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A study on two fourth-order fuzzy problems with fuzzy coefficients

Hülya GÜLTEKİN ÇİTİL*

Giresun University, Faculty of Arts and Sciences, Department of Mathematics, Giresun, Turkey.

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Abstract

This study is on the solutions of two fourth-order fuzzy problems with positive and negative fuzzy number coefficients. The solutions are found using the fuzzy Laplace transform method. Main results are given. Two examples are solved to illustrate the problems. Graphics of the found solutions are drawn for alpha level sets. Also, the graphics are interpreted and conclusions are given.

Keywords: *Fourth-order fuzzy problem, fuzzy function, fuzzy Laplace transform method.*

Fuzzy katsayılı iki dördüncü-mertebeden fuzzy problem üzerine bir çalışma

Öz

Bu çalışma, pozitif ve negatif fuzzy sayı katsayılı iki dördüncü-mertebeden fuzzy problemin çözümleri üzerinedir. Çözümler fuzzy Laplace dönüşüm metodu kullanılarak bulundu. Temel sonuçlar verildi. Problemleri göstermek için iki örnek çözüldü. Alfa seviye setleri için bulunan çözümlerin grafikleri çizildi. Ayrıca, grafikler yorumlandı ve sonuçlar verildi.

Anahtar kelimeler: *Dördüncü-mertebe fuzzy problem, fuzzy fonksiyon, fuzzy Laplace dönüşüm metodu.*

*Hülya GÜLTEKİN ÇİTİL, hulya.citil@giresun.edu.tr, https://orcid.org/0000-0002-3543-033X

1. Introduction

Fuzzy differential equation is useful for solving differential equations in the fields of engineering, physical mathematics, mathematics. So, many resarchers study fuzzy differential equation [1-15]. Fuzzy Laplace transform was introduced by Allahviranloo and Ahmadi in 2010 [16]. They used the strongly generalized differentiability. Allahviranloo et al. obtained a new method for solving fuzzy linear differential equations. [17]. Fuzzy Laplace transform method is practically important method. So, fuzzy Laplace transform method was used by many researchers to solve fuzzy differential equations [18-23].

The aim of this study is to investigate the solutions of two fourth-order fuzzy problems with fuzzy number coefficients and to present the comparison results of the solutions.

In this study, we find the solutions directly with the fuzzy Laplace transform method and see the effect of the coefficients on the solutions.

In this work, we research the problems

$$u^{(iv)}(t) = [\mu]^{\alpha} u''(t), \tag{1}$$

$$u(0) = [\varphi]^{\alpha},\ u'(0) = [\psi]^{\alpha},\ u''(0) = [\chi]^{\alpha},\ u'''(0) = [\omega]^{\alpha} \tag{2}$$

and

$$u^{(iv)}(t) = -[\mu]^{\alpha} u''(t), \tag{3}$$

$$u(0) = [\varphi]^{\alpha},\ u'(0) = [\psi]^{\alpha},\ u''(0) = [\chi]^{\alpha},\ u'''(0) = [\omega]^{\alpha}, \tag{4}$$

by the fuzzy Laplace transform method, where

$$[\mu]^{\alpha} = \left[\underline{\mu}_{\alpha}, \overline{\mu}_{\alpha}\right], [\varphi]^{\alpha} = \left[\underline{\varphi}_{\alpha}, \overline{\varphi}_{\alpha}\right], [\psi]^{\alpha} = \left[\underline{\psi}_{\alpha}, \overline{\psi}_{\alpha}\right], [\chi]^{\alpha} = \left[\underline{\chi}_{\alpha}, \overline{\chi}_{\alpha}\right], [\omega]^{\alpha} = \left[\underline{\omega}_{\alpha}, \overline{\omega}_{\alpha}\right]$$

are symmetric triangular fuzzy numbers, $t > 0$, $u(t)$ is positive fuzzy function, $L(u(t)) = U(s)$ is the Laplace transform of fuzzy function $u(t)$. Throughout the work, u, u', u'', u''' are (i)-differentiable.

2. Preliminaries

Definition 1. [10] A fuzzy number is a mapping $u: \mathbb{R} \to [0,1]$ verifying the following properties:
u is normal, u is upper semi-continuous on $\mathbb{R}$, u is convex fuzzy set and $cl\{x \in \mathbb{R} | u(x) > 0\}$ is compact, where cl denotes the closure of a subset.

