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Some identities of bivariate Pell and bivariate Pell-Lucas polynomials

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Abstract — In this paper, we obtain some identities for the bivariate Pell polynomials and bivariate Pell-Lucas polynomials. We establish some sums and connection formulas involving them. Moreover, we present its two cross two matrices representation and find some of its properties, such as the b^{th} power of the matrix. We finally derive the identities by using Binet's formula, generating function, and induction method.

Keywords: *Bivariate Pell polynomials, bivariate Pell-Lucas polynomials, Binet's formula, generating function*

Subject Classification (2020): 11B37, 11B39

1. Introduction

Sequences and polynomials have a wide range of applications in applied mathematics and physics. Bivariate polynomials are widely used in theoretical physics for modeling physical processes. Catalani [1-3] defined generalized bivariate polynomials from which, by specifying initial conditions, the bivariate Fibonacci and Lucas polynomials are obtained, and many interesting identities are derived. Belbachir and Bencherif [4] generalized to bivariate Fibonacci and Lucas polynomials properties obtained for Chebyshev polynomials. They proved that the coordinates of the bivariate polynomials over an appropriate basis are families of integers satisfying remarkable recurrence relations. Tuğlu et al. [5] presented generalized bivariate Fibonacci and Lucas p-polynomials, which are general forms of the Fibonacci, Lucas, Pell, Jacobsthal, Pell-Lucas, Jacobsthal-Lucas sequences, as well as Fibonacci, Lucas, Pell, Jacobsthal, Pell-Lucas, Jacobsthal-Lucas, bivariate Fibonacci and Lucas, first and second type of Chebyshev polynomials, and many others. Halıcı and Akyüz [6] derived some identities and some sum formulas for the bivariate Pell polynomials using different matrices. Saba and Boussayoud [7] introduced a symmetric function to derive a new generating function of bivariate Pell Lucas polynomials, also derived new symmetric functions, and gave some interesting properties. This study defines the identities of bivariate Pell and bivariate Pell-Lucas polynomials.

2. Preliminaries

For $n \geq 2$, the bivariate Pell polynomials sequence [6] is defined by

$$P_n(x,y) = 2xyP_{n-1}(x,y) + yP_{n-2}(x,y) \tag{2.1}$$

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Therefore, the first few bivariate Pell polynomials are

$$\{P_n(x,y)\} = \{0,1,2xy,4x^2y^2+y,8x^3y^3+4xy^2,16x^4y^4+12x^2y^3+y^2,\cdots\}$$

Binet's formula,

$$P_n(x,y) = \frac{\ell_1^n - \ell_2^n}{\ell_1 - \ell_2}$$

Generating function,

$$P_n(x,y) = \frac{t}{(1-2xyt-yt^2)}$$

For $n \geq 2$, the bivariate Pell-Lucas polynomials sequence [6] is defined by

$$Q_n(x,y) = 2xyQ_{n-1}(x,y) + yQ_{n-2}(x,y) \tag{2.2}$$

Therefore, the first few bivariate Pell-Lucas polynomials are

$$\{Q_n(x,y)\} = \{2,2xy,4x^2y^2+2y,8x^3y^3+6xy^2,16x^4y^4+16x^2y^3+2y^2,\cdots\}$$

Binet's formula,

$$Q_n(x,y) = \ell_1^n + \ell_2^n$$

Generating function,

$$Q_n(x,y) = \frac{2+2xyt(x-1)}{(1-2xyt-yt^2)}$$

The characteristic equation of recurrence relations (2.1) and (2.2) is

$$t^2 - 2xyt - y = 0$$

where $x \neq 0$, $y \neq 0$, and $x^2y^2 + y \neq 0$. This equation has two real roots: $\ell_1 = xy + \sqrt{x^2y^2+y}$ and $\ell_2 = xy - \sqrt{x^2y^2+y}$. Note that $\ell_1 + \ell_2 = 2xy$, $\ell_1\ell_2 = -y$, and $\ell_1 - \ell_2 = \sqrt{x^2y^2+y}$. Moreover, $P_{-n}(x,y) = \frac{-1}{(-y)^n}P_n(x,y)$ and $Q_{-n}(x,y) = \frac{1}{(-y)^n}Q_n(x,y)$. The main objective of this study is to describe sums and connection formulas. Moreover, we introduce the special sums and prove them using Binet's formula.

