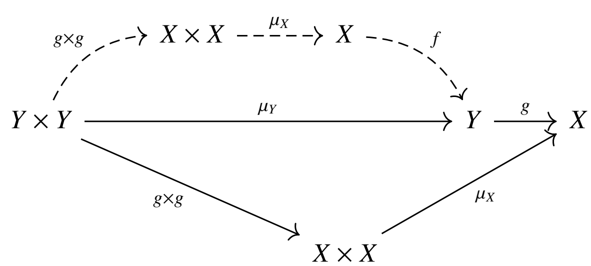
Therefore, f is a fuzzy Hopf homomorphism, shown in the following diagram:



Let show g is a fuzzy Hopf homomorphism:

$$g \circ \mu_Y = g \circ (f \circ \mu_X \circ (g \times g)) \sim 1_X \circ \mu_X \circ (g \times g) = \mu_X \circ (g \times g).$$

Therefore, g is a fuzzy Hopf homomorphism, shown in the following diagram:



□

The following theorem states that the product of two FCH spaces is also an FCH space.

Theorem 4.8. *Let (X, p^r_α, C_X) and (Y, p^r_β, C_Y) be FCH spaces. Then, $X \times Y$ is an FCH space.*

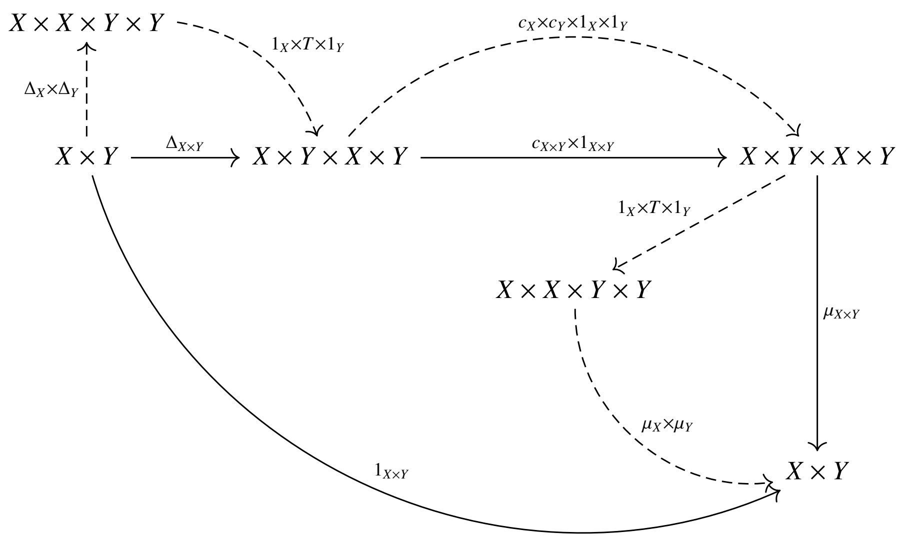
Proof. Let $\Delta_{X\times Y} = (1_X \times T \times 1_Y) \circ (\Delta_X \times \Delta_Y)$, where $T(u, v) = (v, y)$. Define

$$\begin{aligned} \mu_{X\times Y} &= (\mu_X \times \mu_Y) \circ (1_X \times T \times 1_Y) \\ c_{X\times Y} &= c_X \times c_Y. \end{aligned}$$

Then,

$$\begin{aligned} \mu_{X\times Y} \circ (c_{X\times Y} \times 1_{X\times Y}) \circ \Delta_{X\times Y} &= (\mu_X \circ (c_X \times 1_X) \circ \Delta_X) \times (\mu_Y \circ (c_Y \times 1_Y) \circ \Delta_Y) \\ &\sim 1_X \times 1_Y \\ &= 1_{X\times Y}. \end{aligned}$$

Therefore, $(X \times Y, (p^r_\alpha, p^r_\beta), C_{X\times Y})$ is a FCH space.