Definition 2. [10] Let $u \in \mathbb{R}_F$, where $\mathbb{R}_F$ is the space of fuzzy numbers.
$[u]^{\alpha} = \{x \in \mathbb{R} | u(x) \geq \alpha\}$, $0 < \alpha \leq 1$ is the α-level set of u.

Definition 3. [12] A fuzzy number u is a pair $\left[\underline{u}_\alpha, \overline{u}_\alpha\right]$ $0 \leq \alpha \leq 1$, which satisfy the requirements:
$\underline{u}_\alpha$ is right-continuous at $\alpha = 0$ and bounded non-decreasing left-continuous in $(0,1]$,
$\overline{u}_\alpha$ is right-continuous at $\alpha = 0$ and bounded non-increasing left-continuous in $(0,1]$ and $\underline{u}_\alpha \leq \overline{u}_\alpha$, $0 \leq \alpha \leq 1$.

Definition 4. [10] The α-level set of symmetric triangular fuzzy number W is
$[W]^\alpha = \left[\underline{w} + \left(\frac{\overline{w}-\underline{w}}{2}\right)\alpha, \overline{w} - \left(\frac{\overline{w}-\underline{w}}{2}\right)\alpha\right]$,
where $\left[\underline{w}, \overline{w}\right]$ is support of W.

Definition 5. [12] Let $u, v \in \mathbb{R}_F$. If $u = v + w$ such that $w \in \mathbb{R}_F$, then w is the H-difference of u and v. w is denoted as $u \ominus v$.

Definition 6. [22] Let $g: (a,b) \longrightarrow \mathbb{R}_F$ and $t_0 \in (a,b)$. If there exists $g'(t_0) \in \mathbb{R}_F$ such that for all h > 0 sufficiently small, there exist $g(t_0+h) \ominus g(t_0)$, $g(t_0) \ominus g(t_0-h)$ and the limits

$$\lim_{h \longrightarrow 0} \frac{g(t_0+h) \ominus g(t_0)}{h} = \lim_{h \longrightarrow 0} \frac{g(t_0) \ominus g(t_0-h)}{h} = g'(t_0),$$

g is said to be Hukuhara differentiable at t_0.

Definition 7. [22] Let $g: (a,b) \longrightarrow \mathbb{R}_F$ and $t_0 \in (a,b)$. If there exists $g'(t_0) \in \mathbb{R}_F$ such that for all $h > 0$ sufficiently small, there exist $g(t_0+h) \ominus g(t_0)$, $g(t_0) \ominus g(t_0-h)$ and the limits

$$\lim_{h \longrightarrow 0} \frac{g(t_0+h) \ominus g(t_0)}{h} = \lim_{h \longrightarrow 0} \frac{g(t_0) \ominus g(t_0-h)}{h} = g'(t_0),$$

g is (i)-differentiable at t_0.
If there exists $g'(t_0) \in \mathbb{R}_F$ such that for all $h > 0$ sufficiently small, there exist $g(t_0) \ominus g(t_0+h)$, $g(t_0-h) \ominus g(t_0)$ and the limits

$$\lim_{h \longrightarrow 0} \frac{g(t_0) \ominus g(t_0+h)}{-h} = \lim_{h \longrightarrow 0} \frac{g(t_0-h) \ominus g(t_0)}{-h} = g'(t_0),$$

g is (ii)-differentiable.

Theorem 1. [12] Let $g: [a,b] \rightarrow \mathbb{R}_F$ be fuzzy function.
$[g(x)]^\alpha = \left[\underline{g}_\alpha(x), \overline{g}_\alpha(x)\right]$, for each $\alpha \in [0,1]$.
1. If g is (i)-differentiable, $\underline{g}_\alpha$, $\overline{g}_\alpha$ are differentiable,
$\left[g'(x)\right]^\alpha = \left[\underline{g}'_\alpha(x), \overline{g}'_\alpha(x)\right]$,
2. If g is (ii)-differentiable, $\underline{g}_\alpha$, $\overline{g}_\alpha$ are differentiable,
$\left[g'(x)\right]^\alpha = \left[\overline{g}'_\alpha(x), \underline{g}'_\alpha(x)\right]$.