3. Results and Discussions

We first establish sums and relations for bivariate Pell and bivariate Pell-Lucas polynomials. The motivation of this work comes from the study of [8-11].

Proposition 3.1. For $\vartheta, \omega \in \mathbb{Z}$, we get

$$\sum_{b=0}^{\vartheta+\omega} \binom{\vartheta+\omega}{b} (2xy)^b y^{\vartheta+\omega-b} P_b(x,y) = P_{2\vartheta+2\omega}(x,y)$$

Proof.

By Binet's formula,

$$\sum_{b=0}^{\vartheta+\omega}\binom{\vartheta+\omega}{b}(2xy)^b y^{\vartheta+\omega-b}\,P_b(x,y) = \sum_{b=0}^{c+d}\binom{\vartheta+\omega}{b}(2xy)^b y^{\vartheta+\omega-b}\left(\frac{\ell_1^b-\ell_2^b}{\ell_1-\ell_2}\right)$$

$$= \frac{1}{\ell_1-\ell_2}\sum_{b=0}^{\vartheta+\omega}\binom{\vartheta+\omega}{b}(2xy)^b y^{\vartheta+\omega-b}\left(\ell_1^b-\ell_2^b\right)$$

$$= \frac{1}{\ell_1-\ell_2}\sum_{b=0}^{\vartheta+\omega}\binom{\vartheta+\omega}{b}\{(2xy\ell_1)^b-(2xy\ell_2)^b\}y^{\vartheta+\omega-b}$$

$$= \frac{1}{\ell_1-\ell_2}\left\{(2xy\ell_1+y)^{\vartheta+\omega}-(2xy\ell_2+y)^{\vartheta+\omega}\right\}$$

Since ℓ_1 and ℓ_2 are the roots of $t^2-2xyt-y=0$,

$$\sum_{b=0}^{\vartheta+\omega}\binom{\vartheta+\omega}{b}(2xy)^b y^{\vartheta+\omega-b}\,P_b(x,y) = \frac{(\ell_1^2)^{\vartheta+\omega}-(\ell_2^2)^{\vartheta+\omega}}{\ell_1-\ell_2} = P_{2\vartheta+2\omega}(x,y)$$

□

Proposition 3.2. For $\vartheta,\omega\in\mathbb{Z}$, we get

$$\sum_{b=0}^{\vartheta+\omega}\binom{\vartheta+\omega}{b}(2xy)^b(-y)^{\vartheta+\omega-b}\,P_b(x,y) = \sum_{b=0}^{\vartheta+\omega}\binom{\vartheta+\omega}{b}(-2y)^b\,P_{2\vartheta+2\omega-2b}(x,y)$$

Proof.

By Binet's formula,

$$\sum_{b=0}^{\vartheta+\omega}\binom{\vartheta+\omega}{b}(2xy)^b(-y)^{\vartheta+\omega-b}\,P_b(x,y) = \sum_{b=0}^{\vartheta+\omega}\binom{\vartheta+\omega}{b}(2xy)^b(-y)^{\vartheta+\omega-b}\left(\frac{\ell_1^b-\ell_2^b}{\ell_1-\ell_2}\right)$$

$$= \frac{1}{\ell_1-\ell_2}\sum_{b=0}^{\vartheta+\omega}\binom{\vartheta+\omega}{b}(2xy)^b(-y)^{\vartheta+\omega-b}\left(\ell_1^b-\ell_2^b\right)$$

$$= \frac{1}{\ell_1-\ell_2}\sum_{b=0}^{\vartheta+\omega}\binom{\vartheta+\omega}{b}\{(2xy\ell_1)^b-(2xy\ell_2)^b\}(-y)^{\vartheta+\omega-b}$$

$$= \frac{1}{\ell_1-\ell_2}\left\{(2xy\ell_1-y)^{\vartheta+\omega}-(2xy\ell_2-y)^{\vartheta+\omega}\right\}$$

