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Direction curves and construction of developable surfaces in Lorentz 3-space

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Abstract. In this work we investigate singularities for the three types of developable surfaces, introduced by Izumiya and Takeuchi, in Lorentz 3 space and give a local classification in terms of k-order frame. Moreover we search the necessary conditions of being a geodesic for principal direction curves of the rectifying developable surface.

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1. Introduction

A developable surface, useful application tool in cartographic projections and producing flat materials, is defined as a surface that can be developed into flat surfaces without changing the metric on the surface. There are several papers about developable surfaces and some of them are combined with the singularity theory. Zhao et al. investigated the geometric characteristics of developable surfaces with a single parameter that have regular curves [12]. The global behavior of singularities on flat surfaces in Euclidean 3-space is studied by Murata and Umehara [9]. Furthermore, the primary source of inspiration for this paper is the research on developable surface singularities in Euclidean 3-space presented by Izumiya and Takeuchi. They considered three types of developable surfaces named as rectifying developable of a curve, defined as the envelope of the set of rectifying planes along the curve, the second one called Darboux developable which has singularities at the terminal points of the curve's modified Darboux vectors and the third one is tangential Darboux developable which is determined by the space curve's tangent indicatrix's Darboux developable surface. They have shown that these developable surfaces are locally diffeomorphic to the swallowtail, the cuspidal edge or cuspidal cross cap [6, 7].
Moreover Ishikawa and Yamashita provide a comprehensive response to the question of local diffeomorphism categorization in Euclidean 3-space and they give the following theorem;

Theorem 1. *Let ∇ be a torsion free affine connection on a manifold M. Let $\beta : I \longrightarrow M$ be a C^∞ curve from an open interval I.Let $\dim(M) = 3$.*

1) *If $(\nabla_\beta)(s_0)$,$(\nabla^2_\beta)(s_0)$,$(\nabla^3_\beta)(s_0)$ are linearly independent, then the $\nabla-$tangent surface is locally diffeomorphic to the cuspidal edge at $(s_0, 0)$.*
2) *If $(\nabla_\beta)(s_0)$,$(\nabla^2_\beta)(s_0)$,$(\nabla^3_\beta)(s_0)$ are linearly dependent and $(\nabla_\beta)(s_0)$,$(\nabla^2_\beta)(s_0)$,$(\nabla^4_\beta)(s_0)$ are linearly independent then $\nabla-$tangent surface is locally diffeomorphic to the cuspidal crosscap at $(s_0, 0)$.*
3) *If $(\nabla_\beta)(s_0) = 0$ and $(\nabla^2_\beta)(s_0)$,$(\nabla^3_\beta)(s_0)$,$(\nabla^4_\beta)(s_0)$ are linearly independent then $\nabla-$tangent surface is locally diffeomorphic to the swallowtail at $(s_0, 0)$ [5].*

The singular points that are mentioned above theorem can be examined in the following figures [6].

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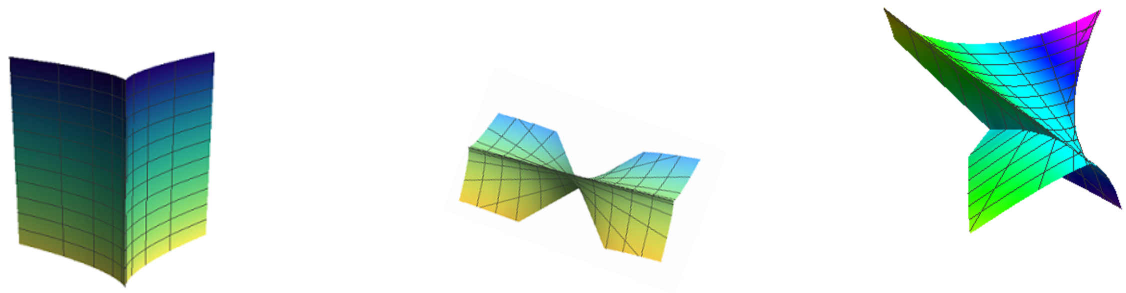


(A) Cuspidaledge (B) Cuspidal crosscap (C) Swallowtail

FIGURE 1. Types of singularities

Furthermore there are some papers about the singularities of surfaces in different spaces. Singularities of surfaces whose mean curvatures are constant, are studied by Brander in Lorentz 3-space [3], Fujimori et al. demonstrate that the cuspidal edges, swallowtails, and cuspidal cross caps are the general components of the singularities of spacelike maximum surfaces in Lorentz 3-space [4]. Kokubu et al. establish that only cuspidal edges and swallowtails are admissible in generic flat fronts in hyperbolic 3-space [8].