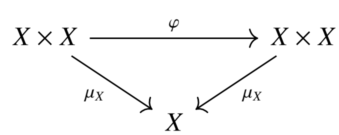


□

Definition 4.9. A FCH space (X, p^r_α, C_X) is called an abelian FCH space if there exists a map

$$\varphi : X \times X \to X \times X,\ \varphi(u, v) = (v, u)$$

makes the following diagram homotopy commutative:



which means $\mu_X \circ \varphi \sim \mu_X$.

Theorem 4.10. *A pointed fuzzy closure space having the same homotopy type as an abelian FCH space is itself an abelian FCH space.*

Proof. Let a FCH space (X, p^r_α, C_X) be abelian, (Y, p^t_β, C_Y) be a pointed fuzzy closure space and $f : X \to Y$, $g : Y \to X$ be homotopy equivalences. Then, (Y, p^t_β, C_Y) is a FCH space with the multiplication $\mu_Y = f \circ \mu_X \circ (g \times g)$, by Theorem 4.4 and $\mu_X \circ \varphi_X \sim \mu_X$ for a map $\varphi_X(u, v) = (v, u)$. Let $\varphi_Y : Y \times Y \to Y \times Y, \varphi_Y(w, z) = (z, w)$. Therefore

$$\begin{aligned} \mu_Y \circ \varphi_Y &= f \circ \mu_X \circ (g \times g) \circ \varphi_Y \\ &= f \circ \mu_X \circ \varphi_X \circ (g \times g) \\ &\sim f \circ \mu_X \circ (g \times g) \\ &= \mu_Y. \end{aligned}$$

Therefore (Y, p^t_β, C_Y) is an abelian FCH space. □

Theorem 4.11. *Let (X, p^r_α, C_X) be a FCH space. The set $[(Y, p^r_\beta, C_Y); (X, p^r_\alpha, C_X)]$ is a monoid for every pointed fuzzy closure space (Y, p^r_β, C_Y).*

Proof. To prove $[(Y, p^r_\beta, C_Y); (X, p^r_\alpha, C_X)]$ is a monoid, let define a product $\circledcirc$ on $[(Y, p^r_\beta, C_Y); (X, p^r_\alpha, C_X)]$ such that $[f] \circledcirc [g] = [\mu_X \circ (f \times g) \circ \Delta]$, for all $[f], [g] \in [(Y, p^r_\beta, C_Y); (X, p^r_\alpha, C_X)]$.

First, prove that $\circledcirc$ is well defined. Let $f \overset{G}{\sim} f'$ and $g \overset{H}{\sim} g'$. Let $F : (Y \times I, C_{(Y\times I)}) \to (X, C_X)$ be defined as $F = \mu_X \circ (G, H)$. Then,

$$\begin{aligned} F(y,0) &= \mu_X \circ (G,H)(y,0) \\ &= \mu_X(G(y,0), H(y,0)) \\ &= \mu_X(f(y), g(y)) \\ &= (\mu_X \circ (f \times g) \circ \Delta)(y), \end{aligned}$$

$$\begin{aligned} F(y,1) &= \mu_X \circ (G,H)(y,1) \\ &= \mu_X(G(y,1), H(y,1)) \\ &= \mu_X(f'(y), g'(y)) \\ &= (\mu_X \circ (f' \times g') \circ \Delta)(y). \end{aligned}$$

Then, $\mu_X \circ (f \times g) \circ \Delta \sim \mu_X \circ (f' \times g') \circ \Delta$. Therefore,

$$[f] \circledcirc [g] = [\mu_X \circ (f \times g) \circ \Delta] = [\mu_X \circ (f' \times g') \circ \Delta] = [f'] \circledcirc [g'].$$

Let $\varepsilon : Y \to X, \varepsilon(y) = \alpha$ for all $y \in Y$. Then, for any $[f] \in [(Y, p^r_\beta, C_Y); (X, p^r_\alpha, C_X)]$,

$$[f] \circledcirc [\varepsilon] = [\mu_X \circ (f \times \varepsilon) \circ \Delta] = [\mu_X \circ (1_X \times c) \circ \Delta \circ f] = [1_X \circ f] = [f],$$

$$[\varepsilon] \circledcirc [f] = [\mu_X \circ (\varepsilon \times f) \circ \Delta] = [\mu_X \circ (c \times 1_X) \circ \Delta \circ f] = [1_X \circ f] = [f].$$