Since ℓ_1 and ℓ_2 are the roots of $t^2-2xyt-y=0$,

$$2xy\ell_1-y=\ell_1^2-2y$$

and

$$2xy\ell_2-y=\ell_2^2-2y$$

Thus,

$$\sum_{b=0}^{\vartheta+\omega}\binom{\vartheta+\omega}{b}(2xy)^b(-y)^{\vartheta+\omega-b}\,P_b(x,y) = \frac{(\ell_1^2-2y)^{\vartheta+\omega}-(\ell_2^2-2y)^{\vartheta+\omega}}{\ell_1-\ell_2}$$

$$= \sum_{b=0}^{\vartheta+\omega}\binom{\vartheta+\omega}{b}(-2y)^b\,P_{2\vartheta+2\omega-2b}(x,y)$$

□

Proposition 3.3. For $\vartheta,\omega \in \mathbb{Z}$, we get

$$\sum_{b=0}^{\vartheta+\omega}\binom{\vartheta+\omega}{b}(2xy)^{\vartheta+\omega-b}(-1)^b\,Q_b(x,y) = Q_{\vartheta+\omega}(x,y)$$

Proof.

By Binet's formula,

$$\sum_{b=0}^{\vartheta+\omega}\binom{\vartheta+\omega}{b}(2xy)^{\vartheta+\omega-b}(-1)^b\,Q_b(x,y) = \sum_{b=0}^{\vartheta+\omega}\binom{\vartheta+\omega}{b}(2xy)^{\vartheta+\omega-b}(-1)^b\left(\ell_1^b+\ell_2^b\right)$$

$$= \sum_{b=0}^{\vartheta+\omega}\binom{\vartheta+\omega}{b}(2xy)^{\vartheta+\omega-b}(-1)^b\left(-\ell_1^b\right) + \sum_{b=0}^{\vartheta+\omega}\binom{\vartheta+\omega}{b}(2xy)^{\vartheta+\omega-b}(-1)^b\left(-\ell_2^b\right)$$

$$= (2xy-\ell_1)^{\vartheta+\omega}+(2xy-\ell_2)^{\vartheta+\omega}$$

$$= \left(\frac{-y}{\ell_1}\right)^{\vartheta+\omega}+\left(\frac{-y}{\ell_2}\right)^{\vartheta+\omega}$$

$$= (-y)^{\vartheta+\omega}\frac{\ell_1^{\vartheta+\omega}+\ell_2^{\vartheta+\omega}}{(\ell_1\ell_2)^{\vartheta+\omega}}$$

$$= Q_{\vartheta+\omega}(x,y)$$

□

Proposition 3.4. For $\vartheta,\omega \in \mathbb{Z}$, we get

$$\sum_{b=0}^{\vartheta+\omega}\binom{\vartheta+\omega}{b}(2xy)^b y^{\vartheta+\omega-b}\,Q_b(x,y) = Q_{2\vartheta+2\omega}(x,y)$$

Proof.

By Binet's formula,

$$\sum_{b=0}^{\vartheta+\omega}\binom{\vartheta+\omega}{b}(2xy)^b y^{\vartheta+\omega-b}\,Q_b(x,y) = \sum_{b=0}^{\vartheta+\omega}\binom{\vartheta+\omega}{b}(2xy)^b y^{\vartheta+\omega-b}\,Q_b(x,y)\left(\ell_1^b+\ell_2^b\right)$$

$$= \sum_{b=0}^{\vartheta+\omega}\binom{\vartheta+\omega}{b}(2xy\ell_1)^b y^{\vartheta+\omega-b} + \sum_{b=0}^{\vartheta+\omega}\binom{\vartheta+\omega}{b}(2xy\ell_2)^b y^{\vartheta+\omega-b}$$

$$= (2xy\ell_1+y)^{\vartheta+\omega}+(2xy\ell_2+y)^{\vartheta+\omega}$$

Since ℓ_1 and ℓ_2 are the roots of $t^2 - 2xyt - y = 0$,

$$\sum_{b=0}^{\vartheta+\omega}\binom{\vartheta+\omega}{b}(2xy)^b y^{\vartheta+\omega-b}\,Q_b(x,y) = (\ell_1^2)^{\vartheta+\omega} + (\ell_2^2)^{\vartheta+\omega}$$

$$= Q_{2\vartheta+2\omega}(x,y)$$

□

Proposition 3.5. If $P_b(x,y)$ and $Q_b(x,y)$ are Bivariate Pell and Bivariate Pell-Lucas polynomials, then for $b \geq \vartheta + \omega$,

$$P_{b+\vartheta+\omega}(x,y) - (-y)^{\vartheta+\omega}P_{b-\vartheta-\omega}(x,y) = P_{\vartheta+\omega}(x,y)Q_b(x,y)$$

Proof.