The primary aim of this study is to generalize the notion of developable surfaces provided by Izumiya and Takeuchi by utilizing the generalized alternative frame developed by [10] in Lorentz 3-space to include technical material of rulings. The alternative frame has the benefit of producing local classifications for the geometric structure of generalized developable surfaces in terms of k-slant helix, N_k-slant helix and conical surfaces. As a supplementary goal, we present the features of singularities when we explore the geometric properties of these generalized developable surfaces by combining the theory of Ishikawa and Yamashita [5]. Additionally, practical discussions on examples are held on the outcomes of theoretical investigations on generalized developable surfaces.

2. Basic Concepts and Notions

Consider the 3-dimensional Lorentz space E_1^3 provided with the following inner product: $\mathbb{R}^3$ endowed via the metric $\langle , \rangle$ as follows:

$$\begin{aligned} \langle , \rangle \quad : \quad & E^3 x E^3 \longrightarrow E \\ & (\xi, \zeta) \longrightarrow \langle \xi, \zeta \rangle = \xi_1\zeta_1 + \xi_2\zeta_2 - \xi_3\zeta_3 \end{aligned}$$

where $\xi = (\xi_1, \xi_2, \xi_3)$ and $\zeta = (\zeta_1, \zeta_2, \zeta_3) \in E^3$.

Let $\beta(s)$ be a differentiable curve of order (n+2) parametrized by arc-length, defined with the Frenet vector fileds $\{T, N, B\}$ and $\beta_0 = \int T ds$ be the tangential direction. In general the $k-$principal direction curve of β is determined as

$$\beta_k(s) = \int N_{k-1} ds, l \leqslant k \leqslant n$$

where the principal normal vector of β_{k-1} is N_{k-1}, β is named as the base curve of β_k. The Frenet vectors with the curvatures of β_k are defined as

$$T_k = N_{k-1}, \ N_k = \frac{N'_{k-1}}{\left\| N'_{k-1} \right\|}, \ B_k = N_k x T_k$$

$$\kappa_k = \sqrt{\left| \ \kappa_{k-1}^2 \pm \tau_{k-1}^2 \right|}, \tau_k = \sigma_{k-1}\kappa_k$$

where $\sigma_{k-1}, \kappa_{k-1}, \tau_{k-1}$ are the geodesic curvature, the first type of curvature and the second type of curvature of β_{k-1} respectively.

The Darboux vector W_k is the vector of angular velocity of the given frame of β_k and holds the following equations with regard to the frame apparatus:

i) Let β_k be a timelike curve, then

$$W_k = -\tau_k T_k - \kappa_k B_k$$

ii) Let β_k is a spacelike curve, then

$$W_k = -\tau_k T_k + \kappa_k B_k$$

Let $\{T_k, N_k, B_k, \kappa_k, \tau_k\}$ be the Frenet frame apparatus of the arc-length parametrized curve β_k in E_1^3. Then the equations below can be given:

$$\begin{pmatrix} T_k' \\ N_k' \\ B_k' \end{pmatrix} = \begin{pmatrix} 0 & \kappa_k & 0 \\ -\epsilon_0\epsilon_1\kappa_k & 0 & \tau_k \\ 0 & -\epsilon_1\epsilon_2\tau_k & 0 \end{pmatrix} \begin{pmatrix} T_k \\ N_k \\ B_k \end{pmatrix}$$

where $\langle T_k, T_k\rangle = \epsilon_0, \langle N_k, N_k\rangle = \epsilon_1, \langle B_k, B_k\rangle = \epsilon_2, \langle T_k, N_k\rangle = \langle T_k, B_k\rangle = \langle N_k, B_k\rangle = 0$ [9]

