Süleyman Demirel Üniversitesi Fen Edebiyat Fakültesi Fen Dergisi https://dergipark.org.tr/sdufeffd
Süleyman Demirel University Faculty of Arts and Sciences Journal of Science e-ISSN: 1306-7575
Research Article, 2024, 19(1): 1-7, DOI: 10.29233/sdufeffd.1396580

L^∞ Spaces of Vector-Valued Functions as Spaces of Continuous Functions

Banu Güntürk[1]

[1]Department of Mechanical Engineering, Faculty of Engineering, Baskent University, 06790, Ankara, TÜRKİYE
https://orcid.org/0000-0002-7728-7228

(Received: 27.11.2023, Accepted: 16.12.2023, Published: 27.05.2024)

Abstract: It is proved that for any decomposable perfect measure space $(Z, \mathcal{A}, \mu)$, the space $L^\infty_{\omega^*}(\mu, E^*)$ of essentially bounded weak* measurable functions on Z to E^* is linearly isometric to the space $C(Z, E^*_*)$ of continuous functions on Z to E^*_*, the latter space is being provided with the supremum norm $\|g\|_\infty = \sup_{z \in Z} \|g(z)\|$, where E^*_* stands for the space E^* endowed with its weak* topology.

Key words: L^∞ Space, Vector-Valued Functions, Perfect Measure, Hyperstonean Space, Continuous Function Spaces

1. Introduction

If μ is a perfect measure on an extremally disconnected compact Hausdorff space X then the Banach space $L^\infty(\mu)$ of essentially bounded scalar measurable functions is linearly isometric to $C(X)$, the space of scalar continuous functions on X provided with the usual supremum norm [1] or [20]. For an arbitrary μ, we may employ the Gelfand-Naimark theorem to achieve the same result, that is, $L^\infty(\mu)$ is isometric to $C(Y)$ where Y denotes the maximal ideal space of $L^\infty(\mu)$, [3], [17] or [12]. In this article, we shall generalize this theorem to L^∞spaces of vector-valued functions. We shall restrict our study to perfect measures, and the range space will be a Banach dual E^* for continuous functions, where E^*_* stands for the dual space E^* provided with its weak* topology.

2. Material and Method

First, let us recall some (not entirely standard) terminology for integration of vector-valued functions. We will call two measure spaces $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \mu)$ *equivalent* if each number $1 \leq p < \infty$ and each Banach space E, the Bochner spaces $L^p(\mu, E)$ and $L^p(\nu, E)$ are linearly isometric. For basic information about these spaces, see references [8,9]. As pointed out in [6] equivalent measure spaces may have different L^∞ spaces.

Following [1] we will call a measure space $(X, \mathcal{A}, \mu)$ *perfect* if X is an extremally disconnected locally compact Hausdorff space, $\mathcal{A}$ contains the Borel algebra and μ is a positive measure on $\mathcal{A}$ such that every nonempty open set contains a clopen set K, where $\mu(K) > 0$, and for every closed set C with empty interior, $\mu(C) = 0$.

In [6], Cengiz proves that an arbitrary measure space (X, Σ, μ) is equivalent to a perfect measure space $(\Omega, \mathcal{A}, \mu)$ with the following additional properties:

i. $\Omega = \sum_{i \in I} \oplus \Omega_i$, where $\{\Omega_i : i \in I\}$ of mutually disjoint extremally disconnected compact Hausdorff spaces Ω_i, $i \in I$,
ii. if a subset S of Ω is measurable then $S \cap \Omega_i$ is measurable for each $i \in I$, and the converse is also true,

iii. the restriction of μ to each Ω_i is a regular Borel measure on Ω_i,
iv. for each $A \in \mathcal{A}$, $\mu(A) = \sum_{i \in I} \mu(A \cap \Omega_i)$,
v. every σ –finite measurable set is contained a.e. (almost everywhere) in the union of a countable subfamily of $\{\Omega_i : i \in I\}$, and
vi. every measurable set A is equivalent to a clopen set C, that is, $\mu(A \Delta C) = 0$.

Note that the measure space $(\Omega, \mathcal{A}, \mu)$ is decomposable [14, p. 317]. Since μ is perfect, for every open set U, $\mu(\overline{U}) = \mu(U)$, and from (iv) it follows that every locally null set is actually null. (Recall that a measurable set is locally null if its intersection with every set of finite measure is null.)