Theorem 2. [12] Let $g': [a,b] \rightarrow \mathbb{R}_F$ be fuzzy function.

$[g(x)]^{\alpha} = \left[\underline{g}_{\alpha}(x), \overline{g}_{\alpha}(x)\right]$, for each $\alpha \in [0,1]$,
g is (i)-differentiable or (ii)-differentiable.
1. If g, $g'$are (i)-differentiable, $\underline{g}'_{\alpha}$, $\overline{g}'_{\alpha}$ are differentiable,
$[g''(x)]^{\alpha} = \left[\underline{g}''_{\alpha}(x), \overline{g}''_{\alpha}(x)\right]$,
2. If g' is (ii)-differentiable and g is (i)-differentiable, $\underline{g}'_{\alpha}$, $\overline{g}'_{\alpha}$ are differentiable,
$[g''(x)]^{\alpha} = \left[\overline{g}''_{\alpha}(x), \underline{g}''_{\alpha}(x)\right]$,
3. If g' is (i)-differentiable and g is (ii)-differentiable, $\underline{g}'_{\alpha}$, $\overline{g}'_{\alpha}$ are differentiable,
$[g''(x)]^{\alpha} = \left[\overline{g}''_{\alpha}(x), \underline{g}''_{\alpha}(x)\right]$,
4. If g and g' are (ii)-differentiable, $\underline{g}'_{\alpha}$, $\overline{g}'_{\alpha}$ are differentiable,
$[g''(x)]^{\alpha} = \left[\underline{g}''_{\alpha}(x), \overline{g}''_{\alpha}(x)\right]$.

Definition 8. [24]

$$G(s) = L(g(t)) = \int_0^{\infty} e^{-st}\, g(t)dt = \left[\lim_{\rho \to \infty} \int_0^{\rho} e^{-st}\, \underline{g}(t)dt\, , \lim_{\rho \to \infty} \int_0^{\rho} e^{-st}\, \overline{g}(t)dt\right]$$

is the fuzzy Laplace transform of fuzzy function g, where

$$G(s,\alpha) = L([g(t)]^{\alpha}) = \left[L\left(\underline{g}_{\alpha}(t)\right), L\left(\overline{g}_{\alpha}(t)\right)\right],$$

$$L\left(\underline{g}_{\alpha}(t)\right) = \int_0^{\infty} e^{-st}\, \underline{g}_{\alpha}(t)dt = \lim_{\rho \to \infty} \int_0^{\rho} e^{-st}\, \underline{g}_{\alpha}(t)dt,$$

$$L\left(\overline{g}_{\alpha}(t)\right) = \int_0^{\infty} e^{-st}\, \overline{g}_{\alpha}(t)dt = \lim_{\rho \to \infty} \int_0^{\rho} e^{-st}\, \overline{g}_{\alpha}(t)dt.$$

Theorem 3. [25] Let $g, g', \ldots, g^{(n-1)}$ be continuous fuzzy-valued functions on $[0, \infty)$ and of exponential order and let $g^{(\mathrm{n})}$ be piecewise continuous fuzzy-valued function on $[0, \infty)$. If $g, g', \ldots, g^{(n-1)}$ are (i)-differentiable,

$$L\left(g^{(n)}(t)\right) = s^n L(g(t)) \ominus s^{n-1} g(0) \ominus s^{n-2} g'(0) \ominus s^{n-3} g''(0) \ominus \ldots \ominus g^{(n-1)}(0),$$

if $g, g', \ldots, g^{(n-2)}$ are (i)-differentiable and $g^{(n-1)}$ is (ii)-differentiable,

$$L\left(g^{(n)}(t)\right) = \ominus \left(g^{(n-1)}(0)\right) \ominus (-s^n) L(g(t)) \ominus s^{n-1} g(0) \ominus s^{n-2} g'(0) \ominus \ldots \ominus s^{n-(n-1)} g^{(n-2)}(0),$$

if $g, g', \ldots, g^{(n-3)}$ are (i)-differentiable and $g^{(n-1)}$, $g^{(n-2)}$ are (ii)-differentiable,

$$L\left(g^{(n)}(t)\right) = \ominus (s^{n-(n-1)} g^{(n-2)}(0)) \ominus g^{(n-1)}(0) \ominus (-s^n) L(g(t)) \ominus s^{n-1} g(0) \ominus s^{n-2} g'(0) \ominus \ldots \ominus (s^{n-(n-2)}) g^{(n-3)}(0).$$