Definition 1. *Let $\beta(s) : I \longrightarrow \mathbb{R}^3$ be an arc-length parametrized curve with the Frenet frame apparatus $\{T, N, B, \kappa, \tau\}$. The curve β is called k-slant helix if the unit vector*

$$\beta_{k+1} = \frac{\beta_k\prime}{\|\beta_k\prime\|}$$

makes a constant angle with a fixed direction. Here $\beta_0 = \beta_s$ and $\beta_1 = \frac{\beta_0\prime}{\|\beta_0\prime\|}$ [1].

Definition 2. *Let $\beta(s) : I \longrightarrow \mathbb{R}^3$ be a differentiable curve of order (n+2) parametrized by arc-length, defined with the Frenet vector fields $\{T, N, B\}$ and $\beta_0 = \int Tds$ be the tangential direction. In general the k-principal direction curve of β is determined as*

$$\beta_k(s) = \int N_{k-1} ds, 1 \leq k \leq n$$

where the principal normal vector of β_{k-1} is N_{k-1}, $\beta_0(s) = \beta(s)$ and $N_0 = N$. Then the curve β is called N_k-slant helix which has the property that the principal normal vector N_k of β_k makes a constant angle with a fixed line. In other words β is called N_k-slant helix if β_k is a slant helix [10].

Thus we can give the following theorem:

Theorem 2. *$\beta(s) : I \longrightarrow \mathbb{R}^3$ is a (k+1)-slant helix if and only if β is a N_k-slant helix.*

Definition 3. *Let $\beta(s)$ be a non-degenerate, arc-length parametrized and differentiable curve of order (n+2), the $k-$principal direction curve of β is β_k in E_1^3. If the principal normal vector of β_k has steadily angle along a fixed axis, then β named as N_k slant helix [11].*

Note that, if k = 0 then the main curve β is named as slant helix, that is the principal normal vectors along β make a constant angle with an axis.

Theorem 3. *Let $\beta(s)$ be an arc-length parametrized, non-degenerate, differentiable curve of order (n+2) and β_k be the $k-$principal direction curve of β in E_1^3. Then β_k is a slant helix if and only if β is a N_k slant helix [11].*

To characterize a N_k slant helix in E_1^3 the following results are stated.

Theorem 4. ***i)*** *Assume that β is an arc-length parametrized timelike curve in E_1^3. Then β is a N_k slant helix if and only if either one of the next two functions is constant.*

$$\sigma_k(s) = \frac{\kappa_k^2}{(\tau_k^2 - \kappa_k^2)^{\frac{3}{2}}}(\frac{\tau_k}{\kappa_k})' \quad or \quad \sigma_k(s) = \frac{\kappa_k^2}{(\kappa_k^2 - \tau_k^2)^{\frac{3}{2}}}(\frac{\tau_k}{\kappa_k})'$$

where $\tau_k^2 - \kappa_k^2 \neq 0$.
ii) *Assume that β is an arc-length parametrized spacelike curve with the Frenet vectors $\{T, N, B, \kappa, \tau\}$ in E_1^3. Then there are two conditions for N_k slant helix case.*
a) *Assume that the unit normal vector of β is spacelike then β is a N_k slant helix if and only if either one of the below functions is constant where $\tau_k^2 - \kappa_k^2 \neq 0$*

$$\sigma_k(s) = \frac{\kappa_k^2}{(\tau_k^2 - \kappa_k^2)^{\frac{3}{2}}}(\frac{\tau_k}{\kappa_k})' \quad or \quad \sigma_k(s) = \frac{\kappa_k^2}{(\kappa_k^2 - \tau_k^2)^{\frac{3}{2}}}(\frac{\tau_k}{\kappa_k})'$$

b) *Assume that the unit normal vector of β is timelike. Then β is a N_k-slant helix if and only if the following function is constant*

$$\sigma_k(s) = \frac{\kappa_k^2}{(\tau_k^2 + \kappa_k^2)^{\frac{3}{2}}}(\frac{\tau_k}{\kappa_k})'$$

[10].