Let Z denote the Stone-Čech compactification of Ω. Then, obviously, Z is extremally disconnected, and since Ω is locally compact, it is open in Z, [11, p.245] or [15, p. 90]. Using these facts, it is easily shown that the extension of μ to the Borel algebra $\mathcal{B}$ of Z, which we will continue to denote by μ, by defining the measure of any B in $\mathcal{B}$ to be the measure of $B \cap \Omega_i$ is indeed a perfect measure on Z. Since $Z \setminus \Omega$ is a null set, we will use Z and Ω interchangeably as the ground set. Hence, every measure is equivalent to a decomposable perfect measure on an extremally disconnected (locally) compact Hausdorff space.

Following [10] we call an exteremally disconnected compact Hausdorff space T *hyperstonean* if the union of the supports of the positive *normal* measures is dense in T, which is equivalent to having a perfect measure on T [1]. (We recall that a regular Borel measure ν on T is normal if $\nu(B) = 0$ for every Borel set of first category.) This condition ensures that $C(T)$ is a dual space [16]. Thus, each measure space is equivalent to a *hyperstonean measure space.*

L^∞ Spaces. Let (X, Σ, μ) be any measure space and E be a Banach space. Let us recall that a function $f: X \to E$ is *strongly measurable (or simply measurable)* if it is the almost everywhere limit in the norm topology of a sequence of measurable simple functions, and *locally measurable* if its restriction to each measurable set of finite measure is measurable. A locally measurable function $f: X \to E$ is *essentially bounded* if for some $\alpha > 0$, the set $\{x \in X: \|f(x)\| > \alpha\}$ is locally null, and the infimum of such numbers α is the essential supremum norm $\|f\|_\infty$ of f. $L^\infty(\nu, E)$ will stand for the Banach space of all essentially bounded locally measurable functions on X to E.

A function $g: X \to E^*$ is *weak* measurable* if for each $e \in E$, the composite function $\hat{e} \circ g$ is measurable, where $\hat{e}$ denotes the image of e in the second dual under the canonical embedding.

Throughout the rest of this paper we will be discussing the L^∞ space of E^*-valued functions rather than E-valued ones and $(\Omega, \mathcal{A}, \mu)$ will denote a fixed perfect measure space with additional properties (i) - (iv) mentioned earlier and Z will stand for the Stone-Čech compactification of Ω. The unique extension of μ to a perfect measure on the Borel algebra $\mathcal{B}$ of Z will still be denoted by μ.

For each $g \in L^\infty(\mu, E^*)$, the mapping ψ_g, defined on $L^1(\mu)$ by

$$\psi_g(f) = \int_\Omega \langle f, g \rangle d\mu$$

for all $f \in L^1(\mu)$, is a bounded linear functional with norm $\|\psi_g\| = \|g\|_\infty$, where $\langle f, g\rangle(\omega) = \langle f(\omega), g(\omega)\rangle = g(\omega)\big(f(\omega)\big)$, $\omega \in \Omega$. This is a well-known result for σ-finite measures [8, p.98], and has been generalized recently to perfect measures [7]. And the isometry $\psi: g \to \psi_g$ from $L^\infty(\mu, E^*)$ into $L^1(\mu, E)^*$ is surjective if and only if E^* has the Radon-Nikodým property (RNP) with respect to μ. It means that, each μ-continuous E^*-valued measure of bounded variation on $\mathcal{A}$ to E^* can be represented (via integral) by an E^*-valued integrable function. (This was first proved by Banach and Taylor [2] for Lebesgue measure on the unit interval $[0,1]$ and generalized to σ-finite measures by Gretsky and Uhl [13], and its generalization to arbitrary perfect measures is due to Cengiz [7]. A nice proof for the σ-finite case can be found in [8].) In particular, for each reflexive Banach space E we have $L^\infty(\mu, E) \simeq L^1(\mu, E)^*$, for such spaces are dual spaces and have the RNP with respect to finite measures [8], and this property can be generalized to perfect measures as the following proposition shows. (If the measure space is not perfect this result may not hold even in the scalar case, [14] or [19].)