Similarly, if g is (ii)-differentiable and $g', \ldots, g^{(n-1)}$ are (i)-differentiable,

$$L\left(g^{(n)}(t)\right) = \ominus \left(s^{n-1}g(0)\right) \ominus (-s^n)L\big(g(t)\big) \ominus s^{n-2}g'(0) \ominus \ldots \ominus g^{(n-1)}(0).$$

Continuing the process until we obtain 2^n system of differential equations,
if $g, g', \ldots, g^{(n-1)}$ are (ii)-differentiable, the last equation is

$$L\left(g^{(n)}(t)\right) = s^n L\big(g(t)\big) \ominus s^{n-1}g(0) \ominus s^{n-2}g'(0) \ominus s^{n-3}g''(0) \ldots - g^{(n-1)}(0).$$

Theorem 4. [16] If $g(t)$, $h(t)$ are continuous fuzzy-valued functions and c_1 and c_2 are constants, then

$$L\big(c_1 g(t) + c_2 h(t)\big) = c_1 L\big(g(t)\big) + c_2 L\big(h(t)\big).$$

3. Main Results

3.1. The problem (1)-(2)

From the equation (1), using the fuzzy Laplace transform method, we have the equations

$$s^4 \underline{U}_\alpha(s) - s^3 \underline{u}_\alpha(0) - s^2 \underline{u}_\alpha'(0) - s\underline{u}_\alpha''(0) - \underline{u}_\alpha'''(0) = \underline{\mu}_\alpha \left(s^2 \underline{U}_\alpha(s) - s\underline{u}_\alpha(0) - \underline{u}_\alpha'(0)\right),$$

$$s^4 \overline{U}_\alpha(s) - s^3 \overline{u}_\alpha(0) - s^2 \overline{u}_\alpha'(0) - s\overline{u}_\alpha''(0) - \overline{u}_\alpha'''(0) = \overline{\mu}_\alpha \left(s^2 \overline{U}_\alpha(s) - s\overline{u}_\alpha(0) - \overline{u}_\alpha'(0)\right).$$

Using the initial conditions (2),

$$\underline{U}_\alpha(s) = \frac{\underline{\varphi}_\alpha}{s} + \frac{\underline{\psi}_\alpha}{s^2} + \frac{\underline{\chi}_\alpha}{s\left(s^2 - \underline{\mu}_\alpha\right)} + \frac{\underline{\omega}_\alpha}{s^2\left(s^2 - \underline{\mu}_\alpha\right)},$$

$$\overline{U}_\alpha(s) = \frac{\overline{\varphi}_\alpha}{s} + \frac{\overline{\psi}_\alpha}{s^2} + \frac{\overline{\chi}_\alpha}{s\left(s^2 - \overline{\mu}_\alpha\right)} + \frac{\overline{\omega}_\alpha}{s^2\left(s^2 - \overline{\mu}_\alpha\right)}$$

are obtained. From this, the solution is

$$\underline{u}_\alpha(t) = \underline{\varphi}_\alpha + \underline{\psi}_\alpha t + \frac{\underline{\chi}_\alpha}{\underline{\mu}_\alpha}\left(\cosh\left(\sqrt{\underline{\mu}_\alpha}t\right) - 1\right) + \frac{\underline{\omega}_\alpha}{\underline{\mu}_\alpha}\left(\frac{sinh\left(\sqrt{\underline{\mu}_\alpha}t\right)}{\sqrt{\underline{\mu}_\alpha}} - t\right),$$

$$\overline{u}_\alpha(t) = \overline{\varphi}_\alpha + \overline{\psi}_\alpha t + \frac{\overline{\chi}_\alpha}{\overline{\mu}_\alpha}\left(\cosh\left(\sqrt{\overline{\mu}_\alpha}t\right) - 1\right) + \frac{\overline{\omega}_\alpha}{\overline{\mu}_\alpha}\left(\frac{sinh\left(\sqrt{\overline{\mu}_\alpha}t\right)}{\sqrt{\overline{\mu}_\alpha}} - t\right),$$