By Binet's formula,

$$P_{b+\vartheta+\omega}(x,y) - (-y)^{\vartheta+\omega}P_{b-\vartheta-\omega}(x,y) = \frac{\ell_1^{b+\vartheta+\omega} - \ell_2^{b+\vartheta+\omega}}{\ell_1 - \ell_2} - (-y)^{\vartheta+\omega}\left(\frac{\ell_1^{b-\vartheta-\omega} - \ell_2^{b-\vartheta-\omega}}{\ell_1 - \ell_2}\right)$$

$$= \frac{\left(\ell_1^{b+\vartheta+\omega} - \ell_2^{b+\vartheta+\omega}\right) - (-y)^{\vartheta+\omega}\left(\ell_1^{b-\vartheta-\omega} - \ell_2^{b-\vartheta-\omega}\right)}{\ell_1 - \ell_2}$$

$$= \frac{\left(\ell_1^{b+\vartheta+\omega} - \ell_2^{b+\vartheta+\omega}\right) - (\ell_1\ell_2)^{\vartheta+\omega}\left(\ell_1^{b-\vartheta-\omega} - \ell_2^{b-\vartheta-\omega}\right)}{\ell_1 - \ell_2}$$

$$= \frac{\left(\ell_1^{b+\vartheta+\omega} - \ell_2^{b+\vartheta+\omega}\right) - \left(\ell_1^{b}\ell_2^{\vartheta+\omega} - \ell_1^{\vartheta+\omega}\ell_2^{b}\right)}{\ell_1 - \ell_2}$$

$$= \left(\frac{\ell_1^{\vartheta+\omega} - \ell_2^{\vartheta+\omega}}{\ell_1 - \ell_2}\right)\left(\ell_1^b + \ell_2^b\right)$$

$$= P_{\vartheta+\omega}(x,y)Q_b(x,y)$$

□

Secondly, we investigate sums for Bivariate Pell and Pell-Lucas polynomials with negative indices.

Theorem 3.6. For $\vartheta \geq 1$ and ω any integer, we get

$$\sum_{i=0}^{b}(-y)^i P_{-i\vartheta-\omega}(x,y) = \begin{cases} \dfrac{P_{\vartheta b+\vartheta+\omega}(x,y) - (-y)^{\vartheta}P_{\vartheta b+\omega}(x,y) - (-y)^{\omega}P_{\vartheta-\omega}(x,y) + P_{\omega}(x,y)}{(-1)^{\vartheta} - Q_{\vartheta}(x,y) + 1}, & \omega < \vartheta \\ \dfrac{P_{\vartheta b+\vartheta+\omega}(x,y) - (-y)^{\vartheta}P_{\vartheta b+\omega}(x,y) + (-y)^{\omega}P_{\vartheta-\omega}(x,y) + P_{\omega}(x,y)}{(-1)^{\vartheta} - Q_{\vartheta}(x,y) + 1}, & \text{otherwise} \end{cases}$$

Proof.