Then, $[\varepsilon]$ is the unit element of $[(Y, p^r_\beta, C_Y); (X, p^r_\alpha, C_X)]$ for $\circledcirc$.
Let show $\circledcirc$ is associative:

$$\begin{aligned} [f] \circledcirc ([g] \circledcirc [h]) &= ([f] \circledcirc (\mu_X \circ (g \times h) \circ \Delta)] \\ &= [\mu_X \circ (f \times (\mu_X \circ (g \times h) \circ \Delta)) \circ \Delta] \\ &= [\mu_X \circ (1_X \times \mu_X) \circ (f \times g \times h) \circ (1_X \times \Delta) \circ \Delta] \\ &= [\mu_X \circ (\mu_X \times 1_X) \circ (f \times g \times h) \circ (1_X \times \Delta) \circ \Delta] \\ &= [\mu_X \circ ((\mu_X \circ (f \times g) \circ \Delta) \times h) \circ \Delta] \\ &= ([\mu_X \circ (f \times g) \circ \Delta] \circledcirc [h]) \\ &= ([f] \circledcirc [g]) \circledcirc [h]. \end{aligned}$$

Consequently, $([(Y, p^r_\beta, C_Y); (X, p^r_\alpha, C_X)], \circledcirc)$ is a monoid. □

The following theorem states that we can construct a Hopf space structure on any set with a surjective function between itself and an FCH space.

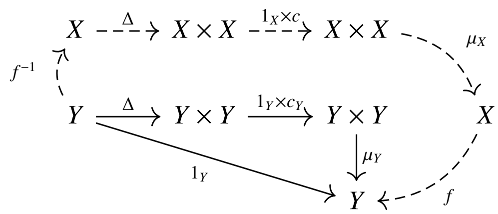
Theorem 4.12. *Let (X, p^r_α, C) be a FCH space and f be a surjective map from X to a nonempty set Y. Then, an FCH space structure can be built on Y with the help of f and C_X.*

Proof. Let $f(p^r_\alpha) = p^r_{f(\alpha)} = p^r_\beta$ and C_f be the quotient fuzzy closure operator on Y. Then, (Y, p^r_α, C_f) is a pointed fuzzy closure space. Define

$$\mu_Y = f \circ \mu_X \circ (f^{-1} \times f^{-1})$$

and $c_Y(y) = \beta$ for all $y \in Y$. Then,

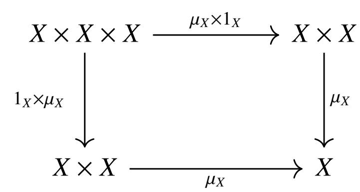
$$\begin{aligned} \mu_Y \circ (1_Y \times c_Y) \circ \Delta &= (f \circ \mu_X \circ (f^{-1} \times f^{-1})) \circ (1_Y \times c_Y) \circ \Delta \\ &= f \circ (\mu_X \circ (1_X \times c)) \circ \Delta \circ f^{-1} \\ &\sim f \circ 1_X \circ f^{-1} \\ &= f \circ f^{-1} \\ &= 1_Y. \end{aligned}$$



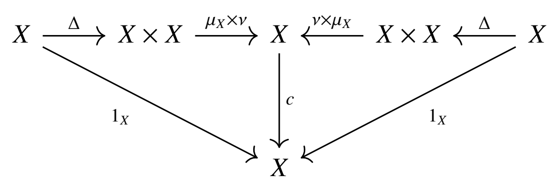
Similarly, $\mu_Y \circ (c_Y \times 1_Y) \circ \Delta = 1_Y$. Therefore, (Y, p^r_β, C_f) is a FCH space. □

4.1. **Fuzzy Closure Hopf Group.** A Hopf group is a group-like structure obtained by Hopf [14] by homotopy. This section constructs the Hopf group structure on pointed fuzzy closure spaces.