$[u(t)]^{\alpha} = [\underline{u}_{\alpha}(t), \overline{u}_{\alpha}(t)].$

Example 1. Consider the problem

$u^{(iv)}(t) = [1]^{\alpha} u''(t),$

$u(0) = [0]^{\alpha} = [-1+\alpha, 1-\alpha],$
$u'(0) = [1]^{\alpha} = [\alpha, 2-\alpha],$
$u''(0) = [2]^{\alpha} = [1+\alpha, 3-\alpha],$
$u'''(0) = [3]^{\alpha} = [2+\alpha, 4-\alpha].$

The solution is

$$\underline{u}_{\alpha}(t) = -1 + \alpha + \alpha t + \left(\frac{1}{\alpha} + 1\right)\left(\cosh(\alpha^{1/2} t) - 1\right) + \left(\frac{2}{\alpha} + 1\right)\left(\frac{sinh(\alpha^{1/2} t)}{\alpha^{1/2}} - t\right), \tag{5}$$

$$\overline{u}_{\alpha}(t) = \left(\frac{1}{2-\alpha} + 1\right)\left(\cosh((2-\alpha)^{1/2} t) - 1\right) + \left(\frac{2}{2-\alpha} + 1\right)\left(\frac{sinh((2-\alpha)^{1/2} t)}{(2-\alpha)^{1/2}} - t\right) + (2-\alpha)t + 1 - \alpha, \tag{6}$$

$$[u(t)]^{\alpha} = [\underline{u}_{\alpha}(t), \overline{u}_{\alpha}(t)]. \tag{7}$$

According to Definition 3 and since $u(t)$ is positive fuzzy function, $u(t)$ is a valid fuzzy function for $t > 0.5051162150589951$ in Figure 1.

3.2. The problem (3)-(4)

From the equation (3), the equations

$$s^4 \underline{U}_{\alpha}(s) - s^3 \underline{u}_{\alpha}(0) - s^2 \underline{u}_{\alpha}'(0) - s \underline{u}_{\alpha}''(0) - \underline{u}_{\alpha}'''(0) = -s^2 \overline{\mu}_{\alpha} \overline{U}_{\alpha}(s) - s \overline{\mu}_{\alpha} \overline{u}_{\alpha}(0) - \overline{\mu}_{\alpha} \overline{u}_{\alpha}'(0),$$

$$s^4 \overline{U}_{\alpha}(s) - s^3 \overline{u}_{\alpha}(0) - s^2 \overline{u}_{\alpha}'(0) - s \overline{u}_{\alpha}''(0) - \overline{u}_{\alpha}'''(0) = -s^2 \underline{\mu}_{\alpha} \underline{U}_{\alpha}(s) - s \underline{\mu}_{\alpha} \underline{u}_{\alpha}(0) - \underline{\mu}_{\alpha} \underline{u}_{\alpha}'(0).$$

are obtained. Using the initial conditions (4), we have the equations

$$s^2 \underline{U}_{\alpha}(s) + \overline{\mu}_{\alpha} \overline{U}_{\alpha}(s) = s \underline{\varphi}_{\alpha} + \underline{\psi}_{\alpha} + \frac{\underline{\chi}_{\alpha}}{s} + \frac{\underline{\omega}_{\alpha}}{s^2} - \frac{\overline{\mu}_{\alpha} \overline{\varphi}_{\alpha}}{s} - \frac{\overline{\mu}_{\alpha} \overline{\psi}_{\alpha}}{s^2}, \tag{8}$$

$$s^2 \overline{U}_{\alpha}(s) + \underline{\mu}_{\alpha} \underline{U}_{\alpha}(s) = s \overline{\varphi}_{\alpha} + \overline{\psi}_{\alpha} + \frac{\overline{\chi}_{\alpha}}{s} + \frac{\overline{\omega}_{\alpha}}{s^2} - \frac{\underline{\mu}_{\alpha} \underline{\varphi}_{\alpha}}{s} - \frac{\underline{\mu}_{\alpha} \underline{\psi}_{\alpha}}{s^2}. \tag{9}$$