Since

$$\sum_{i=0}^{b}(-y)^i P_{-i\vartheta-\omega}(x,y) = -\sum_{i=0}^{b} P_{i\vartheta+\omega}(x,y)$$

By Binet's formula,

$$\sum_{i=0}^{b}(-y)^i P_{-i\vartheta-\omega}(x,y) = -\sum_{i=0}^{b}\frac{\ell_1^{i\vartheta+\omega}-\ell_2^{i\vartheta+\omega}}{\ell_1-\ell_2}$$

$$= \frac{-1}{\ell_1-\ell_2}\left(\ell_1^{\omega}\sum_{i=0}^{b}\ell_1^{\vartheta i}-\ell_2^{\omega}\sum_{i=0}^{b}\ell_2^{\vartheta i}\right)$$

$$= \frac{-1}{\ell_1-\ell_2}\left[\frac{\ell_1^{\vartheta b+\vartheta+\omega}-\ell_1^{\omega}}{\ell_1^{\vartheta}-1}-\frac{\ell_2^{\vartheta b+\vartheta+\omega}-\ell_2^{\omega}}{\ell_2^{\vartheta}-1}\right]$$

$$= \frac{\left(\ell_1^{\vartheta b+\vartheta+\omega}-\ell_2^{\vartheta b+\vartheta+\omega}\right)-(\ell_1\ell_2)^{\vartheta}\left(\ell_1^{\vartheta b+\omega}-\ell_2^{\vartheta b+\omega}\right)-\left(\ell_1^{\vartheta}\ell_2^{\omega}-\ell_1^{\omega}\ell_2^{\vartheta}\right)+\left(\ell_1^{\omega}-\ell_2^{\omega}\right)}{(\ell_1-\ell_2)\left\{(\ell_1\ell_2)^{\vartheta}-\left(\ell_1^{\vartheta}+\ell_2^{\vartheta}\right)+1\right\}}$$

$$= \begin{cases}\dfrac{P_{\vartheta b+\vartheta+\omega}(x,y)-(-y)^{\vartheta}P_{\vartheta b+\omega}(x,y)-(-y)^{\omega}P_{\vartheta-\omega}(x,y)+P_{\omega}(x,y)}{(-1)^{\vartheta}-Q_{\vartheta}(x,y)+1}, & \omega<\vartheta\\[2ex] \dfrac{P_{\vartheta b+\vartheta+\omega}(x,y)-(-y)^{\vartheta}P_{\vartheta b+\omega}(x,y)+(-y)^{\omega}P_{\vartheta-\omega}(x,y)+P_{\omega}(x,y)}{(-1)^{\vartheta}-Q_{\vartheta}(x,y)+1}, & \text{otherwise}\end{cases}$$

□

Theorem 3.7. For $\vartheta \geq 1$ and ω any integer, we get

$$\sum_{i=0}^{b}(-y)^i Q_{-i\vartheta-\omega}(x,y) = \begin{cases}\dfrac{(-y)^{\vartheta}Q_{\vartheta b+\omega}(x,y)-Q_{\vartheta b+\vartheta+\omega}(x,y)-(-y)^{\omega}Q_{\vartheta-\omega}(x,y)+Q_{\omega}(x,y)}{(-1)^{\vartheta}-Q_{\vartheta}(x,y)+1}, & \omega<\vartheta\\[2ex] \dfrac{(-y)^{\vartheta}Q_{\vartheta b+\omega}(x,y)-Q_{\vartheta b+\vartheta+\omega}(x,y)+(-y)^{\omega}Q_{\vartheta-\omega}(x,y)+Q_{\omega}(x,y)}{(-1)^{\vartheta}-Q_{\vartheta}(x,y)+1}, & \text{otherwise}\end{cases}$$

Proof.

Since

$$\sum_{i=0}^{b}(-y)^i Q_{-i\vartheta-\omega}(x,y) = \sum_{i=0}^{b} Q_{i\vartheta+\omega}(x,y)$$

By Binet's formula,

$$\sum_{i=0}^{b}(-y)^i Q_{-i\vartheta-\omega}(x,y) = \sum_{i=0}^{b}\left(\ell_1^{i\vartheta+\omega}+\ell_2^{i\vartheta+\omega}\right)$$

$$= \ell_1^{\omega}\sum_{i=0}^{b}\ell_1^{\vartheta i}+\ell_2^{\omega}\sum_{i=0}^{b}\ell_2^{\vartheta i}$$

$$= \frac{\ell_1^{\vartheta b+\vartheta+\omega}-\ell_1^{\omega}}{\ell_1^{\vartheta}-1}+\frac{\ell_2^{\vartheta b+\vartheta+\omega}-\ell_2^{\omega}}{\ell_2^{\vartheta}-1}$$