3. Results

Proposition 3.1 If a Banach space E has the RNP with respect to any finite measure then it has this property with respect to any perfect measure. Consequently, reflexive spaces have the RNP with respect to perfect measures.

Proof. We will prove this proposition for our fixed perfect measure μ. Let $\lambda : \mathcal{A} \to E$ be a μ-continuous measure has bounded variation. Then for each $i \in I$, there is a μ-integrable function $g_i : \Omega_i \to E$ which vanishes outside Ω_i satisfies the integration

$$\lambda(A) = \int_A g_i d\mu, \qquad \text{for all } A \in \mathcal{A}_i,$$

where $\mathcal{A}_i = \{A \cap \Omega_i : A \in \mathcal{A}\}$. Now let $g = \sum_i g_i$. Then clearly g is locally measurable. We claim that it is actually measurable.

Since λ has bounded variation, $|\lambda|(\Omega) < \infty$. Then we have $|\lambda|(\Omega_i) = 0$ for all but countably many $i \in I$, where $|\lambda|$ states for the total variation of λ. Thus, there exist a countable subset J of I such that the set $N_i = \{x \in \Omega_i : g(x) \neq 0\}$ is null for each $i \in I \setminus J$, and since $\bigcup_{i \in I \setminus J} N_i$ is locally null, it is actually null. Thus, it follows that g is measurable as claimed.

Since

$$|\lambda|(\Omega) = \int_\Omega \|g(.)\| d\mu$$

we conclude that g is integrable, and since the support of g is contained a.e. in $\bigcup_{j \in J} \Omega_j$, more simply we may thus suppose that $I = \{1,2, \dots\}$. Now, we have

$$\lambda(A) = \int_A g d\mu ,$$

for all $A \in \mathcal{A}$, proving our proposition.

The following proposition will be needed later.

Proposition 3.2 Every E^*-valued measurable function is weak* measurable.

Proof. Let $g : \Omega \to E^*$ be measurable. Then by the Pettis measurability theorem, [19] or [8], for each $i \in I$, the restriction g_i of g to Ω_i is weak* measurable. Thus for each $x \in E, \hat{x} \circ g_i$ is measurable, and therefore $\hat{x} \circ g$ is locally measurable, and hence (by Property (ii) of μ), it is measurable. This completes the proof.

Proposition 3.3 If $f : \Omega \to E$ is measurable and $g : \Omega \to E^*$ is weak* measurable then the scalar function $\langle f, g \rangle$ is measurable.

Proof. Let $g : \Omega \to E^*$ be a weak* measurable function, and let $s = x_1\chi_{A_1} + \cdots + x_n\chi_{A_n}$ be a measurable simple function from Ω to E, where for a set S, χ_S denotes the characteristic function of S. Then, since for each $k = 1,2, \dots, n,$ $\hat{x}_k \circ g$ and χ_{A_k} are measurable,

$$\langle s, g \rangle = \sum_{k=1}^{n} (\hat{x}_k \circ g)\, \chi_{A_k}$$

is measurable. Now let $f : \Omega \to E$ be a measurable function and $s_n : \Omega \to E$ be a sequence of measurable simple functions converging a.e. to f in the norm topology on E. Then,

$$\lim_n \langle s_n(\omega), g(\omega) \rangle = \langle f(\omega), g(\omega) \rangle \text{ a.e. on } \Omega,$$

which proves that $\langle f, g \rangle$ is measurable, is claimed.

$C(Z, E_*^*)$ will denote the space of all continuous functions f on Z to E_*^* provided with the supremum norm $\|f\|_\infty = \sup_{z \in Z} \|f(z)\|$.

Corollary 3.4 The elements of $C(Z, E_*^*)$ are weak* measurable.

It is tempting to call $g : \Omega \to E^*$ weak* measurable if $g^{-1}(B)$ is measurable for every weak* Borel subset B of E^*. The following proposition shows that this is true.

Proposition 3.5 Let $g : \Omega \to E^*$ be a function such that $g^{-1}(B)$ is measurable for each weak* measurable subset B of E^*. Then g is weak* measurable.

Proof. For each $x \in E,$ the functional $\hat{x}$ is weak* continuous and so, it is measurable with respect to the weak* Borel algebra on E_*^*. Thus, for each Borel set S in the field of complex numbers $(\hat{x} \circ g)^{-1}(S) = (g)^{-1}(\hat{x}^{-1}(S))$ is measurable. Hence g is weak* measurable.