From the equations (8) and (9), $\underline{U}_{\alpha}(s)$ is obtained as

$$\underline{U}_{\alpha}(s)=\frac{\left(\underline{\mu}_{\alpha}\overline{\mu}_{\alpha}\underline{\psi}_{\alpha}-\overline{\omega}_{\alpha}\overline{\mu}_{\alpha}\right)}{s^{2}\left(s^{4}-\underline{\mu}_{\alpha}\overline{\mu}_{\alpha}\right)}+\frac{\underline{\mu}_{\alpha}\overline{\mu}_{\alpha}\underline{\varphi}_{\alpha}-\overline{\chi}_{\alpha}\overline{\mu}_{\alpha}}{s\left(s^{4}-\underline{\mu}_{\alpha}\overline{\mu}_{\alpha}\right)}+\frac{\underline{\omega}_{\alpha}-2\overline{\mu}_{\alpha}\overline{\psi}_{\alpha}}{\left(s^{4}-\underline{\mu}_{\alpha}\overline{\mu}_{\alpha}\right)}$$

$$+\frac{\left(\underline{\chi}_{\alpha}-2\overline{\mu}_{\alpha}\overline{\varphi}_{\alpha}\right)s}{\left(s^{4}-\underline{\mu}_{\alpha}\overline{\lambda}_{\alpha}\right)}+\frac{\underline{\psi}_{\alpha}s^{2}}{\left(s^{4}-\underline{\mu}_{\alpha}\overline{\mu}_{\alpha}\right)}+\frac{\underline{\varphi}_{\alpha}s^{3}}{\left(s^{4}-\underline{\mu}_{\alpha}\overline{\mu}_{\alpha}\right)}.$$

From this, the lower solution is obtained as

$$\underline{u}_{\alpha}(t)=\frac{1}{2}\left(\frac{\underline{\omega}_{\alpha}-2\overline{\mu}_{\alpha}\overline{\psi}_{\alpha}}{\sqrt{\underline{\mu}_{\alpha}\overline{\mu}_{\alpha}}}+\frac{\overline{\omega}_{\alpha}}{\underline{\mu}_{\alpha}}-\underline{\psi}_{\alpha}\right)\left(\sinh\left(\sqrt[4]{\underline{\mu}_{\alpha}\overline{\mu}_{\alpha}t}\right)-\sin\left(\sqrt[4]{\underline{\mu}_{\alpha}\overline{\mu}_{\alpha}t}\right)\right)$$

$$+\frac{1}{2}\left(\frac{\underline{\chi}_{\alpha}-2\overline{\mu}_{\alpha}\overline{\varphi}_{\alpha}}{\sqrt{\underline{\mu}_{\alpha}\overline{\mu}_{\alpha}}}+\frac{\overline{\chi}_{\alpha}}{\underline{\mu}_{\alpha}}-\underline{\varphi}_{\alpha}\right)\left(\cosh\left(\sqrt[4]{\underline{\mu}_{\alpha}\overline{\mu}_{\alpha}t}\right)-\cos\left(\sqrt[4]{\underline{\mu}_{\alpha}\overline{\mu}_{\alpha}t}\right)\right)$$

$$+\frac{\underline{\psi}_{\alpha}}{2}\left(\sinh\left(\sqrt[4]{\underline{\mu}_{\alpha}\overline{\mu}_{\alpha}t}\right)+\sin\left(\sqrt[4]{\underline{\mu}_{\alpha}\overline{\mu}_{\alpha}t}\right)\right)+\frac{\underline{\varphi}_{\alpha}}{2}\left(\cosh\left(\sqrt[4]{\underline{\mu}_{\alpha}\overline{\mu}_{\alpha}t}\right)+\cos\left(\sqrt[4]{\underline{\mu}_{\alpha}\overline{\mu}_{\alpha}t}\right)\right)$$

$$+\frac{1}{\underline{\mu}_{\alpha}}\left(\left(\underline{\mu}_{\alpha}\underline{\psi}_{\alpha}-\overline{\omega}_{\alpha}\right)t+\left(\underline{\mu}_{\alpha}\underline{\varphi}_{\alpha}-\overline{\chi}_{\alpha}\right)\right).$$