$$= \frac{(\ell_1\ell_2)^{\vartheta}\left(\ell_1^{\vartheta b+\omega}+\ell_2^{\vartheta b+\omega}\right)-\left(\ell_1^{\vartheta b+\vartheta+\omega}+\ell_2^{\vartheta b+\vartheta+\omega}\right)-\left(\ell_1^{\vartheta}\ell_2^{\omega}+\ell_1^{\omega}\ell_2^{\vartheta}\right)+\left(\ell_1^{\omega}+\ell_2^{\omega}\right)}{(\ell_1-\ell_2)\left\{(\ell_1\ell_2)^{\vartheta}-\left(\ell_1^{\vartheta}+\ell_2^{\vartheta}\right)+1\right\}}$$

$$= \begin{cases} \dfrac{(-y)^{\vartheta} Q_{\vartheta b+\omega}(x,y) - Q_{\vartheta b+\vartheta+\omega}(x,y) - (-y)^{\omega} Q_{\vartheta-\omega}(x,y) + Q_{\omega}(x,y)}{(-1)^{\vartheta} - Q_{\vartheta}(x,y) + 1}, & \omega < \vartheta \\ \dfrac{(-y)^{\vartheta} Q_{\vartheta b+\omega}(x,y) - Q_{\vartheta b+\vartheta+\omega}(x,y) + (-y)^{\omega} Q_{\vartheta-\omega}(x,y) + Q_{\omega}(x,y)}{(-1)^{\vartheta} - Q_{\vartheta}(x,y) + 1}, & \text{otherwise} \end{cases}$$

□

Thirdly, we establish some identities for bivariate Pell and bivariate Pell-Lucas polynomials.

Theorem 3.8. For $\vartheta, \omega \in \mathbb{Z}$, we get

$$\sum_{b=0}^{\vartheta+\omega} P_b(x,y) t^{-b} = \frac{1}{t^{\vartheta+\omega}(t^2 - 2xyt - y)} \{t^{\vartheta+\omega+1} - tP_{\vartheta+\omega+1}(x,y) - yP_{\vartheta+\omega}(x,y)\}$$

Proof.

By Binet's formula,

$$\sum_{b=0}^{\vartheta+\omega} P_b(x,y) t^{-b} = \sum_{b=0}^{\vartheta+\omega} \left(\frac{\ell_1^b - \ell_2^b}{\ell_1 - \ell_2}\right) t^{-b}$$

$$= \frac{1}{\ell_1 - \ell_2} \sum_{b=0}^{\vartheta+\omega} \left\{\left(\frac{\ell_1}{t}\right)^b - \left(\frac{\ell_2}{t}\right)^b\right\}$$

$$= \frac{1}{\ell_1 - \ell_2} \left\{\frac{1 - \left(\frac{\ell_1}{t}\right)^{\vartheta+\omega+1}}{1 - \frac{\ell_1}{t}} - \frac{1 - \left(\frac{\ell_2}{t}\right)^{\vartheta+\omega+1}}{1 - \frac{\ell_2}{t}}\right\}$$

$$= \frac{1}{(\ell_1 - \ell_2) t^{\vartheta+\omega}} \left(\frac{t^{\vartheta+\omega+1} - \ell_1^{\vartheta+\omega+1}}{t - \ell_1} - \frac{t^{\vartheta+\omega+1} - \ell_2^{\vartheta+\omega+1}}{t - \ell_2}\right)$$

$$= \frac{1}{(\ell_1 - \ell_2) t^{\vartheta+\omega}} \left\{\frac{t^{\vartheta+\omega+1}(\ell_1 - \ell_2) - t\left(\ell_1^{\vartheta+\omega+1} - \ell_2^{c+d+1}\right) - y\left(\ell_1^{\vartheta+\omega} - \ell_2^{\vartheta+\omega}\right)}{(t - \ell_1)(t - \ell_2)}\right\}$$

$$= \frac{1}{t^{\vartheta+\omega}(t^2 - 2xyt - y)} \{t^{\vartheta+\omega+1} - tP_{\vartheta+\omega+1}(x,y) - yP_{\vartheta+\omega}(x,y)\}$$

□

Theorem 3.9. For $\vartheta, \omega \in \mathbb{Z}$, we get

$$\sum_{b=0}^{\vartheta+\omega} Q_b(x,y) t^{-b} = \frac{2t^2 - xt}{(t^2 - 2xyt - y)} - \frac{1}{t^{\vartheta+\omega}(t^2 - 2xyt - y)} \{tQ_{\vartheta+\omega+1}(x,y) + yQ_{\vartheta+\omega}(x,y)\}$$

Proof.