Definition 4. *A ruled surface in E_1^3 is the transformation $F_{(\gamma,\beta)}(s,u) = \gamma(s) + u\beta(s)$, where I is an open interval and $\gamma : I \longrightarrow E_1^3, \beta : I \longrightarrow E_1^3 \backslash \{0\}$ are smooth mappings. γ is called the base curve and β is called the director curve of the surface. The lines $u \longrightarrow \gamma(s) + u\beta(s)$ are named as rulings.*
A developable surface, also known as a flat surface, is a ruled surface where the Gaussian curvature K is zero everywhere. Suppose that γ be an arc-length parametrized curve in E_1^3 with $\kappa(s) \neq 0$. We handle three types of developable surfaces associated to a non-degenerate space curve in Lorentz 3-space.

1) A ruled surface $F_{(\gamma,\widetilde{W})}(s,u) = \gamma(s) + u\widetilde{W}(s)$ is called the rectifying developable of γ.

2) A ruled surface $F_{(B,T)}(s,u) = B(s) + uT(s)$ is called the Darboux developable of γ .

3) A ruled surface $F_{(\bar{W},N)}(s,u) = \bar{W}(s) + uN(s)$ is called the tangential Darboux developable of γ .

Here $\widetilde{W}(s)$=-$\frac{\tau}{\kappa}(s)T(s) + \epsilon_0 B(s)$ is the modified Darboux vector field of γ , on condition that $\kappa(s) \neq 0$.

Here $\epsilon_0 = -1$ if γ is timelike and $\epsilon_0 = 1$ if γ is spacelike. Also $\bar{W}(s)$ is the unit Darboux vector field of γ [6].

3. Construction of Developable Surfaces by Direction Curves

In this part we give a generalization of developable surfaces in terms of k-order frame in E_1^3 and obtain some results.

Definition 5. *Assume that $\gamma(s)$ is an arc-length parametrized non-degenerate, differentiable curve of order (n+2) and γ_k is the $k-$principal direction curve of $\gamma, 1 \leqslant k \leqslant n$ and the Darboux vector of γ_k is W_k in E_1^3.*

1) The k-rectifying developable of γ is defined as the ruled surface given by $F_{(\gamma,\widetilde{W}_k)}(s,u) = \gamma(s)+u\widetilde{W}_k(s)$.

2) The k-Darboux developable of γ is represented by the ruled surface $F_{(B_k,T_k)}(s,u) = B_k(s) + uT_k(s)$.

3) The k-tangential Darboux developable of γ is characterized by the ruled surface $F_{(\bar{W}_k,N_k)}(s,u) = \bar{W}_k(s) + uN_k(s)$.

Here $\widetilde{W}_k(s)$=$-\frac{\tau_k}{\kappa_k}(s)T_k(s) + \epsilon_0 B_k(s)$ is the modified Darboux vector field of γ, under the condition that $\kappa_k(s) \neq 0$. Here ϵ_0 is -1 if γ is timelike and 1 if γ is spacelike. And $\bar{W}_k(s)$ is the unit Darboux vector of γ_k .