Similarly, we obtain the upper solution as

$$\overline{u}_{\alpha}(t)=\frac{1}{2}\left(\frac{\overline{\omega}_{\alpha}-2\underline{\mu}_{\alpha}\,\underline{\psi}_{\alpha}}{\sqrt{\underline{\mu}_{\alpha}\overline{\mu}_{\alpha}}}+\frac{\underline{\omega}_{\alpha}}{\overline{\mu}_{\alpha}}-\overline{\psi}_{\alpha}\right)\left(\sinh\left(\sqrt[4]{\underline{\mu}_{\alpha}\overline{\mu}_{\alpha}t}\right)-\sin\left(\sqrt[4]{\underline{\mu}_{\alpha}\overline{\mu}_{\alpha}t}\right)\right)$$

$$+\frac{1}{2}\left(\frac{\overline{\chi}_{\alpha}-2\underline{\mu}_{\alpha}\underline{\varphi}_{\alpha}}{\sqrt{\underline{\mu}_{\alpha}\overline{\mu}_{\alpha}}}+\frac{\underline{\chi}_{\alpha}}{\overline{\mu}_{\alpha}}-\overline{\varphi}_{\alpha}\right)\left(\cosh\left(\sqrt[4]{\underline{\mu}_{\alpha}\overline{\mu}_{\alpha}t}\right)-\cos\left(\sqrt[4]{\underline{\mu}_{\alpha}\overline{\mu}_{\alpha}t}\right)\right)$$

$$+\frac{\overline{\psi}_{\alpha}}{2}\left(\sinh\left(\sqrt[4]{\underline{\mu}_{\alpha}\overline{\mu}_{\alpha}t}\right)+\sin\left(\sqrt[4]{\underline{\mu}_{\alpha}\overline{\mu}_{\alpha}t}\right)\right)+\frac{\overline{\varphi}_{\alpha}}{2}\left(\cosh\left(\sqrt[4]{\underline{\mu}_{\alpha}\overline{\mu}_{\alpha}t}\right)+\cos\left(\sqrt[4]{\underline{\mu}_{\alpha}\overline{\mu}_{\alpha}t}\right)\right)$$

$$+\frac{1}{\overline{\mu}_{\alpha}}\left(\left(\overline{\mu}_{\alpha}\overline{\psi}_{\alpha}-\underline{\omega}_{\alpha}\right)t+\left(\overline{\mu}_{\alpha}\overline{\varphi}_{\alpha}-\underline{\chi}_{\alpha}\right)\right).$$

Consequently, the solution is

$[u(t)]^\alpha = [\underline{u}_\alpha(t), \overline{u}_\alpha(t)].$

Example 2. Consider the problem

$u^{(iv)}(t) = -[1]^\alpha u''(t),$

$u(0) = [0]^\alpha, u'(0) = [1]^\alpha, u''(0) = [2]^\alpha, u'''(0) = [3]^\alpha.$

The solution of the problem is

$$\underline{u}_\alpha(t) = \frac{1}{2}\left(\frac{9\alpha - 2\alpha^2 - 6}{(\alpha(2-\alpha))^{1/2}} - \alpha + \frac{4}{\alpha} - 1\right)\left(\sinh\left((\alpha(2-\alpha))^{1/4} t\right) - \sin\left((\alpha(2-\alpha))^{1/4} t\right)\right)$$

$$+\frac{1}{2}\left(\frac{7\alpha - 2\alpha^2 - 3}{(\alpha(2-\alpha))^{1/2}} + \frac{3}{\alpha} - 1\right)\left(\cosh\left((\alpha(2-\alpha))^{1/4} t\right) - \cos\left((\alpha(2-\alpha))^{1/4} t\right)\right)$$

$$+\frac{\alpha}{2}\left(\sinh\left((\alpha(2-\alpha))^{1/4} t\right) + \sin\left((\alpha(2-\alpha))^{1/4} t\right)\right)$$

$$+\left(\frac{\alpha-1}{2}\right)\left(\cosh\left((\alpha(2-\alpha))^{1/4} t\right) + \cos\left((\alpha(2-\alpha))^{1/4} t\right)\right)$$

$$+\frac{1}{\alpha}\left((\alpha^2 + \alpha - 4)t + \alpha^2 - 3\right), \quad (10)$$

$$\overline{u}_\alpha(t) = \frac{1}{2}\left(\frac{4 - \alpha - 2\alpha^2}{\sqrt{\alpha(2-\alpha)}} + \frac{2\alpha}{2-\alpha} + \alpha - 1\right)\left(\sinh\left((\alpha(2-\alpha))^{1/4} t\right) - \sin\left((\alpha(2-\alpha))^{1/4} t\right)\right)$$