By Binet's formula,

$$\sum_{b=0}^{\vartheta+\omega} Q_b(x,y) t^{-b} = \sum_{b=0}^{\vartheta+\omega} \left(\ell_1^b + \ell_2^b\right) t^{-b}$$

$$= \sum_{b=0}^{\vartheta+\omega} \left\{ \left(\frac{\ell_1}{t}\right)^b + \left(\frac{\ell_2}{t}\right)^b \right\}$$

$$= \frac{1-\left(\frac{\ell_1}{t}\right)^{\vartheta+\omega+1}}{1-\frac{\ell_1}{t}} + \frac{1-\left(\frac{\ell_2}{t}\right)^{\vartheta+\omega+1}}{1-\frac{\ell_2}{t}}$$

$$= \frac{1}{t^{\vartheta+\omega}} \left(\frac{t^{\vartheta+\omega+1} - \ell_1^{\vartheta+\omega+1}}{t-\ell_1} + \frac{t^{\vartheta+\omega+1} - \ell_2^{\vartheta+\omega+1}}{t-\ell_2} \right)$$

$$= \frac{2t^{\vartheta+\omega+2} - t\left(\ell_1^{\vartheta+\omega+1} + \ell_2^{\vartheta+\omega+1}\right) - t^{\vartheta+\omega+1}(\ell_1+\ell_2) + \ell_1\ell_2\left(\ell_1^{\vartheta+\omega} + \ell_2^{\vartheta+\omega}\right)}{t^{\vartheta+\omega}(t-\ell_1)(t-\ell_2)}$$

$$= \frac{2t^2 - xt}{(t^2 - 2xyt - y)} - \frac{1}{t^{\vartheta+\omega}(t^2-2xyt-y)} \{tQ_{\vartheta+\omega+1}(x,y) + yQ_{\vartheta+\omega}(x,y)\}$$

□

Fourthly, we define identities involving common factors of bivariate Pell and Pell-Lucas polynomials.

Theorem 3.10. If $P_b(x,y)$ and $Q_b(x,y)$ are Bivariate Pell and Pell-Lucas polynomials, then holds for every b and s,

i. $P_{2b+s}(x,y)Q_{2b+1}(x,y) = P_{4b+s+1}(x,y) + (-y)^{2b+1}P_{s-1}(x,y)$

ii. $P_{2b+s}(x,y)Q_{2b+2}(x,y) = P_{4b+s+2}(x,y) + y^{2b+2}P_{s-2}(x,y)$

iii. $P_{2b+s}(x,y)Q_{2b}(x,y) = P_{4b+s}(x,y) + y^{2b}P_s(x,y)$

iv. $P_{2b-s}(x,y)Q_{2b+1}(x,y) = P_{4b-s+1}(x,y) + (-y)^{2b+1}P_{-s-1}(x,y)$

v. $P_{2b-s}(x,y)Q_{2b-1}(x,y) = P_{4b-s-1}(x,y) + (-y)^{2b-1}P_{1-s}(x,y)$

vi. $P_{2b-s}(x,y)Q_{2b}(x,y) = P_{4b-s}(x,y) + (-y)^{2b}P_{-s}(x,y)$

vii. $P_{2b}(x,y)Q_{2b+s}(x,y) = P_{4b-s}(x,y) - (-y)^{2b}P_s(x,y)$

viii. $(x^2y^2+y)P_{2b}(x,y)Q_{2b+s}(x,y) = Q_{4b+s}(x,y) - (-y)^{2b}Q_s(x,y)$

ix. $Q_{2b}(x,y)Q_{2b+s}(x,y) = Q_{4b+s}(x,y) + (-y)^{2b}Q_s(x,y)$

Proof.