Theorem 5. *Assume that $\gamma(s)$ is an arc-length parametrized (n+2)-differentiable curve and γ_k and γ_{k-1} be the $k-$principal direction curve and $(k-1)$ $-$principal direction curve of γ respectively in E_1^3.*
i) *Let γ_{k-1} and γ_k be non-degenerate curves in E_1^3. Then there is a diffeomorphism between the k-Darboux developable of γ and the cuspidal edge at $F_{(B_k,T_k)}(s_0,u_0)$ if and only if $\sigma_{k-1}(s_0) \neq 0, (\frac{\tau_k}{\kappa_k})' \neq 0$ and $u_0 = -\epsilon\frac{\tau_k}{\kappa_k}(s_0)$.*
ii) *Let γ_{k-1} and γ_k be non-degenerate curves in E_1^3. Then the k-Darboux developable of γ is diffeomorphic to the swallowtail at $F_{(B_k,T_k)}(s_0,u_0)$ if and only if $\sigma_{k-1}(s_0) \neq 0, (\frac{\tau_k}{\kappa_k})' = 0$ $(\frac{\tau_k}{\kappa_k})'' \neq 0$ and $u_0 = -\epsilon\frac{\tau_k}{\kappa_k}(s_0)$.*
iii) *Let γ_{k-1} and γ_k be non-degenerate curves in E_1^3. Then a diffeomporhism exists between the Darboux developable of γ and the cuspidal crosscap at $F_{(B_k,T_k)}(s_0,u_0)$ if and only if $\sigma_{k-1}(s_0) = 0, (\frac{\tau_k}{\kappa_k})' \neq 0$ and $u_0 = 0$.*
Here $\epsilon = -1$ if γ_k is timelike and $\epsilon = 1$ if γ_k is spacelike.

Proof. Because of other cases are similar we only give the first proof.
Let γ_{k-1} be a timelike curve. Then $\gamma_k(s) = \int N_{k-1}ds$ is a spacelike curve and the Frenet frame apparatus

of γ_k is $\{T_k, N_k, B_k, \kappa_k, \tau_k\}$.Thus we have the Frenet formulas

$$\begin{bmatrix} T_k' \\ N_k' \\ B_k' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_k & 0 \\ -\kappa_k & 0 & \tau_k \\ 0 & \tau_k & 0 \end{bmatrix} \begin{bmatrix} T_k \\ N_k \\ B_k \end{bmatrix}$$

The k-Darboux developable of γ is $F_{(B_k,T_k)}(s,u) = B_k + uT_k$ and a straight forward computation gives us the singular point of the k-Darboux developable as $u_0 = -\frac{\tau_k}{\kappa_k}(s_0)$.

The cuspidal edge singularities are obtained along points where $\gamma', \gamma'', \gamma'''$ are linearly independent. So if we do the necessary computations with the value of $u_0 = -\frac{\tau_k}{\kappa_k}(s_0)$, there is a diffeomorphism between the k-Darboux developable of γ and the cuspidal edge when $\sigma_{k-1}(s_0) \neq 0, (\frac{\tau_k}{\kappa_k})'(s_0) \neq 0$. □

As an extension of theorem 1, the following theorem yields a local charazterization for the k-tangential Darboux developable of a space curve.

Theorem 6. *Assume that $\gamma(s)$ is an arc-length parametrized, non-degenerate, differentiable curve of order (n+2) and γ_k and γ_{k-1} be the $k-$principal direction curve and $(k-1)$ $-$principal direction curve of γ respectively in E_1^3.*
i) *Let γ_{k-1} and γ_k be non-degenerate curves in E_1^3. Then there is a diffeomorphism between the k-tangential Darboux developable of γ and the cuspidal edge at $F_{(\bar{W}_k,N_k)}(s_0,u_0)$ if and only if $\sigma_k(s_0) \neq 0, \sigma_k'(s_0) \neq 0$ and $u_0 = \sigma_k(s_0)$.*
ii) *Let γ_{k-1} and γ_k be non-degenerate curves in E_1^3. Then there exists a diffeomorpism between k-tangential Darboux developable of γ and the swallowtail at $F_{(\bar{W}_k,N_k)}(s_0,u_0)$ if and only if $\sigma_k(s_0) \neq 0, \sigma_k''(s_0) \neq 0$ and $\sigma_k'(s_0) = 0$ and $u_0 = \sigma_k(s_0)$.*
iii) *Let γ_{k-1} and γ_k be non-degenerate curves in E_1^3. Then the k-tangential Darboux developable of γ is diffeomorphic to the cuspidal crosscap at $F_{(\bar{W}_k,N_k)}(s_0,u_0)$ if and only if $\sigma_k(s_0) = 0, \sigma_k'(s_0) \neq 0$ and $u_0 = 0$.*