$$+\frac{1}{2}\left(\frac{3 + \alpha - 2\alpha^2}{\sqrt{\alpha(2-\alpha)}} + \frac{1+\alpha}{2-\alpha} + \alpha - 1\right)\left(\cosh\left((\alpha(2-\alpha))^{1/4} t\right) - \cos\left((\alpha(2-\alpha))^{1/4} t\right)\right)$$

$$+\left(1 - \frac{\alpha}{2}\right)\left(\sinh\left((\alpha(2-\alpha))^{1/4} t\right) + \sin\left((\alpha(2-\alpha))^{1/4} t\right)\right)$$

$$+\left(\frac{1-\alpha}{2}\right)\left(\cosh\left((\alpha(2-\alpha))^{1/4} t\right) + \cos\left((\alpha(2-\alpha))^{1/4} t\right)\right)$$

$$+\left(\frac{1}{2-\alpha}\right)\left((\alpha^2-5\alpha+2)t+\alpha^2-4\alpha+1\right), \tag{11}$$

$$[u(t)]^\alpha=\left[\underline{u}_\alpha(t),\overline{u}_\alpha(t)\right]. \tag{12}$$

According to Definition 3 and since $u(t)$ is positive fuzzy function, $u(t)$ is a valid fuzzy function for $t>2.364610903068273$ in Figure 2.

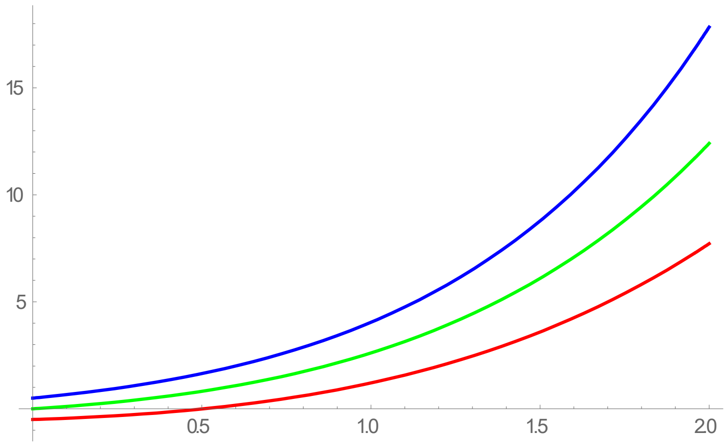

Figure 1. Graphic of solution (5)-(7) for $\alpha=0.5$

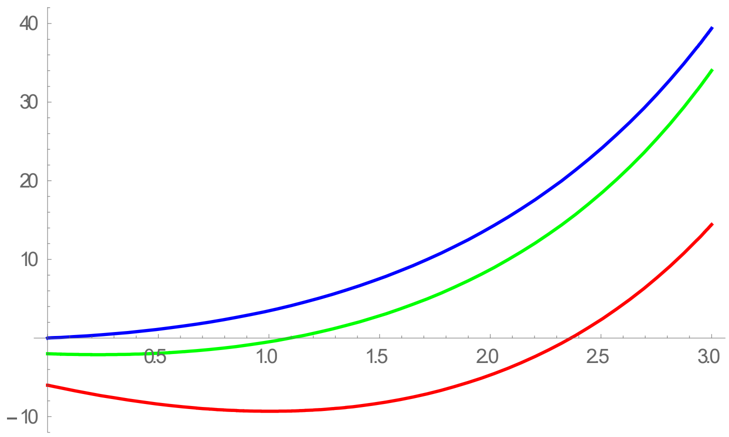

Figure 2. Graphic of solution (10)-(12) for $\alpha=0.5$

Red $\rightarrow \underline{u}_\alpha(t)$, Green $\rightarrow \overline{u}_1(t)=\underline{u}_1(t)$, Blue $\rightarrow \overline{u}_\alpha(t)$.

4. Conclusions

In this paper, we research two different fourth-order fuzzy problems. The fuzzy Laplace transform method is used. Solutions are found directly by the fuzzy Laplace transform method. Comparison results of the solutions are given. We give two examples. We draw graphics of the found solutions for alpha level sets. It is seen that the solutions are valid fuzzy functions in different intervals for each of α-level sets. Also, we see that the fuzzy problem with positive fuzzy coefficient is a valid fuzzy function over a wider interval than the fuzzy problem with negative fuzzy coefficient.

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