Using Binet's formula of Bivariate Pell and Pell-Lucas polynomials and Principle of Mathematical Induction (PMI) on b and s, the proof is clear. □

Finally, we present two cross two matrix for bivariate Pell and Pell-Lucas polynomials by $B = \begin{bmatrix} 2xy & 1 \\ y & 0 \end{bmatrix}$. Then, we can write, $B^n = \begin{bmatrix} P_{n+1}(x,y) & P_n(x,y) \\ yP_n(x,y) & yP_{n-1}(x,y) \end{bmatrix}$ and we get $det\ (B^n) = (-1)^n(y^n)$ (Cassini's identity).

Many authors introduce and present matrices properties and identities of bivariate polynomials [1,2,4].

Theorem 3.11. Let $b \in \mathbb{N}$. Then,

$$\begin{bmatrix} P_{b+1}(x,y) \\ yP_b(x,y) \end{bmatrix} = B \begin{bmatrix} P_b(x,y) \\ yP_{b-1}(x,y) \end{bmatrix}$$

Proof.

Let $b \in \mathbb{N}$. For $b = 1$,

$$\begin{bmatrix} P_2(x,y) \\ yP_1(x,y) \end{bmatrix} = B \begin{bmatrix} P_1(x,y) \\ yP_0(x,y) \end{bmatrix} = B \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The identity is valid for $b = 1$.

For the mathematical induction on b, suppose that the identity is true for b. Thus,

$$\begin{bmatrix} P_{b+2}(x,y) \\ yP_{b+1}(x,y) \end{bmatrix} = \begin{bmatrix} 2xyP_{b+1}(x,y) + yP_b(x,y) \\ yP_{b+1}(x,y) \end{bmatrix}$$

$$= \begin{bmatrix} 2xy & 1 \\ y & 0 \end{bmatrix} \begin{bmatrix} P_{b+1}(x,y) \\ yP_b(x,y) \end{bmatrix}$$

$$= \begin{bmatrix} 2xy & 1 \\ y & 0 \end{bmatrix} \begin{bmatrix} 2xy & 1 \\ y & 0 \end{bmatrix} \begin{bmatrix} P_b(x,y) \\ yP_{b-1}(x,y) \end{bmatrix}$$

$$= \begin{bmatrix} 2xy & 1 \\ y & 0 \end{bmatrix} \begin{bmatrix} 2xyP_b(x,y) + yP_{b-1}(x,y) \\ yP_b(x,y) \end{bmatrix}$$

$$= \begin{bmatrix} 2xy & 1 \\ y & 0 \end{bmatrix} \begin{bmatrix} P_{n+1}(x,y) \\ yP_n(x,y) \end{bmatrix}$$

$$= B \begin{bmatrix} P_{b+1}(x,y) \\ yP_b(x,y) \end{bmatrix}$$

□

Theorem 3.12. Let $b \in \mathbb{N}$. Then,

$$\begin{bmatrix} Q_{b+1}(x,y) \\ yQ_b(x,y) \end{bmatrix} = B \begin{bmatrix} Q_b(x,y) \\ yQ_{b-1}(x,y) \end{bmatrix}$$

Theorem 3.13. Let $b \in \mathbb{N}$. Then,

$$\begin{bmatrix} P_{b+1}(x,y) \\ yP_b(x,y) \end{bmatrix} = B^b \begin{bmatrix} P_1(x,y) \\ yP_0(x,y) \end{bmatrix}$$

Theorem 3.14. Let $b \in \mathbb{N}$. Then,

$$\begin{bmatrix} Q_{b+1}(x,y) \\ yQ_b(x,y) \end{bmatrix} = B^b \begin{bmatrix} Q_1(x,y) \\ yQ_0(x,y) \end{bmatrix}$$

4. Conclusion

In this paper, we present sums of bivariate Pell and Pell-Lucas polynomials. Moreover, we describe sums with negative indices, some connection formulas, and two cross two matrix representation and give several interesting identities involving them.

Author Contributions

The author read and approved the final version of the paper.

Conflict of Interest

The author declares no conflict of interest.

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