Proof. Because of other cases are similar we only give the first proof.
Let γ_{k-1} be a timelike curve. Then $\gamma_k(s) = \int N_{k-1}ds$ is a spacelike curve and the Frenet frame apparatus of γ_k is $\{T_k, N_k, B_k, \kappa_k, \tau_k\}$. Thus we have the Frenet formulas

$$\begin{bmatrix} T_k' \\ N_k' \\ B_k' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_k & 0 \\ -\kappa_k & 0 & \tau_k \\ 0 & \tau_k & 0 \end{bmatrix} \begin{bmatrix} T_k \\ N_k \\ B_k \end{bmatrix}$$

The k-tangential Darboux developable of γ is $F_{(\bar{W}_k,N_k)}(s,u) = \bar{W}_k(s) + uN_k(s)$ and a straightforward computation gives us the singular point of the k-tangential Darboux developable as $u_0 = \sigma_k(s_0)$.

As we mentioned before the cuspidal edge singularities are obtained at the points where $\gamma', \gamma'', \gamma'''$ are linearly independent. So if we do the necessary computations with the value of $u_0 = \sigma_k(s_0)$, the k-tangential Darboux developable of γ is diffeomorphic to the cuspidal edge when $\sigma_k(s_0) \neq 0, \sigma_k'(s_0) \neq 0$. □

Theorem 7. *Suppose that $\gamma(s)$ is an ac-length parametrized, non-degenerate, differentiable curve of order (n+2) and γ_k and γ_{k-1} be the $k-$principal direction curve and $(k-1)$ $-$principal direction curve of γ respectively in E_1^3.*
i) *Let γ_{k-1} and γ_k be non-degenerate curves in E_1^3. Then there is a diffeomorpism between k-rectifying developable of γ and the cuspidal edge at $F_{(\gamma,\tilde{W}_k)}(s_0,u_0)$ if and only if $(\frac{\tau_k}{\kappa_k})'(s_0) \neq 0, (\frac{\tau_k}{\kappa_k})''(s_0) \neq 0$ and $u_0 = \frac{1}{(\frac{\tau_k}{\kappa_k})'(s_0)}$.*
ii) *Let γ_{k-1} and γ_k be non-degenerate curves in E_1^3. Then the k-rectifying developable of γ is diffeomorphic to the swallowtail at $F_{(\gamma,\tilde{W}_k)}(s_0,u_0)$ if and only if $(\frac{\tau_k}{\kappa_k})'(s_0) \neq 0, (\frac{\tau_k}{\kappa_k})''(s_0) = 0$ and $u_0 = \frac{1}{(\frac{\tau_k}{\kappa_k})'(s_0)}$.*

Proof. With the use of Theorem 3, the proof can be obtained easily. □

Theorem 8. *Suppose that $\gamma(s)$ is an arc-length parametrized, non-degenerate, differentiable curve of order (n+2) and γ_k the $k-$principal direction curve of γ in E_1^3. The equivalent cases hold as follows:*
i) *The k-tangential Darboux developable of γ is a conical surface.*
ii) *γ_k is a slant helix.*
iii) *γ is a N_k slant helix.*
iv) *γ is a $(k+1)-$slant helix.*

Proof. The singular locus of the k-tangential Darboux developable is given by $\beta_k(s) = \bar{W}_k(s)+\sigma_k(s)N_k(s)$. Thus $F_{(\bar{W}_k,N_k)}$ is a conical surface if and only if $\beta_k'(s) = 0$. Let $F_{(\bar{W}_k,N_k)}$ be a conical surface. Then we have $\bar{W}_k'(s) - \sigma_k'(s)N_k(s) - \sigma_k(s)N_k'(s) = 0$ and we can easily see that $\bar{W}_k'(s) = \sigma_k(s)N_k'(s)$. Thus $\sigma_k'(s) = 0$ and γ_k is a slant helix and γ is a N_k slant helix.