Definition 4.13. Let (X, p^r_α, C_X) be a FCH space with the multiplication μ_X. If $\mu_X \circ (\mu_X \times 1_X) \sim \mu_X \circ (1_X \times \mu_X)$, i.e. the following diagram is homotopy commutative:



then μ_X is called associative multiplication. A continuous function $\nu : X \to X$ such that $\mu_X \circ (\nu, 1_X) \sim c \sim \mu_X \circ (1_X, \nu)$, is called homotopy inverse, making the following diagram homotopy commutative:

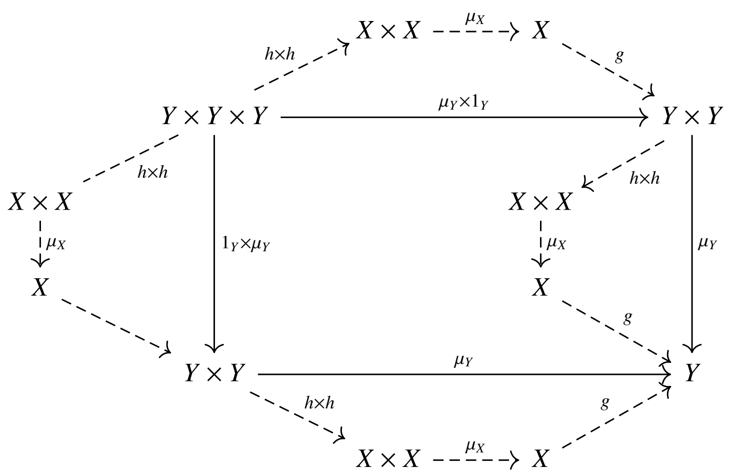


A fuzzy closure Hopf group (briefly FCH-group) is an FCH space with a homotopy associative multiplication and homotopy inverse.

Theorem 4.14. *Let (X, p^r_α, C_X) be a FCH group and (Y, p^r_β, C_Y) has the same homotopy type with (X, p^r_α, C_X). Then, (Y, p^r_β, C_Y) is a FCH group.*

Proof. Let g and h are homotopy equivalences and $\mu_Y = g \circ \mu_X \circ (h \times h)$ be continuous multiplication of (Y, p^r_β, C_Y). Then, (Y, p^r_β, C_Y) is a FCH space by Theorem 4.4.
Now, let show that μ_Y is homotopy associative:

$$\begin{aligned}
\mu_Y \circ (\mu_Y \times 1_Y) &= \Big(g \circ \mu_X \circ (h \times h)\Big) \circ \Big((g \circ \mu_X \circ (h \times h)) \times 1_Y\Big) \\
&= (g \circ \mu_X) \circ (h \circ g) \circ \Big((\mu_X \circ (h \times h) \times h\Big) \\
&\sim (g \circ \mu_X) \circ \Big((\mu_X \circ (h \times h)) \times h\Big) \\
&= g \circ (\mu_X \circ (\mu_X \times 1_X)) \circ (h \times h \times h) \\
&\sim g \circ (\mu_X \circ (1_X \times \mu_X)) \circ (h \times h \times h) \\
&= \Big(g \circ \mu_X \circ (h \times h)\Big) \circ \Big(1_Y \times (g \circ \mu_X \circ (h \times h))\Big) \\
&= \mu_Y \circ (1_Y \times \mu_Y).
\end{aligned}$$



Now, let show that (Y, p^r_β, C_Y) has a homotopy inverse:
Let ν be the homotopy inverse of (X, p^r_α, C_X) and $\nu' = g \circ \nu \circ h$. Then,