Theorem 3.6 For our perfect measure space, $L_{\omega^*}^\infty(\mu, E^*) \simeq C(Z, E_*^*) \simeq L^1(\mu, E)^*$.

Proof. Let $g \in C(Z, E_*^*)$. Then for each $f \in L^1(\mu, E)$, the function $\langle f, g \rangle$ is measurable and since $|\langle f(.), g(.) \rangle| \leq \|f(.)\| \|g\|_\infty$, it is also integrable, and the mapping ψ_g defined on $L^1(\mu, E)$ by

$$\psi_g(f) = \int_\Omega \langle f, g \rangle d\mu \,, \qquad f \in L^1(\mu, E)$$

is a bounded functional with norm $\leq \|g\|_\infty$. Actually, since $(Z, \mathcal{B}, \mu)$ is a perfect measure space, by a theorem of Cambern and Greim [5] the mapping $g \to \psi_g$ is a linear isometry from $C(Z, E_*^*)$ onto $L^1(\mu, E)^*$. (The known proof of mentioned theorem depends on the observation that $C(Z, E_*^*)$ is isometric to the space $\mathcal{L}(E; C(Z))$ of bounded operators on E to $C(Z)$ which is proved explicitly in [4], therefore a more direct proof of the inequality $\|g\|_\infty \leq \|\psi_g\|$, will be welcome.) So, the space $L^\infty(\mu, E^*)$ is isometric to a subspace of $C(Z, E_*^*)$ and, this isometry is surjective if and only if E^* has the RNP with respect to μ.

A weak* measurable function $g : \Omega \to E^*$ may not be essentially bounded in the usual sense, or better, the definition of essential boundedness may not apply to g, for the function $\|g(.)\|$ need not be measurable, and therefore, the definition of essential boundedness for weak* measurable functions should be different. But, in view of Proposition 3.2, Corollary 3.4, and the fact that $\|g\|_\infty$ is the same as the norm of the operator ψ_g when g is either in $L^\infty(\mu, E)$ or $C(Z, E_*^*)$, what can be more natural than calling a weak* measurable function g essentially bounded if

$$\psi_g(f) = \int_\Omega \langle f, g \rangle d\mu$$

defines a bounded functional on $L^1(\mu, E)$, that is, for each $f \in L^1(\mu, E)$, $\langle f, g \rangle$ is integrable and there is a constant $k > 0$ such that

$$\left| \int_\Omega \langle f, g \rangle d\mu \right| \leq k \|f\|_1 \text{ for all } f \in L^1(\mu, E),$$

in which case, we define the essential supremum norm $\|g\|_\infty$ of g as the norm of the functional ψ_g on $L^1(\mu, E)$.

$L^\infty_{\omega^*}(\mu, E^*)$ will denote the normed space of all essentialy bounded weak* measurable functions on Ω to E^*, provided with the essential supremum norm.

For two normed spaces E and F, the notation $E \simeq F$ will indicate that they are linearly isometric.

We can identify $C(Z, E_*^*)$ with a subspace of $L^\infty_{\omega^*}(\mu, E^*)$ in the most natural way, and since the mapping $g \to \psi_g$ maps $C(Z, E_*^*)$ onto, and $L^\infty_{\omega^*}(\mu, E^*)$ into $L^1(\mu, E)^*$ we conclude that $C(Z, E_*^*) \simeq L^\infty_{\omega^*}(\mu, E^*)$. Hence we have completed the proof of the theorem.

Corollary 3.7 $L^\infty(\mu, E)^* = L^\infty_{\omega^*}(\mu, E^*)$ if and only if E^* has the RNP.

Corollary 3.8 $L^\infty(\mu, E)$ isometric to a subspace of $C(Z, E_*^{**})$, where E_*^{**} denotes the second dual with its weak* topology.

Proof. We identify $L^\infty(\mu, E)$ with the subspace $L^\infty(\mu, \hat{E})$ of $L^\infty(\mu, E^{**}) \backsimeq C(Z, E^{**}_*)$ where $\hat{E}$ denotes the image of E in E^{**} under the canonical embedding.