Conversely if γ is a N_k slant helix then γ_k is a slant helix. Thus $\sigma_k'(s) = 0$ and $\beta_k'(s) = \bar{W}_k'(s) - \sigma_k(s)N_k'(s)$. Since $\bar{W}_k'(s) = \sigma_k(s)N_k'(s)$ we have $\beta_k'(s) = 0$ and $F_{(\bar{W}_k,N_k)}$ is a conical surface. □

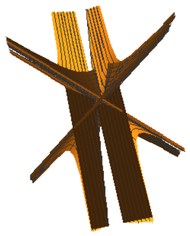
Theorem 9. *Assume that $\gamma(s)$ is a unit speed, differentiable curve of order (n+2) and γ_k is the $k-$principal direction curve of γ in E_1^3. Then the following cases are equivalent.*
i) *The k-rectifying developable of γ is a conical surface.*
ii) *γ_k is a conical geodesic curve.*

Proof. The singular locus of the k-rectifying developable of γ is given by $\beta_k(s) = \gamma_k + \frac{1}{(\frac{\tau_k}{\kappa_k})'}\widetilde{W}_k(s)$. Let $F_{(\gamma,\widetilde{W}_k)}$ be a conical surface. According to the k-frames formulas we have $\beta_k'(s) = -\frac{(\frac{\tau_k}{\kappa_k})''}{(\frac{\tau_k}{\kappa_k})'}\widetilde{W}_k(s)$. Therefore $\beta_k'(s) = 0$ if and only if $(\frac{\tau_k}{\kappa_k})'' = 0$. This completes the proof. □

Now let us explain the concepts of the present paper via some examples, thus we can show the connection between the role of the k order frame and determining the type of singularity.

Example 1. *Let $\alpha(s) = (\frac{1}{24}\sin 8s + \frac{2}{3}\sin 2s,\ -\frac{1}{24}\cos 8s + \frac{2}{3}\cos 2s,\ \frac{4}{15}\sin 5s) \in \mathbb{E}_1^3$ be an arc-length parametrized spacelike curve in $\mathbb{E}_1^3$. Then the developable surfaces of α and corresponding singular points are formed in terms of first and second order frames as follow:*

i) *The 1-Rectifying developable surface of α is $F_{(\alpha,\tilde{W}_1)} = \alpha + u\tilde{W}_1$ determined with the modified Darboux vector $\tilde{W}_1 = (\frac{5}{3}\cos 3s\csc 5s,\ (5+10\cos 2s)/(3+6\cos 2s+6\cos 4s),\ \frac{4}{3}\csc 5s)$ where $\left(\frac{\tau_1}{\kappa_1}\right)'(s_0) \neq 0$ and $\left(\frac{\tau_1}{\kappa_1}\right)''(s_0) \neq 0$ at $s_0 \neq \frac{1}{10}(2\pi n+\pi)$, $n \in \mathbb{Z}$, Theorem 7 explains that $F_{(\alpha,\tilde{W}_1)}$ is locally diffeomorphic to the cuspidal edge at the points $u_0 = \frac{1}{5}\sin[5s_0]^2$ for all $s_0 \neq \frac{1}{10}(2\pi n+\pi)$, and otherwise is locally diffeomorphic to the swallowtail. This implies the points are given by $u_0 = \frac{1}{5}\sin[5s_0]^2$ and $s_0 = \frac{1}{10}(2\pi n+\pi)$.*
ii) *The 1-Darboux developable surface of α, $F_{(B_1,T_1)} = B_1 + uT_1$ is obtained as where $\sigma = 5/4$ and $\left(\frac{\tau_1}{\kappa_1}\right)'(s_0) \neq 0$ for all s_0, then $F_{(B_1,T_1)}$ is just locally diffeomorphic to the cuspidal edge at the points $u_0 = -\cot[5s_0]$ and $\frac{5s_0}{\pi} \notin \mathbb{Z}$ from Theorem 5.*



(A) 1-Rectifying developable surface of α

(B) 1-Darboux developable surface of α

FIGURE 2. First order developable surfaces of a spacelike curve

iii) *The 1-Tangential Darboux developable surface of* α *is* $F_{(\overline{W}_1,N_1)} = \overline{W}_1 + uN_1$ *is obtained as where we have only one singular point at* $u_0 = 5$ *as can be seen in Figure 3.*