$$\begin{aligned}
\mu_Y \circ (1_Y \times \nu') \circ \Delta &= (g \circ \mu_X \circ (h \times h)) \circ (1_Y \times (g \circ \nu \circ h)) \circ \Delta \\
&= (g \circ \mu_X) \circ (h \times (h \circ g \circ \nu \circ h)) \circ \Delta \\
&\sim (g \circ \mu_X) \circ (h \times (\nu \circ h)) \circ \Delta \\
&= g \circ (\mu_X \circ (1_X \times \nu)) \circ (h \times h) \circ \Delta \\
&\sim g \circ (\mu_X \circ (\nu \times 1_X)) \circ (h \times h) \circ \Delta \\
&= (g \circ \mu_X) \circ (\nu \circ (h \times h)) \circ \Delta \\
&\sim (g \circ \mu_X) \circ (h \circ g \circ \nu \circ (h \times h)) \circ \Delta \\
&= (g \circ \mu_X \circ (h \times h)) \circ ((g \circ \nu \circ h) \times 1_Y) \circ \Delta \\
&= \mu_Y \circ (\nu' \times 1_Y) \circ \Delta.
\end{aligned}$$

Therefore, (Y, p^r_β, C_Y) is a FCH-group. □

Theorem 4.15. *Let (X, p^r_α, C_X) be a FCH group. Then, the set $[(Y, p^r_\beta, C_Y); (X, p^r_\alpha, C_X)]$ is a group, for every pointed fuzzy closure space (Y, p^r_β, C_Y). If μ_X is abelian, then $[(Y, p^r_\beta, C_Y); (X, p^r_\alpha, C_X)]$ is also abelian.*

Proof. $[(Y, p^r_\beta, C_Y); (X, p^r_\alpha, C_X)]$ is a monoid by the Theorem 4.11. Let ν be the homotopy inverse of (X, p^r_α, C_X).
For any $[f] \in [(Y, p^r_\beta, C_Y); (X, p^r_\alpha, C_X)]$,

$$[f] \circledcirc [\nu \circ f] = [\mu_X \circ (f \times (\nu \circ f)) \circ \Delta] = [\mu_X \circ (1_X \times \nu) \circ \Delta \circ f] = [c \circ f] = [e],$$

$$[\nu \circ f] \circledcirc [f] = [\mu_X \circ ((\nu \circ f) \times f) \circ \Delta] = [\mu_X \circ (\nu \times 1_X) \circ \Delta \circ f] = [c \circ f] = [e].$$

So $[\nu \circ f]$ is the homotopy inverse of $[f]$. Therefore, $[(Y, p^r_\beta, C_Y); (X, p^r_\alpha, C_X)]$ is a group. Let μ_X be abelian. Then, $\mu_X \circ \varphi \sim \mu_X$ for a map $\varphi : X \times X \to X \times X$. Then,

$$\begin{aligned}
[f] \circledcirc [g] &= [\mu_X \circ (f \times g) \circ \Delta] \\
&= [\mu_X \circ \varphi \circ (f \times g) \circ \Delta] \\
&= [\mu_X \circ (g \times f) \circ \Delta] \\
&= [g] \circledcirc [f].
\end{aligned}$$

Then, $\circledcirc$ is abelian. □

Definition 4.16. The category whose objects are pointed fuzzy closure spaces and the set of morphisms is $[(X, p^r_\alpha, C_X), (Y, p^r_\beta, C_Y)]$ is called the homotopy category of the pointed fuzzy closure spaces, denoted by $\mathcal{FCH}$. The composition of the morphisms is the product defined by Theorem 4.11.

Definition 4.17 ([28])**.** A contravariant functor $\wp$ from the category $\mathcal{F}$ to the category $\mathcal{G}$ is a function, which maps each object A of $\mathcal{F}$ to an object B of $\mathcal{G}$ and each morphism $f \in hom(A, B)$ of $\mathcal{F}$ to a morphism $\wp(f) : \wp(B) \to \wp(A)$, such that $\wp(1_A) = 1_{\wp(A)}$ and $\wp(g \circ f) = \wp(f) \circ \wp(g)$.