Corollary 3.9 Z is the maximal ideal space of $L^\infty(\mu)$.

Remark For a perfect measure space $(X, \mathcal{A}, \nu)$ with X compact, $L^\infty(\mu) \backsimeq C(X)$ was already known, [1] or [20].

4. Conclusion

In this paper, we prove an important isometry between the L^∞ space of vector-valued functions and the space of continuous functions on Z to E^*_*, where Z is the Stone-Čech compactification of the hyperstonean space Ω. Hyperstonean spaces are very important spaces with several properties [21] and they are also huge indeed. So, this relation is very crucial between the functional analysis and measure theory. Hence, the results obtained will shed light on important studies to be conducted on this subject in the future.

Authorship contribution statement

B. Güntürk: Conceptualization, Methodology, Investigation, Software, Review and Editing.

Declaration of competing interest

As the author of this study, I declare that I do not have any conflict of interest statement.

Acknowledgment

I am grateful to my teacher, the late Prof. Bahaettin Cengiz, who made a valuable contribution to this article.

Ethics Committee Approval and/or Informed Consent Information

As the author of this study, I declare that I do not have any ethics committee approval and/or informed consent statement.

References

[1] E. Behrends, et al, L^p-structure in real Banach spaces, Lecture Notes in Mathematics, 613, Berlin-New-York, Springer-Verlag, 1977.
[2] S. Bochner and R.E. Taylor, “Linear functionals on certain spaces of abstractly-valued functions”, *Annals of Mathematics*, 39(2), 913-944, 1938.
[3] F. Bonsall and J. Duncan, Complete Normed Algebras, Berlin-Heidelberg-New York, Springer-Verlag, 1973.
[4] M. Cambern and P. Greim, “The bidual of $C(X, E)$”, *Proceedings of the American Mathematical Society*, 85, 53-58, 1982.
[5] M. Cambern and P. Greim, “The dual of a space of vector measures”, *Mathematische Zeitschrift*, 180, 373-378, 1982.
[6] B. Cengiz, “On the duals of Lebesgue-Bochner L^p spaces”, *Proceedings of the American Mathematical Society*, 114, 923-926, 1992.
[7] B. Cengiz, “The isometries of the Bochner space $L^p(\mu, H)$ ”, *Turkish Journal of Mathematics*, 23(3), 1999.

[8] J. Diestel and J. J. Uhl Jr., Vector Measures, Mathematical Surveys and Monographs no.15, *American Mathematical Society*., Providence, Rhode Island, 1977.

[9] Dinculeanu N., Vector Measures, New-York, Pergamon Press, 1967.

[10] J. Dixmier, "Sur certains espaces considérés par M.H. Stone, *Summa Brasiliensis Math*ematicae, 2(11) , 151-182, 1951.

[11] J. Dugundji, Topology, Boston, Allyn and Bacon Inc., 1966.

[12] N. Dunford and J. T. Schwartz, Linear Operators, Part I, New York, Interscience, 1958.

[13] N. E. Gretsky and J. J. Uhl Jr., "Bounded linear operators on Banach function spaces of vector-valued functions", *Transactions American Mathematical Society*, 167, 263-277, 1972.

[14] E. Hewitt , K. Stromberg, Real and Abstract Analysis, New York, Springer-Verlag, 1965.

[15] L. Gilman and M. Jerison, Rings of Continuous Functions, Princeton-Toronto-London-Melbourne, D. Van Nostrand Company, 1960.

[16] H. E. Lacey, The Isometric Theory of Classical Banach Spaces, Berlin- Heidelberg-New York, Springer-Verlag, 1974.

[17] M. A. Naimark, Normed Algebras, The Netherlands, Wolters-Noordhoff Publishing, 1972.

[18] B. J. Pettis, "On integration in vector measures", *Transactions American Mathematical Society*, 44, 277-304, 1938.

[19] H. L. Royden, Real Analysis, 3rd ed., London-New York, Collier-Macmillan, 1988.

[20] H. H. Schaefer, Banach Lattices and Positive Operators, Berlin-Heidelberg-New York, Springer-Verlag, 1974.

[21] B. Güntürk, B. Cengiz, "On some properties of hyperstonean spaces", *Turkish Journal of Mathematics*, 5(42), 2288-2295, 2018.