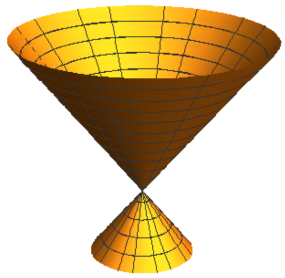


FIGURE 3. 1-Tangential Darboux developable surface of α

Now we observe the singular points of developable surfaces composed of the 2*-order frame* $\{T_2, N_2, B_2\}$ *of* α *with the curvatures* τ_2, κ_2*. Moreover, since the 2-tangent vector satisfies the equality such as* $\langle T_2, T_2\rangle < 0$, α *is a timelike curve with respect to the* 2*-order frame.*

(iv) *The 2-Rectifying developable surface of* α, $F_{(\alpha,\tilde{W}_2)} = \alpha + u\tilde{W}_2$ *is determined with the modified Darboux vector* $\tilde{W}_2 = (0, 0, -3/4)$ *, then we have* $\left(\frac{\tau_1}{\kappa_1}\right)'(s_0) = 0$ *and* $\left(\frac{\tau_1}{\kappa_1}\right)''(s_0) = 0$ *for all* $s_0 \in \mathbb{R}$*, thus 2-Rectifying developable surface of* α *has no singular points.*

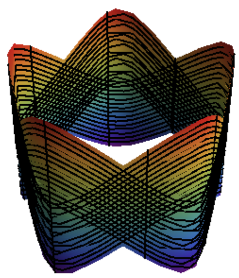


FIGURE 4. 2-Rectifying developable surface of α

(v) *The 2-Darboux developable surface of* α, $F_{(B_2,T_2)} = B_2 + uT_2$ *is can be obtained as where 2-Darboux developable surface of* α *is isometric to 1-Tangent Darboux developable surface of* α *then their points have the same type of singularity.*

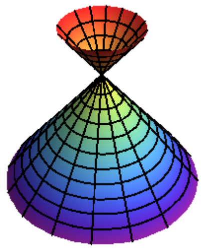


FIGURE 5. 2-Darboux developable surface of α

(vi) *The 2-Tangential Darboux developable surface of* α *is obtained as a plane defined by* $(u\sin[3s], u\cos[3s], -1)$, *hence there are no singular points and the trace of the points of the 2-Tangential Darboux developable surface* $F_{(\overline{W}_2,N_2)}$ *can be seen as follows: When we continue to express the spa-*

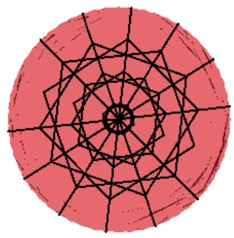


FIGURE 6. 2-Tangential Darboux developable surface of α

tial curve $\alpha \in \mathbb{E}_1^3$ *with respect to the other vector triples of the* $k-$*order process, we can eliminate the singularity for each Darboux developable surface.*

4. Conclusion

In differential geometry, developable surfaces are particular kind of surfaces that can be flattened onto a plane without distortion, that is, they can be unfurled into a flat shape without ripping or stretching. Particularly significant are these surfaces in the domains of computer graphics, manufacturing and architecture. An additional important tool in differential geometry are the alternative frames, which are the coordinate systems relative to a curve in the curve theory that provide different ways to describe the geometry and motion along the curve. In disciplines like computer graphics, robotics and physics these frames are especially helpful. They provide other viewpoints for interpreting and analyzing curves other than the widely used Serret-Frenet frame.

In this study we supply a generalized definition for developable surfaces by using an alternative frame produced via the direction curves through their rulings. A dynamic structure is resulted from the generalization and the theory of Ishikawa and Yamashita is utilized to characterize the singular points of these structures. The examples given in the article demonstrate the dynamic structure that arises from the alternative frame. Emotionally and raitonally, it is plainly observed that when the degree of the alternative frame utilized rise, the approach to the developable surfaces goes towards the light cone and the Lorentzian plane. This makes it easier to detect the evolution of singular points.

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