Theorem 4.18. *Let (X, p^r_α, C_X) be a FCH group. Then, there exists a contravariant functor from $\mathcal{FCH}$ to the category of groups and homomorphisms, denoted by $\mathcal{G}$.*

Proof. Define a map $\wp^Y$ from $\mathcal{FCH}$ to the category of sets and functions, denoted by $\mathcal{S}$ such that

$$\wp^Y(Y, p^r_\beta, C_Y) = [(Y, p^r_\beta, C_Y), (X, p^r_\alpha, C_X)],$$

$$\wp^Y([g]) = g^* : [(Z, p^r_\eta, C_Z), (X, p^r_\alpha, C_X)] \to [(Y, p^r_\beta, C_Y), (X, p^r_\alpha, C_X)],$$

where $g^*([f]) = [f \circ g]$, $[g] \in [(Y, p^r_\beta, C_Y), (Z, p^r_\eta, C_Z)]$.
Let $[f], [h] \in [(Z, p^r_\eta, C_Z), (Y, p^r_\beta, C_Y)]$.

$$\begin{aligned} g^*([f] \circledcirc [h]) &= g^*([\mu_Y \circ (f \times h) \circ \Delta]) \\ &= [(\mu_Y \circ (f \times h) \circ \Delta) \circ g] \\ &= [\mu_Y \circ (f \circ g \times h \circ g)] \\ &= [f \circ g] \circledcirc [h \circ g] \\ &= g^*([f]) \circledcirc g^*([h]). \end{aligned}$$

Then, g^* is a homomorphism. Also, by the Theorem 4.11,

$$\wp^Y(X, p^r_\alpha, C_X) = [(Y, p^r_\beta, C_Y), (X, p^r_\alpha, C_X)]$$

is a group with the binary operation $\circledcirc$.
Now, let us show that $\wp^Y$ is a contravariant functor.
Let $[1_X] \in [(Y, p^r_\beta, C_Y), (Y, p^r_\beta, C_Y)]$ be the unit morphism of $\mathcal{FCH}$. Then,

$$\wp^Y([1_X] = 1^*_X : [(Y, p^r_\beta, C_Y), (X, p^r_\alpha, C_X)] \to [(Y, p^r_\beta, C_Y), (X, p^r_\alpha, C_X)]$$

and for any morphism $[f] \in [(Y, p^r_\beta, C_Y), (X, p^r_\alpha, C_X)]$, $1^*_X([f]) = [f \circ 1_X] = [f]$. So, $\wp^Y([1_X])$ is the unit morphism.
Let $[g] \in [(Y, p^r_\beta, C_Y), (X, p^r_\alpha, C_X)]$. For any morphism $[h] \in [(X', x'_0, C_{X'}), (Z, p^r_\eta, C_Z)]$,

$$\begin{aligned} \wp^Y([g \circ f])([h]) &= [h \circ (g \circ f)] = [(h \circ g) \circ f] \\ &= \wp^Y([f])([h \circ g]) \\ &= \wp^Y([f])(\wp^Y([g])([h])) \\ &= (\wp([f]) \circ \wp^Y([g]))([h]). \end{aligned}$$

Then, $\wp^Y([g \circ f]) = \wp([f]) \circ \wp^Y([g])$. Therefore, $\wp^Y$ is a contravariant functor since it preserves the composition and the identity. We conclude that there exists a contravariant functor $\wp^Y$ from $\mathcal{FCH}$ to the category of abelian groups and homomorphisms for an abelian FCH group (Y, p^r_β, C_Y). □

5. Conclusion

In this study, we have introduced the concepts of fuzzy closure Hopf spaces and fuzzy closure Hopf groups within the theoretical framework of fuzzy closure spaces, utilizing principles from homotopy theory. We have explored the interconnections between the fuzzy closure Hopf group and its homotopy equivalence.

Furthermore, we have elucidated the existence of a contravariant functor from the category of fuzzy closure Hopf spaces and continuous functions to the category of groups and homomorphisms. This demonstration highlights the structural parallels between fuzzy closed Hopf spaces and conventional groups, providing a better understanding of their algebraic properties. Our research also revealed that homotopy function classes between fuzzy closure Hopf groups form a group. By establishing the foundations of fuzzy closure Hopf spaces and groups, we have laid the groundwork for further exploration and applications in diverse areas of mathematics and beyond. These concepts offer promising avenues for future research, with potential implications in fields ranging from topology to mathematical modeling.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The author has read and agreed to the published version of the manuscript.

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